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(expecting experimentalists as an audience)

One-particle motion in nuclear many-body problem

(The 3rd lecture, V.3)

Giant resonances (GR) and sum rules in stable and unstable nuclei

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The figures with figure-numbers but without reference, are taken from

When IVGDR was found in photo-neutron cross sections, it had a resonance shape, but the width was typically of the order of $5\,\text{MeV}$, which was an order of magnitude larger than the resonances known in nuclei at that time (~ 1960 ies). Thus, it was called “Giant Resonance”.

\[
\text{Photon energy resolution } = \text{ several hundreds } (< 500) \, \text{keV}.
\]
The hydrodynamical model consists of incompressible neutron and proton fluids.

Nuclei consist of nucleons, and the population of GR by, for example, $\gamma$-absorption is via one-particle operator. Thus, the presence of quantum-mechanical shell-structure of one-particle levels in nuclei sets a limitation on the applicability of the hydrodynamical model.

If a collective mode consumes an appropriate sum-rule

$\rightarrow$ Possibility of being approximately described by a macroscopic hydrodynamical model

Taking the simplest model for nuclei, namely harmonic-oscillator model, the above possibility exists if all one-particle excitations by a given operator have the same energy, one-frequency.

ex. IVGDR is this example; all one-particle excitations have $\Delta N=1$, $\Delta E = \hbar \omega_0$

GQR, since $\Delta N=0$ and 2.

The hydrodynamical model is not directly applicable to spin-dependent modes.
GRs in heavier nuclei have often a better resonance shape than those in lighter nuclei.

This may be due to the fact that in heavier nuclei:

(a) GR can be more collective, since many more $1p-1h$ configurations are available.
(b) The spread of energies of $1p-1h$ excitations ($\sim A^{-1/3}$) is smaller, while
the p-h interaction which couples $1p-1h$ excitations with different energies
has no strong $A$-dependence.

Thus, building up a collective state out of available $1p-1h$ configurations is easier.

Lighter nuclei have less clear distinction between the surface and the inside.
This makes a difference, for example, when a probe used is sensitive only to the surface
or GR is of a surface type.
Neutron-excess gives an essential difference in charge-exchange GR, \( t_\pm \) GR, from the case of N=Z nuclei.

ex. Some \( t_\pm \) GR may disappear due to the Pauli principle.

ex. \( E_x(t_-\text{GR}) > E_x(t_+\text{GR}) \) in the presence of neutron excess.

**Neutron excess** \( \rightarrow \) Excitations made by **Isoscalar** (i.e. isospin-independent) operators carry an **isovector transition density** \( \delta \rho_n - \delta \rho_p \neq 0 \).
Giant resonances and sum rules

7.1. Introduction

7.2. Sum rules

7.2.1. Sum rules for \( (1 \text{ or } t_z) \) excitations
- Classical oscillator sum (= energy-weighted sum)
- Sum-rule in axially-symmetric quadrupole-deformed nuclei

7.2.2. Sum rules for \( (t_{\pm}) \) charge-exchange excitations
- Difference, \( S_- - S_+ \), of non-energy-weighted sums

7.3. Giant resonances of IS or \( t_z \) type (excitations within the same nuclei)

7.3.1. Isovector giant dipole resonance (IVGDR)

7.3.2. Isoscalar and isovector giant quadrupole resonance (ISGQR and IVGQR)

7.3.3. Isoscalar giant monopole resonance (ISGMR) - compression mode
7.4. Giant resonances of charge-exchange (n→p or p→n) type  
(excitations to the neighboring nuclei)

7.4.1. Fermi transitions (IAS)
7.4.2. Gamow-Teller (GT) resonance (incl. magnetic giant dipole resonance)
7.4.3. Isovector spin giant monopole resonance (IVSGMR)
7.4.4. Isovector spin giant dipole resonance (IVSGDGR)

7.5. Giant resonances in nuclei far away from the stability line

7.5.1. ISGQR of nuclei with weakly-bound neutrons  
- an example of threshold strength

7.5.2. $\beta$-decay to GTGR in drip line nuclei

$\beta^-$ decay to GTGR$_-$ in very neutron-rich light nuclei
$\beta^+$ decay to GTGR$_+$ in medium-heavy (N>Z) proton-drip-line nuclei

References:


7. Collective motion based on particle-hole excitations
   - giant resonances and sum-rules

7.1. Introduction

Collective motion:

Many nucleons participate coherently in the motion so that a given observable
(transition) is much enhanced compared with a single-particle estimate.

The best-established collective motion in nuclei is rotational motion of deformed nuclei.

The properties of very low-energy collective states are sensitive both to pair correlations
and to the shell-structure around the Fermi levels. Only those particles close to the Fermi levels contribute to the pair correlation.

In contrast, many (if not all) particles in a nucleus participate in giant resonances (GR), so that
(a) the properties of GR are almost independent of the shell-structure around the Fermi level,
(b) depend on the bulk properties, and
(c) are expressed as a smooth function of $Z$, $N$ and $A$. 
The total transition strength should be limited by a "sum rule", which depends on the ground-state properties. Due to the collective nature, GR consumes the major part of the sum rule that is defined for respective collectivity.

Then, GR may correspond to a classical picture of collective motion.

Usefulness of sum-rules

If an observed peak consumes the major part of the sum-rule, the peak expresses a collective mode. Moreover, there are almost no other collective excitations carrying the strength of the same operator $F$, while the mode created with the operator acting on the ground state is approximately an eigenstate of the Hamiltonian.
Examples of Giant Resonances experimentally studied in $\beta$-stable nuclei are

(a) Excitations in the same nuclei (IS = Isoscalar, IV = Isovector)

<table>
<thead>
<tr>
<th>spin-parity</th>
<th>operator</th>
<th>observed peak energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>IS GMR$^*$</td>
<td>$0^+$ $\sum_k r_k^2$</td>
<td>$80 \ A^{-1/3} \text{ MeV (for } A &gt; 90)$</td>
</tr>
<tr>
<td>IS GDR$^*$</td>
<td>$1^-$ $\sum_k r_k^3 Y_{1\mu}(\hat{r}_k)$</td>
<td>$79 \ A^{-1/3} \text{ MeV (for } A &gt; 50)$</td>
</tr>
<tr>
<td>IV GDR</td>
<td>$1^-$ $\sum_k \tau_z(k)\hat{r}_k$</td>
<td>$79 \ A^{-1/3} \text{ MeV (for } A &gt; 50)$</td>
</tr>
<tr>
<td>IV GQR</td>
<td>$2^+$ $\sum_k r_k^2 Y_{2\mu}(\hat{r}_k)$</td>
<td>$63 \ A^{-1/3} \text{ MeV (for } A &gt; 60)$</td>
</tr>
<tr>
<td>IV spin GR</td>
<td>$1^+$ $\sum_k \tau_z(k)\vec{\sigma}_k$</td>
<td>GRs have width of several MeV (except IAS) and exhaust the major part of respective sum-rule.</td>
</tr>
</tbody>
</table>

(b) Excitations to neighboring nuclei

<table>
<thead>
<tr>
<th>spin-parity</th>
<th>operator</th>
<th>* compression mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>IAS</td>
<td>$0^+$ $\sum_k t_{\pm}(k)$</td>
<td></td>
</tr>
<tr>
<td>GT GR</td>
<td>$1^+$ $\sum_k t_{\pm}(k)\vec{\sigma}_k$</td>
<td></td>
</tr>
<tr>
<td>IV GQR</td>
<td>$2^+$ $\sum_k t_{\pm}(k)r_k^2 Y_{2\mu}(\hat{r}_k)$</td>
<td></td>
</tr>
<tr>
<td>IV spin GMR$^*$</td>
<td>$1^+$ $\sum_k t_{\pm}(k)\vec{\sigma}<em>k (r_k^2 - \langle r^2 \rangle</em>{\text{excess}})$</td>
<td></td>
</tr>
<tr>
<td>IV spin GDR</td>
<td>$0^-$, $1^-$, $2^-$ $\sum_k t_{\pm}(k)r_k(Y_1(\hat{r}_k)\vec{\sigma}_k)_J^\pi$</td>
<td></td>
</tr>
</tbody>
</table>

* compression mode

---

$\sum$ denotes summation over the continuum.
Examples of selection rules in spherically-symmetric harmonic-oscillator potential

1) Operator $rY_{1\mu}$ (or $x, y, z$) $\Rightarrow \Delta N=1$ ($E_x = \hbar \omega_0$) excitations

Closed-shell configuration

Partially-occupied $N_F$ shell

2) Operator $r^2 Y_{2\mu}$ (or $x^2, y^2, z^2$) $\Rightarrow \Delta N=0$ or $2$ ($E_x = 0\hbar \omega_0$ or $2\hbar \omega_0$) excitations

Closed-shell configuration

Partially-occupied $N_F$ shell

In realistic potentials the above selection rules do not exactly work, but work approximately.
Observed one-particle energies are not well reproduced by Hartree-Fock calculations using Skyrme interactions with $m^* (= (0.6-0.8) \ m)$.  

In contrast, observed energy of ISGQR are often reproduced by RPA based on the Hartree-Fock calculation with the same Skyrme interaction (so-called self-consistent RPA).

Note that the parameters related to ISGQR are well taken care of, when Skyrme parameters are determined.

