One-particle motion in nuclear many-body problem

(The 2nd lecture, V.2)

In this second lectures, V.2, first, the effective one-particle operators with $e_{\text{eff}}(E\lambda)$ and $g_{\text{eff}}(M\lambda)$ of electromagnetic transitions in the spherical case are reviewed. Then, the energies and electromagnetic moments in the laboratory system are examined, when the shape in the body-fixed system is deformed.

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The figures with figure-numbers but without reference, are taken from

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6. Energy and electromagnetic observables of one-particle states

6.1. **spherical** case – effective one-particle operators (E2, M1, E1)

The deviation of \( \frac{e_{pol}(E\lambda)}{e} \) and \( \frac{g^{eff}(M\lambda)}{g^{bare}} \) from unity depends on the multipole \( \lambda \), one-particle orbits, and the size of the configuration space included in the construction of wave functions.

For example, if the wave functions are constructed taking into account the whole degrees of freedom of all nucleons in a given nucleus, the “effective” operators should be the same as the bare operators, except the renormalization coming from possible non-nucleon degree of freedom.

In this section we assume that all possible configuration mixing within one major shell is already taken into account in the construction of wave functions of states. This includes so-called one-particle states (= one-particle + closed-shell core). Then, the renormalization of one-particle operators comes from the core polarization involving virtual excitations of giant resonances, besides the possible contribution by non-nucleon degree of freedom.

In other words, the major components of wave functions are explicitly taken into account in the construction of wave functions. The effect of some small components on the matrix element of a particular operator, which appreciably contribute to the matrix-element in spite of small admixed probabilities in wave functions, is expressed by renormalizing one-particle operators. → effective operators
Core polarization

If the relevant interaction is **attractive**, one-particle moments **increase**.

If the relevant interaction is **repulsive**, one-particle moments are **reduced**.

For spin polarization of the core the density above should be replaced by spin density.

\[
\Delta E_{tr} \ll \hbar \omega_{\text{core}} \quad (\Delta E_{tr} \text{: transition energy, } \hbar \omega_{\text{core}} \text{: energy of core excitations}), \text{ and } \\
[\text{mixed probability of core excitations into one-particle wave-functions}] \ll 1,
\]

the effect of admixed components can be expressed by the renormalization of one-particle operator → **static** polarization and **effective** one-particle operators.
1) one-particle energy, $\varepsilon_{ij}$, obtained for the potential is identified as an observed one-particle energy.

Or, alternatively one-particle energy can be calculated in the Hartree-Fock approximation, if the two-body interaction is sufficiently known, and the one-particle energy is identified as an observed one-particle energy.

In shell model calculations one-particle energies are often just parameters.
2) **Electric quadrupole moment operator**

\[ eQ_{op} = e \sum_p r_p^2 (3 \cos^2 \theta_p - 1) \]

For a single-particle in an orbit \((n\ell j)\)

\[ Q_{sp} = \langle n\ell j, m = j | r^2 (3 \cos^2 \theta - 1) | n\ell j, m = j \rangle = -\frac{2j-1}{2j+2} \langle n\ell j | r^2 | n\ell j \rangle \]

where \( \langle n\ell j | r^2 | n\ell j \rangle \equiv \int r^4 R_{n\ell j}^2(r)dr \)

**E2 transition operator**

\[ M(E2, \mu = 0) = \sqrt{\frac{5}{16\pi}} eQ_{op} \]

The reduced E2 transition probability

\[ B(E2; I_1 \rightarrow I_2) = \sum_{\mu M_2} | \langle I_2 M_2 | M(E2, \mu) | I_1 M_1 \rangle |^2 = \frac{1}{2I_1 + 1} \left| \langle I_2 \| M(E2) \| I_1 \rangle \right|^2 \]

For E2 transitions of a single particle

\[ B_{sp}(E2; n_1\ell_1 j_1 \rightarrow n_2\ell_2 j_2) = \frac{5}{4\pi} e^2 \left( C(j_1, j_2; 1/2, 0, 1/2) \langle n_2\ell_2 j_2 | r^2 | n_1\ell_1 j_