In this lecture we do not further go into detail of [Skyrme H.F. + RPA] calculation.

Instead, we try to understand GRs, sometimes using the result of [Skyrme H.F. + RPA] calculation, but mostly using the models which are as simple as possible.
Shape oscillations - typical vibrational excitations when nuclear matter is incompressible.

Compression modes → information on nuclear compressibility

\[ \rho_{n0}^{\mu}(\vec{r}) \equiv \left\langle n \sum_{k=1}^{A} \delta(\vec{r} - \vec{r}_k) \right| 0 \right\rangle \equiv \rho_{\lambda\mu}^r(r) Y_{\lambda\mu}(\hat{r}) \]

radial transition density

\[ 3\rho_0(r) + r \frac{d\rho_0(r)}{dr} \]

G.F. Bertsch and S.F. Tsai
Physics Reports 18, (1975) 125.

Tassie model

\[ r \frac{d\rho_0(r)}{dr} \]

Fig. 10. Monopole transition density in $^{208}$Pb. The solid curve is the result for the transition density operator $R^2 (2 \mu \rho(r) - \rho)$, for the SkI interaction between the ground and the 20 MeV state. Units are fm$^{-1}$. The dotted curve is the prediction of the Tassie model, eq. (43b). Here, the overall magnitude of the curve has no significance.

Fig. 9. Isoscalar quadrupole transition density ($4 \lambda^2$) in $^{208}$Pb with the SkI interaction to the giant state predicted at 11.4 MeV. The solid curve is the calculation; the dashed curve is the collective model, eq. (43a).
In heavier nuclei GR may show a resonance (Lorentzian ?) shape and the properties can be systematic, while those of GR in medium weight and light nuclei are more individual. In very light nuclei GR strength distribution is split into several fragments.

∴ In lighter nuclei the collectivity is weaker, or a number of p-h configurations to contribute to GR is smaller.

In lighter nuclei the difference of the relevant p-h excitation energies may be large compared with the interaction between them,

Transition densities of GR with good accuracy is not experimentally available.

Example of transition density of IS shape oscillation ;

3− state of 208 Pb at Ex = 2.61 MeV


I.H., P.L. 66B (1977) 410
7.2. Sum rules

In this lecture we treat nucleons as elementary particles, neglecting possible contributions from internal degree of freedom of nucleons.

⇒ Valid in the energy interval, well below internal excitations of nucleons.

7.2.1. Sum rules for (1 or \( t_z \)) excitations

**Classical oscillator sum** - sum of energy-weighted transition strength

\[
S(F_{\lambda}) = \sum_a (E_a - E_0) B(F_{\lambda}; 0 \rightarrow aI_a) = \sum_a (E_a - E_0) \left| \langle a | F_{\lambda} | 0 \rangle \right|^2 \\
= \frac{1}{2} \langle 0 | [ F_{\lambda}, [H, F_{\lambda}] ] | 0 \rangle \quad \text{where} \quad H = \sum_i t_i + \sum_{i<j} v_{ij} \quad H | 0 \rangle = E_0 | 0 \rangle \quad H | a \rangle = E_a | a \rangle
\]

If \( v_{ij} \) does not explicitly depend on the momentum of particles, \( v_{ij}(\vec{p}_k) \), and if one-particle operator \( F_{\lambda} \) depends only on \( \vec{r}_k \)

\[
\left[ \sum_{i<j} v_{ij}, F_{\lambda} \right] = 0 \quad \text{Thus,} \quad [H, F_{\lambda}] \Rightarrow \left[ \sum_i t_i, F_{\lambda} \right]
\]

Then,

\[
S(F_{\lambda}) = \langle 0 | \sum_k \frac{\hbar^2}{2m_k} (\nabla_k F_{\lambda}(\vec{r}_k))^2 | 0 \rangle
\]

Note that the sum is expressed as a ground-state expectation value of one-body operator - insensitive to the many-body correlation in the ground state – “model independent”.

Sums with other energy weightings involve two- or many-body operators.
In particular, if \( F_{\lambda\mu} = f(r)Y_{\lambda\mu}(\hat{r}) \), we obtain

\[
S(F_{\lambda})_{\text{class}} = \frac{2\lambda + 1}{4\pi} \frac{\hbar^2}{2m} A \left\langle \left( \frac{df}{dr} \right)^2 + \lambda(\lambda + 1) \left( \frac{f}{r} \right)^2 \right\rangle
\]

where \( \left\langle \right\rangle \) expresses the average per particle in the ground state of \( A \) particles.

For \( E\lambda \) transitions with \( \lambda \geq 2 \), neglecting the correction due to the center of mass motion,

\[
F_{\lambda\mu} \Rightarrow er^2 Y_{\lambda\mu}(\hat{r}) \quad \text{only for protons, then,}
\]

\[
S(E\lambda)_{\text{class}} = \frac{\lambda(2\lambda + 1)\hbar^2}{4\pi} \frac{\hbar^2}{2m} Ze^2 \left\langle r^{2\lambda-2} \right\rangle_{\text{proton}}
\]

radial average for protons in the ground state

For \( E0 \) transitions,

\[
F_{\lambda\mu} \Rightarrow er^2 \quad \text{only for protons, then,}
\]

\[
S(E0)_{\text{class}} = \frac{2\hbar^2}{m} Ze^2 \left\langle r^2 \right\rangle_{\text{proton}}
\]
For $E1$ transitions;

Since isoscalar dipole operator corresponds to the center of mass motion that must not create an excitation, the dipole operator which creates excitations is necessarily of isovector character.

For example, electric-dipole excitation operator (in the direction of z-axis) should be

$$e \sum_i^p z_i \rightarrow e \sum_i^p \left( z_i - \frac{1}{A} \left( \sum_j^p z_j + \sum_k^n z_k \right) \right) = e \sum_i^p z_i - \frac{e}{A} Z \left( \sum_j^p z_j + \sum_k^n z_k \right)$$

$$= \frac{N}{A} e \sum_i^p z_i - \frac{Z}{A} e \sum_k^n z_k$$

spin-parity = $1^-$

where (p) and (n) express the sum over protons and neutrons.

Then, using (7.1), the classical oscillator sum, which should be the sum rule for IVGDR, when the interaction has $\vec{p}_k$, $(\vec{r} \cdot \vec{r})$, and $(\vec{\sigma} \cdot \vec{\sigma})$.

$$S(E1)_{class} = \sum_{a\mu} (E_a - E_0) \left| (a) \frac{N}{A} e \sum_i^p (rY_1\mu)_i - \frac{Z}{A} e \sum_k^n (rY_1\mu)_k |0\right|^2 = \frac{9 \hbar^2}{4\pi 2m} e^2 \left( Z \left( \frac{N}{A} \right)^2 + N \left( \frac{Z}{A} \right)^2 \right)$$

$$= \frac{9 \hbar^2}{4\pi 2m} e^2 \frac{NZ}{A}$$

$$S(F_\lambda = (erY_1)_{\ p}) - S(E1)_{class} = \frac{9 \hbar^2}{4\pi 2m} Z e^2 - \frac{9 \hbar^2}{4\pi 2m} e^2 \frac{NZ}{A}$$

$$= \frac{9 \hbar^2}{4\pi 2m} Z e^2 \frac{Z^2}{A} ; \text{oscillator strength associated with the center of mass motion}$$
ex. center of mass motion for $E2$ operator

Total $E2$ moment measured with respect to the center of mass

$$\sum_{k=1}^{A} e_k \left[ 2(x_k - X_c)^2 - (y_k - Y_c)^2 \right]$$

where $Z_c \equiv \frac{1}{A} \sum_{k=1}^{A} z_k$, etc. and $e_k = \begin{cases} e & \text{for proton} \\ 0 & \text{for neutron} \end{cases}$

$$= \sum_{k=1}^{A} e_k \left[ 2z_k^2 - x_k^2 - y_k^2 \right] - 4Z_c \sum_{k=1}^{A} e_k z_k + 2X_c \sum_{k=1}^{A} e_k x_k + 2Y_c \sum_{k=1}^{A} e_k y_k + \sum_{k=1}^{A} e_k (2Z_c^2 - X_c^2 - Y_c^2)$$

$$= \sum_{k=1}^{A} \left[ e_k \left( 1 - \frac{2}{A} \right) + e \frac{Z}{A^2} \right] (2z_k^2 - x_k^2 - y_k^2) + \frac{1}{A} \sum_{j<k} \left[ -2e_j - 2e_k + 2 \frac{Ze}{A} \right] (2z_j z_k - x_j x_k - y_j y_k)$$

(a)

For a single-particle configuration

$$e_k \rightarrow \begin{cases} e \left( 1 - \frac{2}{A} + \frac{Z}{A^2} \right) & \text{for proton} \\ e \frac{Z}{A^2} & \text{for neutron} \end{cases}$$

(b)

For harmonic oscillator wave-functions and low-energy transitions; matrix elements of (a) receive no contributions from the recoil term, namely, the correction term in (b) is exactly cancelled by the 2-body term in (a).
Distribution of $S(E2)$ in axially-symmetric deformed nuclei

(1) Low-energy IS ($r^2 Y_{2\nu}$) excitations ($\Delta N \approx 0$) 

\begin{align*}
\text{rotational excitation (} & \nu = \pm 1 \text{ )} \\
\text{gamma-vibration (} & \nu = \pm 2 \text{ )} \\
\text{beta-vibration (} & \nu = 0 \text{) (one-particle excitations)}
\end{align*}

a) rotation excitation – main $E2$ oscillator strength in the low-energy region

For even-even nuclei 

\begin{align*}
(E_2 - E_0)B(E2; K = 0, I = 0 \rightarrow K = 0, I = 2) &= \frac{3h^2}{3} \frac{5}{16\pi} e^2 Q_0^2
\end{align*}

For odd-A nuclei 

\begin{align*}
\sum_I [E(K_0, I) - E(K_0, I_0)]B(E2; K_0I_0 \rightarrow K_0I) &= \frac{3h^2}{3} \frac{5}{16\pi} e^2 Q_0^2
\end{align*}

Observed moments of inertia  $\mathcal{I}_{\text{obs}} \approx 5\mathcal{I}_{\text{irrot}} \rightarrow S(E2)_{\text{rot}} \approx (0.05)S(E2)_{\text{class}}$

comparable to low-energy 2+ mode in spherical nuclei

b) gamma vibration ($r^2Y_{2\pm 2}$ type surface vibration) takes less than a few % of $S(E2)_{\text{class}}$

beta vibration ($r^2Y_{20}$ Type surface vibration) takes less than $(0.01)S(E2)_{\text{class}}$
Figure 6-3 Quadrupole shape oscillations in a spheroidal nucleus. The figure shows projections of the nuclear shape in directions perpendicular and parallel to the symmetry axis.
(2) High-energy ($ΔN≈2$) excitations

E2 strength will split depending on the quantum number $|ν|$ of $Y_{2ν}$.

The E2 strength of GR with $ν = 0, ±1, ±2$ is expected to be approximately $1 : 2 : 2$

---

**OBS.** L-S doubly-closed spherical nuclei (such as 40-Ca) have only high-energy ($ΔN≈2$) collective quadrupole excitations.

Spherical vibrating nuclei have both low- and high-energy collective quadrupole excitations. The low-energy ($ΔN≈0$) IS quadrupole modes have enhanced E2 transitions due to the attractive quadrupole interaction, but carry less than $(0.10) S(E2)_{class}$.
ex. an extra contribution to $S(E1)$ from an exchange interaction

Increase of energy-weighted sum-rules, $S(E1)$, from $S(E1)_{\text{class}}$ due to the presence of attractive Majorana space-exchange interaction

ex. A proton-neutron pair with 2-body space-exchange $(\vec{r}_i \leftrightarrow \vec{r}_j)$ interaction

$$v_{ij} = f(|\vec{r}_i - \vec{r}_j|)P^M \quad \text{PM : Majorana space-exchange operator}$$

$$P^M = -\frac{1+(\vec{r}_i \cdot \vec{r}_j)}{2} \frac{1+(\vec{\sigma}_i \cdot \vec{\sigma}_j)}{2}$$

$$\left[v_{ij}, z_i\right] = fP^M z_i - z_i fP^M = (z_j - z_i) fP^M \quad z_i : \text{proton coordinate}$$

$$\left[z_i, \left[v_{ij}, z_i\right]\right] = \left[z_i, (z_j - z_i) fP^M\right]$$

$$= z_i(z_j - z_i) fP^M - (z_j - z_i) fP^M z_i$$

$$= z_i(z_j - z_i) fP^M - (z_j - z_i) z_j fP^M$$

$$= - (z_j - z_j)^2 fP^M$$

$$= - \frac{1}{3} (r_{ij})^2 fP^M$$

An attractive Majorana interaction makes an extra contribution to $S(E1)$. 

∴

An attractive Majorana interaction makes an extra contribution to $S(E1)$. 

7.2.2. Sum rules for (t±) charge-exchange excitations

There is no sum-rule, which corresponds to the classical oscillator strength for IS operators. Instead, model-independent and non-energy weighted sum-rules, for the difference between t– and t+ transitions.

| Isospin of nucleon, | t = \frac{1}{2} | t_\pm | n \rangle = \frac{1}{2} | n \rangle | t_\pm | p \rangle = -\frac{1}{2} | p \rangle | t_\pm \equiv t_x \pm it_y |
|-------------------|----------------|-------------------|-----------------|-------------------|
| t_– | n \rangle = | p \rangle | t_+ | p \rangle = | n \rangle | t_+ | n \rangle = 0 | t_- | p \rangle = 0 |

\[
\left[ \sum_{k}^{A} t_+(k), \sum_{j}^{A} t_-(j) \right] = 2 \sum_{k}^{A} t_z(k) \\
2 \langle N, Z | \sum_{i}^{A} t_z(i) | N, Z \rangle = N - Z \\
\sum_{\mu=1}^{3} \sigma_{\mu}^{2} = 4 \sum_{\mu=1}^{3} s_{\mu}^{2} = 4(\bar{s})^{2} = 4 \frac{1}{2} \left( \frac{1}{2} + 1 \right) = 3
\]

: Basic formulas

(a) charge-exchange non-spin-flip excitations : \( \hat{O}_\pm = \sum_{k} t_\pm(k) f(r_k) \)

\[
\sum_{m} |\langle m | \hat{O}_- | 0 \rangle |^2 - \sum_{n} |\langle n | \hat{O}_+ | 0 \rangle |^2 = N \langle (f(r))^2 \rangle_n - Z \langle (f(r))^2 \rangle_p
\]

(b) charge-exchange spin-dependent excitations : \( \hat{O}_\pm = \sum_{\mu} \sum_{k} t_\pm(k) \sigma_{\mu}(k) f(r_k) \)

\[
\sum_{m} |\langle m | \hat{O}_- | 0 \rangle |^2 - \sum_{n} |\langle n | \hat{O}_+ | 0 \rangle |^2 = 3 \left[ N \langle (f(r))^2 \rangle_n - Z \langle (f(r))^2 \rangle_p \right]
\]
7.3. Giant resonance of isoscalar (IS) or $t_z$ type

7.3.1. Isovector giant dipole resonance (IVGDR) : the oldest and best known Giant Resonance

Systematics of observed IVGDR frequency

![Graph showing systematics of observed IVGDR frequency](image)

Observed $79 \, A^{-1/3} \, \text{MeV}$ is in good agreement with the value estimated in the harmonic-oscillator model.

For light nuclei with $A < 50$ a deviation from the systematics is observed.

--- Other types of GR show the same tendency.

Note that unpertrubed $I^\pi = 1^-$ p-h energies in realistic potentials are approximately degenerate and close to $41 \, A^{-1/3} \, \text{MeV}$ (!) also in drip-line nuclei !
Well-established IVGDR – observed peak(s) in photo absorption cross section

Photo absorption cross section of $^{197}$Au

Additional dipole strength is observed on the high-energy side, that appears to be associated with short-range (velocity-dependent) correlations between nucleons.

\[
\int_0^{140\text{MeV}} \sigma_{photoabs} dE_{\gamma} \rightarrow 2S(E1)_{class}
\]

25 MeV

\[
\int_0^{25\text{MeV}} \sigma_{photoabs} dE_{\gamma} \rightarrow S(E1)_{class}
\]

where

\[
S(E1)_{class} = \frac{9}{4\pi} \frac{\hbar^2}{2M} \frac{N}{A} e^2
\]

\[
S(E1)_{obs} = S(E1)_{class} (1 + x)
\]

\(x\) comes from the velocity- and \((\tau \cdot \tau)\)-dependent terms in the nucleonic interactions.

Figs. 6-18 and 6-26 of A. Bohr & B. R. Mottelson, Nuclear Structure, vol. II
IVGDR is the giant resonance, of which the semi-classical picture is possible.

Goldhaber-Teller model

Steinwedel-Jensen model

Neutron and proton fluids are oscillating within a sphere, keeping the total density constant.

For simplicity, assuming [IVGDR ~ a standing wave in a nucleus with a fixed boundaries],

the frequency $\omega \propto R^{-1}$

Then, in contrast to one-peak structure
in spherical nuclei,

in axially-symmetric quadrupole-deformed nuclei

$$\omega_z \propto R_z^{-1}$$
$$\omega_\perp \propto R_\perp^{-1}$$

the strength distribution would have two peaks corresponding to an oscillation of neutrons vs protons along the long and short axes, as observed in $^{150}$Nd$_{90}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6-21}
\caption{Photoabsorption cross section for even isotopes of neodymium.}
\end{figure}
For a prolate (oblate) shape the integrated cross section associated with the vibration along the symmetry axis (= longer (shorter) axis), which has lower (higher) frequency, should be about a half of the one along two shorter (longer) axes.

The energy splitting is proportional to the deformation $\delta$

$$\frac{\omega_\perp - \omega_z}{\bar{\omega}} \approx \frac{\Delta R}{R} \approx \delta$$

Thus, the ground state of $^{150}$Nd$_{90}$ is prolately deformed!
Harmonic oscillator potential

\[ \langle N_f \mid x, y, z \mid N_i \rangle \neq 0 \quad \text{only for} \quad N_f = N_i \pm 1 \]

one-particle energy

\[ N_i + 1 \quad \hbar \omega_0 \]

unoccupied

occupied

All excitations are from the last-filled major shell to the next major shell, with excitation energies,

\[ \Delta E = \hbar \omega_b \approx 41A^{-1/3} \text{ MeV}. \]

Many degenerate particle-hole (p-h) excitations, especially in heavier nuclei.

A schematic model:

(for IVGDR)


After including a separable repulsive interaction between the p-h excitations, only one collective state is pushed up, while all other states remain at the unperturbed excitation-energy, and the collective state absorbs all transition strength, if one takes

separable interaction \[\longleftrightarrow\] relevant transition operator
Taking the strength of IV dipole-dipole interaction from the symmetry term of the phenomenological nuclear one-body potential, in the harmonic oscillator potential model we obtain

\[ \frac{1}{3} \omega_0 = 41 A^{-1/3} \text{ MeV} \]

\[ \rightarrow 80 A^{-1/3} \text{ MeV} \]

for the excitation energy of the collective state, (= IVGDR)

in agreement with the observed systematics in medium-heavy nuclei.

\[ |e_{\text{eff}}^p (E1)| \text{ and } |e_{\text{eff}}^n (E1)| \text{ for low-energy E1 transitions are much smaller} \]

than the values of \((N/A) e\) and \((Z/A) e\), respectively (see Sect. 6.1).
In the self-consistent calculations plus RPA for spherical nuclei, the strength of IVGDR is split into several peaks even for heavier nuclei. The transition density of lower-lying peaks appears to be closer to the Steinwedel-Jensen prediction, while that of higher-lying peaks looks more like the Goldhaber-Teller one.

Radial transition density:

Goldhaber-Teller model

\[ \rho_{GT}^r(r) \propto \frac{d\rho_0(r)}{dr} \]

Steinwedel-Jensen model

\[ \rho_{SW}^r(r) \propto r\rho_0(r) \]

Examples of self-consistent calculations plus RPA

\[ \text{IVGDR} \]

\[ \text{ISGDR} \]

I.H., H.Sagawa and X.Z.Zhang, PRC 57, R1064 (1998)
ex. “Pigmy dipole resonances” observed at much lower energy than IVGDR of the $A \approx 140$ region consume less than 1% of $S(E1)_{\text{class}}$.

obtained from by folding with a Lorentzian with a width of 500 keV

(from A. Zilges, 2007)
### 7.3.2. Isoscalar and isovector giant quadrupole (ISGQR and IVGQR) resonance

<table>
<thead>
<tr>
<th>Operator</th>
<th>spin-parity</th>
<th>observed peak energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>ISGQR</td>
<td>$\sum_k r_k^2 Y_{2\mu} (\hat{r}_k)$</td>
<td>2+</td>
</tr>
<tr>
<td>IVGQR</td>
<td>$\sum_k \tau_z(k) r_k^2 Y_{2\mu} (\hat{r}_k)$</td>
<td>2+</td>
</tr>
</tbody>
</table>

#### A schematic model: (for ISGQR)

- **many degenerate p-h excitations**
- **a collective state**
- **ground state**

After including a **separable attractive** interaction between the p-h excitations, only one collective state is pushed down, while all other states remain at the unperturbed excitation-energy, and the collective state obtains all transition strength, if one takes **separable interaction** ↔ **relevant transition operator**
Taking the strength of IS quadrupole-quadrupole interaction from the self-consistent condition that the eccentricity of the potential is the same as that of the density, in the harmonic oscillator potential model we obtain

\[ 2\hbar \omega_0 = 82 \text{ A}^{-1/3} \text{ MeV} \]

\[ \sqrt{2} \hbar \omega_0 = 58 \text{ A}^{-1/3} \text{ MeV} \]

for the excitation energy of the collective state (= ISGQR)

In stable nuclei the estimate based on the above h-o potential model works well, because most one-particle levels in the major shell \( (N_i + 2) \) are narrow resonances in realistic potentials.

In the schematic harmonic oscillator model

\[ \langle N_f | x^2, y^2, z^2 | N_i \rangle \neq 0 \]

only for \( N_f = N_i \pm 2 \) or \( N_f = N_i \)
Using (7.1),

\[ S(IS, \lambda = 2)_{\text{class}} = \frac{50}{4\pi} \frac{\hbar^2}{2m} A\langle r^2 \rangle \quad \text{Energy Weighted Sum Rule (EWSR)} \]

The classical sum-rule for IS giant resonances should work when the interaction is

In the harmonic-oscillator potential model,

\[ \langle N_f | x^2, y^2, z^2 | N_i \rangle \neq 0 \quad \text{only for} \quad N_f = N_i \quad \text{and} \quad N_f = N_i \pm 2 \]

Thus, in contrast to IVGDR, the quadrupole operator has \( N \rightarrow N \) matrix elements, in addition to \( N \rightarrow N+2 \) matrix elements. And, the IS (attractive) coupling between the two kinds of modes, \( \Delta N = 0 \) and \( 2 \), shifts some transition strength to lower-energy modes.

In open-shell nuclei the \( N \rightarrow N \) transitions are possible within the last filled major shell, while in medium-heavy nuclei the transitions are present even in the closed-shell nuclei, due to the presence of the spin-orbit splitting. For example, in the doubly-closed shell nucleus \( _{82}^{126}\text{Pb} \) one finds 4 low-energy excitations; 2 proton-excitations, \( 1h_{11/2} \rightarrow 1h_{9/2} \), \( 2f_{7/2} \) and 2 neutron-excitations, \( 1i_{13/2} \rightarrow 1i_{11/2} \), \( 2g_{9/2} \).

Nevertheless, since the sum-rule considered here is the energy-weighted sum-rule, the observed IS lower-lying quadrupole excitations exhaust only up till 15 percent of EWSR.
Some summary of the observed properties of ISGQR of medium-heavy nuclei

M.N. Harakeh & A. van der Woude, Giant Resonances, 2001

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$E_x$ (MeV)</th>
<th>$\Gamma$ (MeV)</th>
<th>EWSR (%)</th>
<th>Reference</th>
<th>$E_x A^{1/3}$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{90}$Zr</td>
<td>14.05 ± 0.25</td>
<td>4.0 ± 0.25</td>
<td>49</td>
<td>(BUE84)</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>14.0 ± 0.2</td>
<td>3.4 ± 0.2</td>
<td>66 ± 17</td>
<td>(YOU81)</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>14.0 ± 0.4</td>
<td>3.0 ± 0.5</td>
<td>111 ± 25</td>
<td>(BOR89)</td>
<td>63</td>
</tr>
<tr>
<td>$^{112}$Sn</td>
<td>13.65 ± 0.2</td>
<td>3.6 ± 0.2</td>
<td>55</td>
<td>(BUE84)</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td>13.51 ± 0.13</td>
<td>3.15 ± 0.23</td>
<td>123 ± 26</td>
<td>(SHA88)</td>
<td>63</td>
</tr>
<tr>
<td>$^{116}$Sn</td>
<td>13.15 ± 0.25</td>
<td>3.6 ± 0.3</td>
<td>60</td>
<td>(BUE84)</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>13.2 ± 0.3</td>
<td>3.3 ± 0.2</td>
<td>84 ± 25</td>
<td>(YOU81)</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>13.39 ± 0.14</td>
<td>2.94 ± 0.31</td>
<td>134 ± 28</td>
<td>(SHA88)</td>
<td>65</td>
</tr>
<tr>
<td>$^{120}$Sn</td>
<td>12.75 ± 0.25</td>
<td>3.75 ± 0.3</td>
<td>82</td>
<td>(BUE84)</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>12.7 ± 0.4</td>
<td>3.5 ± 0.4</td>
<td>80</td>
<td>(YOU81)</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>13.24 ± 0.13</td>
<td>2.88 ± 0.20</td>
<td>135 ± 27</td>
<td>(SHA88)</td>
<td>65</td>
</tr>
<tr>
<td>$^{124}$Sn</td>
<td>12.35 ± 0.25</td>
<td>3.6 ± 0.3</td>
<td>88</td>
<td>(BUE84)</td>
<td>61</td>
</tr>
<tr>
<td></td>
<td>12.3 ± 0.4</td>
<td>3.1 ± 0.3</td>
<td>78 ± 25</td>
<td>(YOU81)</td>
<td>61</td>
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<tr>
<td></td>
<td>13.02 ± 0.13</td>
<td>2.80 ± 0.30</td>
<td>127 ± 31</td>
<td>(SHA88)</td>
<td>65</td>
</tr>
<tr>
<td>fit 1</td>
<td>12.7 ± 0.35</td>
<td>3.4 ± 0.5</td>
<td>190 ± 75</td>
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<tr>
<td>fit 2</td>
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<td>3.8 ± 0.5</td>
<td>225 ± 75</td>
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<td>61</td>
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<tr>
<td>$^{144}$Sm</td>
<td>12.25 ± 0.2</td>
<td>2.5 ± 0.2</td>
<td>50</td>
<td>(BUE84)</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>12.2 ± 0.2</td>
<td>2.4 ± 0.2</td>
<td>45 ± 15</td>
<td>(YOU81)</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>12.70 ± 0.14</td>
<td>2.62 ± 0.20</td>
<td>123 ± 29</td>
<td>(SHA88)</td>
<td>66</td>
</tr>
<tr>
<td>$^{150}$Sm</td>
<td>12.3 ± 0.2</td>
<td>3 ± 0.2</td>
<td>76</td>
<td>(BUE84)</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>12.75 ± 0.17</td>
<td>2.85 ± 0.36</td>
<td>132 ± 50</td>
<td>(SHA88)</td>
<td>68</td>
</tr>
<tr>
<td>$^{152}$Sm</td>
<td>11.95 ± 0.2</td>
<td>3 ± 0.2</td>
<td>81</td>
<td>(BUE84)</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>12.78 ± 0.17</td>
<td>3.63 ± 0.42</td>
<td>183 ± 80</td>
<td>(SHA88)</td>
<td>68</td>
</tr>
<tr>
<td>$^{208}$Pb</td>
<td>10.60 ± 0.25</td>
<td>2.8 ± 0.25</td>
<td>100</td>
<td>(BUE84)</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>11 ± 0.2</td>
<td>2.7 ± 0.3</td>
<td>105 ± 25</td>
<td>(YOU81)</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>10.9 ± 0.3</td>
<td>3.1 ± 0.3</td>
<td>120 ± 170</td>
<td>(BRA85)</td>
<td>64</td>
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<tr>
<td></td>
<td>11.0 ± 0.3</td>
<td>3.3 ± 0.3</td>
<td>100–150</td>
<td>(BRA85)</td>
<td>65</td>
</tr>
</tbody>
</table>

Observation of the IS giant quadrupole resonance (ISGQR) - one of the first observations of a giant resonance other than the well-known IVGDR

R. Pitthan, Z. Phys. 260 (1973) 283
Experimental information on isovector giant quadrupole resonance (IVGQR) is very limited.

The reason for this can be:

(a) Due to the high frequency mode, large background and possible overlap with many other excitations;
(b) Large width and relatively small excitation cross section;
(c) Lack of a selective experimental tool to excite IVGQR

For reference, the result of a self-consistent HF+RPA calculation is shown below.

\( ^{20}\text{Ca}_{20} \) is a stable nucleus, while \( ^{20}\text{Ca}_{40} \) is possibly a neutron-drip-line nucleus.

In both nuclei ISGQR appears as a clean collective peak, while IVGQR spreads over several peaks with varying form factors. The ‘threshold strength’ in \( ^{60}\text{Ca} \) comes from the presence of weakly-bound neutrons in the ground state, which are not present in stable nuclei.

p-h energies, $82 A^{-1/3} \text{ MeV}$ → collective ISGQR at $58 A^{-1/3} \text{ MeV}$, which consumes the major part of IS quadrupole strength.

due to the attractive p-h interaction

means; ISGQR makes a considerable amount of positive contribution to $e_{pol}(E2)$ of low-energy E2 transitions.
7.3.3. Isoscalar giant monopole resonance (ISGMR) - compression mode

In $^{208}\text{Pb}$, observed ISGMR ($E_{\text{ISGMR}} \approx 14 \text{ MeV}, \Gamma \approx 3 \text{ MeV}$) exhausts about 100% of the energy-weighted sum rule.

![Observed properties of ISGMR](image)

Examples of experimental data of ISGMR

S. Shlomo and D. H. Youngblood, PRC 47, 529 (1993)

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Nucleus</th>
<th>Corrected* for systematic difference</th>
<th>Values adopted for calculations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$E_x$ (MeV)</td>
<td>$\sigma(E_x)$ (MeV)</td>
<td>$E_x$ (MeV)</td>
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<td>24</td>
<td>Mg</td>
<td>16.71</td>
<td>0.23</td>
</tr>
<tr>
<td>28</td>
<td>Si$^b$</td>
<td>19.06</td>
<td>0.50</td>
</tr>
<tr>
<td>40</td>
<td>Ca</td>
<td>14.11</td>
<td>0.23</td>
</tr>
<tr>
<td>58</td>
<td>Ni</td>
<td>17.00</td>
<td>0.40</td>
</tr>
<tr>
<td>64</td>
<td>Zn</td>
<td>18.20</td>
<td>0.50</td>
</tr>
<tr>
<td>66</td>
<td>Zn</td>
<td>18.40</td>
<td>0.70</td>
</tr>
<tr>
<td>90</td>
<td>Zr</td>
<td>16.20</td>
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</tr>
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<td>90</td>
<td>Zr</td>
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<td>112</td>
<td>Sn</td>
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<td>0.30</td>
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<td>114</td>
<td>Sn</td>
<td>15.59</td>
<td>0.27</td>
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<td>116</td>
<td>Sn</td>
<td>15.60</td>
<td>0.30</td>
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<tr>
<td>118</td>
<td>Sn</td>
<td>15.50</td>
<td>0.28</td>
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<td>120</td>
<td>Sn</td>
<td>15.20</td>
<td>0.50</td>
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<tr>
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<td>Sn</td>
<td>15.23</td>
<td>0.27</td>
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<td>Sn</td>
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<td>0.20</td>
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<tr>
<td>148</td>
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<td>0.20</td>
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<td>0.29</td>
</tr>
<tr>
<td>154</td>
<td>Sm</td>
<td>14.90</td>
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</tr>
<tr>
<td>208</td>
<td>Pb</td>
<td>13.70</td>
<td>0.40</td>
</tr>
<tr>
<td>208</td>
<td>Pb$^c$</td>
<td>13.63</td>
<td>0.38</td>
</tr>
<tr>
<td>232</td>
<td>Th</td>
<td>13.80</td>
<td>0.40</td>
</tr>
<tr>
<td>238</td>
<td>U</td>
<td>13.70</td>
<td>0.40</td>
</tr>
</tbody>
</table>
Measured energy of ISGMR (= “breathing mode”), $E_{\text{ISGMR}}$, → information on the compressibility of nuclear matter ($K_{nm}$).

Nuclear compressibility is an important information on the equation of state of nuclear matter.

ex. shape of the density distribution, values of the radii, the strength of shock wave following the collapse of supernovae, etc.

However, the relation, $E_{\text{ISGMR}} \leftrightarrow K_{nm}$ is model-dependent!

$K_{nm}$ is defined by

$$K_{nm} = 9\rho_0^2 \frac{d^2(E/A)}{d\rho^2}\bigg|_{\rho=\rho_0}$$

An effective compression modulus, $K_A$, for a nucleus with mass number $A$ in terms of $E_{\text{ISGMR}}(A)$ is defined by

$$K_A = \frac{m(E_{\text{ISGMR}}(A))^2 \langle r^2 \rangle_m}{\hbar^2}$$

where $\langle r^2 \rangle_m$ is the mean-square mass radius.

Writing

$$K_A = K_{\text{vol}} + K_{\text{surf}} A^{-1/3} + K_{\text{sym}} \left(\frac{N-Z}{A}\right)^2 + K_{\text{Coul}} \frac{Z^2}{A^{4/3}} + \cdots \quad (\$)$$

$$\lim_{A \to \infty} K_A = K_{\text{vol}} = K_{nm}$$

if the mode corresponds to a radial scaling of the ground-state density.

$$\lim_{A \to \infty} K_A = (7/10) K_{nm}$$

from a Hartree-Fock calculation with a constraint on the r.m.s. radius.
Moreover, various $K_i$ values in ($) are **poorly determined**, since the variations of $K_A$ with $N$ and $Z$ are very small for available nuclei.

Thus, some experts state (for example, Blaizot et al., *NPA* **591** (1995) 435):

**Phenomenological expansion** ($) using measured $E_{ISGMR}(A)$ values cannot be used to obtain $K_{nm}$.

Microscopic calculations remain the most reliable tool for determining $K_{nm}$ from measured $E_{ISGMR}(A)$ values.

$$K_{nm} = 210 \pm 30 \text{ MeV}$$
Comparison of calculated ISGMR using self-consistent Hartree-Fock calculations plus RPA with various Skyrme interactions, which have different $K_{nm}$ values. $K_{nm} = 217, 256$ and $355$ MeV for SkM*, SGI and SIII, respectively.

I.H., H.Sagawa & X.Z.Zhang, PRC 56, 3121 (1997)

Calculated ISGMR in medium weight and light nuclei usually does not have a clean one-peak shape.

Calculated ISGMR in heavy nuclei is obtained as a well defined resonance and exhausts the sum rule.

(In the above calculation the particle decay width of GR is fully taken into account, while the spreading width, coming from the coupling to 2p-2h configurations, is not included.)
The effective charge of E0 transitions, $e_{\text{eff}}(E0)$, for low-energy E0 transitions has not really been studied.

In heavier nuclei self-consistent calculations plus RPA produce a relatively clean resonance peak. Nevertheless, the calculated peak energy is not so different from averaged unperturbed p-h energies in the potential based on harmonic-oscillator.

$\rightarrow$ Calculated values of E0 polarization charge, $e_{\text{pol}}(E0)$, for low-energy E0 transitions due to ISGMR may not be large and may depend sensitively on the models and parameters used.

ex. A recent information on $e_{\text{eff}}^n(E0)$ from the data on $^{12}_{4}Be_8$


\[
\begin{align*}
2.251 & \quad 0^+_2 \\
\downarrow & \quad 0^+_1 \\
& \quad \text{Measured partial life } \tau(0^+_2 \rightarrow 0^+_{g.s.}) = 402 \pm 16 \text{ ns} \\
& \quad \langle 0^+_2 | e_{\text{eff}}(E0)r^2 | 0^+_1 \rangle = 0.87 \text{ e fm}^2 \\
& \quad e_{\text{eff}}^n(E0) = e_{\text{pol}}^n(E0) = 0.076 \text{ e}
\end{align*}
\]

The presence of weakly-bound neutrons in the deformed potential is duly taken into account.

OBS. The polarization charge for E0 transitions obtained from subtracting the center of mass motion is analogous to that of E2 transitions described in Sect.7.2.1. and is

\[
e_{\text{eff}}^n(E0) = (Z/A^2)e = (0.028) \text{ e} \quad \text{for } ^{12}_{4}Be
\]
Comparison of IS and IVGR in $^{208}$Pb calculated by self-consistent Hartree-Fock plus RPA using SKM* interaction.

Particle decay width is fully taken into account, though spreading width coming from the coupling to 2p-2h configurations is not included.


Figure 3. Unperturbed and RPA response functions of $^{208}$Pb to monopole ($r^2Y_{00}$ and $\tau r^2Y_{00}$ for the IS and IV operator, respectively), compression dipole ($r^2Y_{10}$ and $\tau r^2Y_{10}$) and quadrupole ($r^2Y_{20}$ and $\tau r^2Y_{20}$) operators. The SKM* interaction is used both in HF and RPA.
7.4. Giant Resonances of charge-exchange type ($\Delta T_z = \pm 1$)

Various isospin states, which can be excited by acting an isovector excitation operator on a nucleus with $T = T_z \neq 0$.

Excitation strengths $\propto \left[ C(T_0, T; T_0, \Delta T_z, T_0 + \Delta T_z) \right]^2$

Excitation strengths

$$\propto \frac{T_0 + 1}{(2T_0 + 1)(T_0 + 1)}$$

$$\propto \frac{T_0}{T_0 + 1}$$

$$\propto \frac{T_0 - 1}{2T_0 - 1}$$

$$\propto \frac{T_0 + 1}{2T_0 + 1}$$

$T_0 = T_z = \frac{N - Z}{2}$

$T_0 + 1, N+1$

$T_0, N$

$T_0 - 1, N-1$

$T_z = \frac{N - Z}{2} + 1$

$T_0 = T_z = \frac{N - Z}{2}$

$T_0 = T_z = \frac{N - Z}{2} - 1$

$Z-1, N+1$

$Z \neq N$

$Z+1, N-1$
\[ E_X(t_+ \text{ GR}) < E_X(t_- \text{ GR}) \] in the presence of neutron excess

The p-h excitation energy \( E_X \) measured from the ground state of mother nuclei

\[ E_X(t_+) = E_0 - \Delta E \]
\[ E_X(t_-) = E_0 + \Delta E \]

\[
\begin{aligned}
E_0 &\sim \alpha \hbar \omega_0 \sim A^{-1/3} \\
\Delta E &\sim A^{2/3} \text{ for neutron-excess in stable neutron-rich nuclei}
\end{aligned}
\]

\[ \therefore E_X(t_+) \text{ becomes monotonically smaller as } A \rightarrow \text{ larger}. \]

This relation, \( E_X(t_+ \text{ GR}) < E_X(t_- \text{ GR}) \), is present in all charge-exchange \((t_{\pm})\) GRs.

In \( N>Z \) nuclei towards neutron-drip-line
\[ E_x(t_+ \text{ GR}) \ll E_x(t_- \text{ GR}) \]
\( E_x \) in respective final nuclei
Some **expected** features unique in **spin-isospin** \( (\sigma_\mu t_\pm) \) **Giant Resonances**

1) \([t_\pm \sigma_\mu]\) \(\rightarrow\) **Not** almost all strength under the GR peak.

Instead, a considerable amount of **high-energy tail** above the peak is expected, with the **tensor correlation** responsible for the highest energy components.

**Dependence** of the high-energy tail on respective GRs?

2) \([\sigma_\mu]\) \(\rightarrow\) **Relatively large width** (or **large spread**) of GR

\[ \therefore a) \] Unperturbed 1p-1h excitations have already an energy spread of \(2 \Delta E_{ls}\), where the spin-orbit splitting of high-j orbit is expressed by \(\Delta E_{ls}\) (\(\approx 7\)–9 MeV), except for GTGR and some IVSGDR where the spread \(\approx \Delta E_{ls}\).

\[ b) \] Due to the **same sign** of the couplings to a particle and a hole; the coupling of 1p-1h to 2p-2h configurations is **strong**, in contrast to **spin-independent** modes.
Examples of charge-exchange Giant Resonances studied in $\beta$-stable nuclei

<table>
<thead>
<tr>
<th>spin-parity</th>
<th>operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>IAS</td>
<td>0+</td>
</tr>
<tr>
<td>GT GR</td>
<td>1+</td>
</tr>
<tr>
<td>IV GQR</td>
<td>2+</td>
</tr>
<tr>
<td>IV spin GMR*</td>
<td>1+</td>
</tr>
<tr>
<td>IV spin GDR</td>
<td>0−, 1−, 2−</td>
</tr>
</tbody>
</table>

Direct and systematic experimental data are available only for IAS and GTGR.

IAS = Isobaric Analogue State
IV = IsoVector
GMR = Giant Monopole Resonance
GDR = Giant Dipole Resonance
Allowed $\beta$ decay; \[ \sum_k A_t^\pm(k) \equiv F_\pm \] Fermi transitions: \[ (\ell, j)_p \leftrightarrow (\ell, j)_n \]
\[ \sum_k A_\sigma^\pm(k) t^\pm(k) \equiv GT_\pm \] Gamow-Teller transitions:
\[ (\ell, j = \ell \pm \frac{1}{2})_p \leftrightarrow (\ell, j = \ell \pm \frac{1}{2})_n \]
and \[ (\ell, j = \ell \mp \frac{1}{2})_n \]

<table>
<thead>
<tr>
<th>Isospin of nucleon, ( t = \frac{1}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t^-</td>
</tr>
<tr>
<td>( t^+</td>
</tr>
<tr>
<td>( t^+_n = 0 )</td>
</tr>
<tr>
<td>( t^-_p = 0 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ex. In the L-S doubly-closed-shell ( N=Z ) nucleus, (^{40}<em>{20}\text{Ca}</em>{20} ), one expects</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_k A_t^\pm(k)</td>
</tr>
<tr>
<td>and</td>
</tr>
<tr>
<td>( \sum_k A_\sigma^\mu(k)t^\pm(k)</td>
</tr>
</tbody>
</table>

\( F_\pm \) operators are raising and lowering operators of the z-component of total isospin \( T \) without changing the total isospin, \( \Delta T=0 \):
\[ \sum_k A_t^\pm(k) = T_\pm \]
\[ T_\pm|T, T_z\rangle = |T, T_z \pm 1\rangle \]
In particular, \[ T_\pm|T = T_z = 0\rangle \approx 0 \]

\( GT_\pm \) operators may change the total isospin, \( \Delta T = -1, 0, +1 \), but \( T = 0 \Rightarrow T = 0 \)
\textbf{β-decay} can populate only the states with \( Ex \leq Q_{β±} \) in daughter nuclei.

That means, in \textbf{β-stable} nuclei, β-decays of ground states can \textit{populate only} the low-energy tail of GTGR in daughter nuclei. Thus, those β-decays are considerably hindered.

In contrast, using \textit{charge-exchange reactions} on \textit{mother} nuclei,

\begin{align*}
(p, n), & \quad (^{3}\text{He}, t) \quad \text{for} \; t_- \; (n \rightarrow p \; \text{in target nuclei}) \\
(n, p), & \quad (d, ^{2}\text{He}), \quad (t, ^{3}\text{He}) \quad \text{for} \; t_+ \; (p \rightarrow n \; \text{in target nuclei})
\end{align*}

the response is obtained up till high excitation energy in \textit{daughter} nuclei. The price which one must pay is; the \textit{analysis of data} to obtain nuclear matrix elements is much more complicated than in β decays.

In those charge-exchange reactions, \textit{Gamow-Teller Giant Resonance (GTGR)} was \textit{found}!
7.4.1. Fermi transitions; \( F_\pm = \) \( \hat{O}_\pm = \sum_k t_\pm(k) \)

\[
\sum_m |\langle m | \hat{O}_- | 0 \rangle|^2 - \sum_n |\langle n | \hat{O}_+ | 0 \rangle|^2
\]

\( \equiv S_- - S_+ ) = (N - Z) \)

The sum rule for Fermi transitions is usually exhausted by the transition to the Isobaric Analogue State (IAS), which has a very narrow width.

\[
|IAS\rangle = T_\pm | 0 \rangle
\]

That means, Isospin is a good quantum number, in general, in both light nuclei and medium-heavy nuclei with neutron excess.

Isospin of the ground state is maximum broken for \( N=Z \) nuclei with \( Z \rightarrow \) large.

ex. For \( N>Z \)

\[
\begin{align*}
\text{IAS} & \quad T = (N-Z)/2 \\
\text{g.s.} \quad T = (N-Z)/2-1 \\
(N,Z) & \quad (N-1, Z+1) \\
T_z = (N-Z)/2 & \quad T_z = (N-Z)/2-1
\end{align*}
\]

In this example \( F_+ | T = T_z = (N - Z) / 2 \rangle = 0 \)

\[
\therefore | T = (N - Z) / 2 \rangle \quad \text{cannot have component.}
\]

\[
\therefore S_- - S_+ = N - Z
\]
7.4.2. Gamow-Teller resonance; \( \langle \hat{O} \rangle = \sum \frac{t_{\pm}(k)}{\sigma_{\mu}(k)} \)

\[
\sum_{\mu=1}^{3} \sum_{m} |\langle m|\hat{O}_{-}|0\rangle|^2 - \sum_{\mu=1}^{2} \sum_{n} |\langle n|\hat{O}_{+}|0\rangle|^2 \quad (\equiv S_{-} - S_{+}) = 3(N - Z)
\]

Some experimental observation

In order to observe GTGR, the incident energy of proton or \(^3\text{He}\) beams must be chosen carefully. (The population of spin-isospin modes relative to excitations of other types depends on the incident energy.)


The 0° \(^{71}\text{Ga}(^{3}\text{He},t)^{71}\text{Ge}\) spectrum at 450 MeV.


The observed properties of IAS and GTGR in $^{208}_{82}Pb_{126}$ $(^{3}He,t)^{208}_{83}Bi_{125}$ with $E(^{3}He) = 450$ MeV were studied by H. Akimune et al., PRC 52, 604 (1995).

<table>
<thead>
<tr>
<th>Ex (MeV)</th>
<th>Width (MeV)</th>
<th>Sum rule (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T = 22)</td>
<td>(T = 21)</td>
<td></td>
</tr>
<tr>
<td>“IAS”</td>
<td>“GTGR”</td>
<td></td>
</tr>
<tr>
<td>18.8</td>
<td>19.2</td>
<td>100</td>
</tr>
<tr>
<td>0.232</td>
<td>3.7</td>
<td>~ 60</td>
</tr>
</tbody>
</table>

“IAS” = $T \cdot |\text{gr of }^{208}\text{Pb}\rangle$

“GTGR” = $GT \cdot |\text{gr of }^{208}\text{Pb}\rangle$

From the $(^{3}He,t)$ reaction; only the GTGR peak region is included and $S_{+} = 0$ was assumed due to Pauli blocking.

Missing (GT) strength used to be a problem in 1980s.

1) Back-ground subtraction problem;
   - broad GT bump is located on top of a continuum.
   - including this continuum or not makes a large difference in the extracted strength.
   - GTGR has a clean resonance shape?

2) $S_{+}$ may not be negligible even for medium-heavy nuclei.

3) Possible missing GT strength is carried by the excitation, [nucleon $\rightarrow \Delta$ resonance at 1232 MeV]?

The direct measurement of $S_-$ and $S_+$, performing both $(p,n)$ and $(n,p)$ reactions;


$^{90}$Zr $(p,n)$ $E_p = 295$ MeV
$^{90}$Zr $(n,p)$ $E_n = 293$ MeV

$S_+$ was carefully measured!

A multipole decomposition technique was applied to extract the GT component from the continuum.

GT quenching factor extracted from $E_X < 50$ MeV:

$$Q \equiv \frac{S_- - S_+}{3(N - Z)} = 0.88 \pm 0.06$$

1) The coupling to non-nucleonic degrees of freedom (ex. $\Delta$-resonance !?) in nuclei is presumably very small.

2) An appreciable amount of GT strength is found in the energy region much higher than the peak energy of GTGR.


Due to the spin-isospin character of GT operator, some unperturbed 1p-1h GT strength is shifted to the higher-lying (10-45 MeV) 2p-2h states, with the tensor correlation responsible for the highest energy components.
1) The T=6 part of GTGR\(_-\) is only a fraction, 1/66, of the total GTGR\(_-\) strength.

2) All GTGR\(_+\) strength in \(^{90}\text{Y}\) has T=6, which is however expected to be very small.

3) The total strength \(S_+\) of IVSM\(_+\) (all with T=6) on \(^{90}\text{Zr}\) is not small and about 70\% of that \(S_-\) of IVSM\(_-\) on \(^{90}\text{Zr}\).

\[
\Delta_{np} \equiv [\Delta M(n) - \Delta M(^1\text{H})]c^2 = (m_n - m_p - m_e)c^2 = 0.78 \text{ MeV}
\]
The energy of GTGR is pushed up from unperturbed (proton-hole) (neutron) or (proton) (neutron-hole) energies, due to the repulsive interaction in the $\hat{\sigma} \hat{T}$ channel.

⇒ Effective GT operator, $(\text{GT})_{\text{eff}} \approx (0.6 - 0.7) (\text{GT})_{\text{free}}$

Spin-dependent part of magnetic dipole (M1) operator is approximately

$$(M1) \propto \sum_{\mu=1}^{3} \sum_{k} t_{z}(k) \sigma_{\mu}(k) = [\Delta T_{z} = 0] \text{ part of GT operator, } (\text{GT})_{\pm} = \sum_{\mu=1}^{3} \sum_{k} t_{\pm}(k) \sigma_{\mu}(k)$$

$$(\text{GT})_{\pm} = \sum_{k} \pm \frac{\Delta T_{z}}{2}$$

M1GR

In heavy nuclei the strength of M1 GR is highly fragmented.

ex. $^{208}\text{Pb}$ (a j-j closed shell nucleus)

\[
\begin{align*}
\text{neutron} : & \quad \left( i_{13/2}^{1} i_{11/2}^{1/2} \right)_{1+} \quad \varepsilon_{p-h} = 5.57 \text{ MeV} \\
\text{proton} : & \quad \left( h_{11/2}^{1-1} h_{9/2}^{1/2} \right)_{1+} \quad \varepsilon_{p-h} = 5.85 \text{ MeV}
\end{align*}
\]

⇒ Giant M1 resonance centered around 7.3 MeV, with a full width of about 1 MeV.

⇒ $g_{s}^{\text{eff}} \approx (0.7) g_{s}^{\text{free}}$ for low-energy M1 transitions.

Cf. In $^{12}_{6}C_{6}$ ($S_{p}=15.96$, $S_{n}=18.72$ MeV)

$E_{x}(1^{+}, T=0) = 12.7$, $E_{x}(1^{+}, T=1) = 15.1$ MeV

M1 strength for $E_{x} < S_{n} (= 7.37$ MeV) measured by $^{208}\text{Pb}(\vec{\gamma}, \gamma)$ using highly polarized tagged photons

7.4.3. IsoVector Spin Giant Monopole Resonance (IVSGMR);

\[ \hat{O}_\pm = \sum_k t_\pm(k)\sigma_\mu(k)r_k^2 \]

\[ \sum_m |\langle m|\hat{O}_-|0\rangle|^2 - \sum_n |\langle n|\hat{O}_+|0\rangle|^2 = 3\left( N\langle r^4 \rangle_n - Z\langle r^4 \rangle_p \right) \]

spin-parity of the operator = \(1^+\)

This IVSM operator has the same spin, isospin and parity as those of GT operator, though IVSM mode is a compression mode while GT is not. Moreover, the GT strength extends to the continuum energy region much higher than that of the main peak, in the high energy region it may be experimentally difficult to differentiate IVSM strength from higher-lying GT strength.

Taking into account the orthogonality to GT operator, theoretically one needs to use

\[ \hat{O}_{IVSM} = \sum_k t_\pm(k)\sigma_\mu(k)\left(r_k^2 - \langle r^2 \rangle \right) \]

in order to obtain only the strength of IV Spin Monopole mode.


However, IVSM mode has a form factor quite different from that of GT transitions.

\( \rightarrow (^3He,t) \) with appropriate incident energies may excite IVSMR more easily than \((p,n)\)?

The dependence of cross sections on incident energies or a comparison of (p,n) with (^3He,t) may differentiate the strength of IVSM from that of GT.
In nuclei with a larger neutron excess, \( E_x(GR_—) > E_x(GR_+) \)

less (if not zero) GT\(_+\) strength is expected due to Pauli blocking (namely, the neutron level in \( p \rightarrow n \) by GT\(_+\) operator is already occupied).

\( \leftrightarrow \) Excitation energy of IVSGMR\(_+\) in daughter nuclei becomes considerably lower, compared with that of IVSGMR\(_-\) in daughter nuclei.

[ Maximum energy of relevant p-h configurations estimated from the ground state of mother nuclei]

(The collective peak may appear just above the max p-h energy, when unperturbed p-h excitations are spread over a broad energy region, compared with the strength of relevant p-h interactions.)

For stable nuclei  \( (N-Z)_{\beta-\text{stable}} \approx 6 \times 10^{-3} A^{5/3} \implies (N_F^n - N_F^p) \hbar \omega_0 \approx 0.183 A^{2/3} \) MeV

\[ \begin{align*}
E_x (IVSM_+) &= 2 \hbar \omega_0 - 0.183 A^{2/3} + \Delta E_{ls} \\
E_x (IVSM_-) &= 2 \hbar \omega_0 + 0.183 A^{2/3} + \Delta E_{ls}
\end{align*} \]

\[ \therefore [E_x (IVSGMR_-) - E_x (IVSGMR_+)] > 2 \times 0.183 A^{2/3}, \]

since IVSGMR\(_-\) is more collective than IVSGMR\(_+\) due to the neutron excess.

\[ 0.183 A^{2/3} = 6.42 \text{ MeV} \approx \hbar \omega_0 \text{ for } ^{208}Pb \]
The relation \[E_x(t_+ GR) < E_x(t_- GR)\] in nuclei with neutron excess is valid for all types of \(t_\pm\) GRs, though the actual energy difference depends also on the collectivity of modes.

In nuclei which are much more neutron-rich than \(\beta\)-stable nuclei, one has

1) \((N - Z) > 6 \times 10^{-3} A^{5/3}\)

2) The ground state of \(t_+\) daughter nuclei becomes much higher than that of mother nuclei.

Then, possible IVSGMR\(_+\) may have even lower \(E_x\) in daughter nuclei.

Or, some appreciable 1\(^+\) strength may be found at lower \(E_x\), when GT\(_+\) transitions should be forbidden.

One may try reactions such as \((n,p)\) or \((t, ^3\text{He})\) on such neutron-rich nuclei in the inverse kinematics, and find out the lower-lying spin-dependent strength?

Some comments:

1) Knowing that even the simplest compression mode, ISGMR, has not a simple resonance shape in the light-medium mass region, IVSM strength may not be concentrated on one collective resonance.

In the schematic harmonic oscillator model;
unperturbed p-h excitations for ISGMR are totally degenerate at \(2\hbar \omega_0\), while
those for IVSGMR are spread over \(2\hbar \omega_0 \pm \Delta E_{\ell 5} \approx 80 A^{-1/3} \pm 8 \text{ MeV}\).

2) Similar to GTGR or the GT strength distribution, IVSGMR may have a considerable amount of strength tail at the energy higher than the major peak, since it is also a spin-isospin mode.
Ex. of calculated charge-exchange spin monopole ($t_{\pm}$ SMR) modes

\[
\begin{align*}
^{208}_{82}Pb_{126} & \rightarrow ^{208}_{81}Tl_{127} & t_+ \text{ mode} \\
^{208}_{82}Pb_{126} & \rightarrow ^{208}_{83}Bi_{125} & t_- \text{ mode}
\end{align*}
\]

HF plus TDA with a Skyrme interaction

Response functions

Radial part of transition density of IVSGMR$_{\pm}$

(compression modes !)

$E_x$ is measured from the ground state of the mother nucleus

Possible high-energy tail of the strength is not obtained in this kind of calculations (namely, [HF plus TDA] or [HF plus RPA]).
7.4.4. IsoVector Spin Giant Dipole Resonance (IVSGDR) ;  
\[ \hat{O}_\pm = \sum_k t_\pm(k) r_k (Y_1(\hat{r}_k) \otimes \tilde{\sigma}(k))_{J\pi} \]
(There are a considerable amount of experimental data.)

Defining  
\[ S_{J_\pi}^{J_\pi} = \sum_m \left| \sum_k t_\pm(k) r_k (Y_1(\hat{r}_k) \otimes \tilde{\sigma}(k))_{J\pi} \right|^2 \]
where  \( J\pi = 0-, 1- \) and 2- ,

one obtains

\[ S_{-J} - S_{+J} = \frac{2J + 1}{4\pi} \left[ N \left\langle r_n^2 \right\rangle - Z \left\langle r_p^2 \right\rangle \right] \]

\[ \sum_{J=0,1,2} (S_{-J} - S_{+J}) = \frac{9}{4\pi} \left[ N \left\langle r_n^2 \right\rangle - Z \left\langle r_p^2 \right\rangle \right] \quad (\&) \]

ex. Using experimental data from \(^{90}\text{Zr}(p,n)\) and \(^{90}\text{Zr}(n,p)\) on the l.h.s. of (\&), the difference between \( \left\langle r_n^2 \right\rangle \) and \( \left\langle r_p^2 \right\rangle \) can be obtained, if \( \left\langle r_p^2 \right\rangle \) is known from the observed charge radius. (Harakeh & Woude, Giant Resonances, 2001)


The ground state of \(^{40}\text{Zr}_{50}\) has \( T = T_0 (= T_2) = (50 - 40) / 2 = 5 \)

\[ \text{IVSD}_- : \quad T = 4, 5, 6 \quad \text{in} \quad ^{90}\text{Zr}_{50}(p,n)_{40}^{90}\text{Nb}_{49} \]

\[ \text{IVSD}_+ : \quad T = 6 \quad \text{in} \quad ^{90}\text{Zr}_{50}(n,p)_{39}^{90}\text{Y}_{51} \]

A multipole decomposition analysis at \( \theta = 4.6^\circ \) (= max of SD mode) was performed, and the SD strengths up to 40 MeV in the left figure were included. \[ N \left\langle r_n^2 \right\rangle - Z \left\langle r_p^2 \right\rangle = 207 \pm 17 \text{ fm}^2 \]

neutron skin thickness :

\[ \sqrt{\left\langle r_n^2 \right\rangle} - \sqrt{\left\langle r_p^2 \right\rangle} = 0.07 \pm 0.04 \text{ fm} \]
7.5. Giant resonances in nuclei far away from the stability line

drip-line nuclei —— very different $N/Z$ ratio, compared to stable nuclei with the same $A$, in addition to the presence of weakly-bound nucleons.

Since the Fermi levels for protons and neutrons are very different in drip line nuclei, this binding energy difference of least-bound protons and neutrons will produce interesting phenomena in charge-exchange reactions or $\beta$ decays.
7.5.1. ISGQR of nuclei with weakly-bound neutrons

(an example of weakly-bound neutrons → threshold strength)

Ex. Calculated GQR of β-stable nuclei

Calculated GQR of neutron-drip-line nuclei

Increase of energy-weighted sum-rules, 
\[ S(IS, \lambda = 2)_{\text{class}} = \frac{50}{4\pi} \frac{\hbar^2}{2m} A \langle r^2 \rangle \], by the threshold strength

← extra contribution by weakly-bound neutrons in the ground state to \( \langle r^2 \rangle \).

Threshold strength couples very little with other p-h configurations

→ threshold strength contributes very little to \( e_{\text{pol}}(E2) \).
Ex. ISGQR of a possibly neutron-drip-line nucleus with weakly-bound neutrons, $^{60}_{20}Ca_{40}$

(calculated results only)

Compared with ISGQR in $\beta$-stable nuclei, the frequencies of possible neutron p-h configurations are lower, while the frequencies of proton p-h configurations remain nearly the same or become larger.

However, collective correlation structure

transition density

are similar to those of $\beta$-stable nuclei.

---

I.H., H.Sagawa and X.Z.Zhang, PRC 64, 024313 (2001).
7.5.2. $\beta$-decay to GTGR in drip line nuclei

The diagram illustrates the beta decay to ground state rotational (GTGR) nuclei in the drip line. The decay can occur through two pathways:

- **$\beta^+$ decay to GTGR$^+$**
- **$\beta^-$ decay to GTGR$^-$**

Key elements and isotopes highlighted include:

- $\text{(Ca) 20}$
- $\text{(Ni) 28}$
- $\text{(Sn) 50}$
- $\text{(Pb) 82}$
- $\text{(O) 8}$
- $\text{(He) 2}$

The stability line is marked along with the neutron drip line and magic number 126. The diagram uses arrows and labels to denote the decay paths and isotopes.
1) $\beta^-$ decay in nuclei with $N > Z$

**$\beta$ stable nuclei**

$$T_0 = \frac{N+1-Z}{2}$$

very neutron-rich light ($Z < 7$) nuclei

The relative energy between IAS and GTGR is a function of $(N-Z)/A$. The larger $(N-Z)/A$, the lower GTGR.

$$\Delta_{np} = (\Delta M(n) - \Delta M(^1H)) c^2 = 0.78 \text{ MeV}$$

$$\Delta E_{Coul}(Z+1) = E_{Coul}(Z+1) - E_{Coul}(Z) \propto ((Z+1)^2 - Z^2) A^{-1/3} \propto Z A^{-1/3}$$

Energy difference of different $T$ states in a given nucleus

$$E(A, T+1, M_T = T) - E(A, T, M_T = T) \approx 4b_{sym} \frac{T+1/2}{A} \propto \frac{|N-Z|}{A}$$
Dependence of the energy difference between \textit{IAS} and \textit{GTGR} on \(\frac{(N-Z)}{A}\)

\[
E(\text{GT}) - E(\text{IAS}) = 7.0 - 57.8 \frac{N-Z}{2A}
\]

Fig. 1. Plot of \(E_{\text{GT}} - E_{\text{IAS}}\) versus \(\frac{(N-Z)}{A}\). The experimental data were taken from ref. [81] for \(^{90}\text{Zr}\), from ref. [10] for

\begin{align*}
\text{K.Nakayama, A.Pio Galeao and F.Krmpotic, PLB114, 217 (1982)}
\end{align*}
2) $\beta^+$ decay in nuclei with $N > Z$

$\beta$ stable nuclei

$T_0 + 1$

$T_0$

$\beta^+$

$g.s.$

$(Z+1, N)$

$(Z, N+1)$

$T_0 = \frac{N - Z - 1}{2}$

medium-heavy proton-drip-line nuclei ($Z > 50$)

$\Delta E_{\text{Coul}} - \Delta_{np}$

$\Delta p$

$M(Z+1,N) - M(Z,N+1) + \Delta_{np}$

$(Z+1,N)$

$(Z, N+1)$

$(Z-1, N+1)$

The mass difference, $M(Z+1,N) - M(Z,N+1)$, increases rapidly, as stable $\rightarrow$ proton-drip-line nuclei.

$\Rightarrow$ GTGR$_+$ comes easily into the scope of $\beta^+$ decays, namely below the ground state of mother nuclei.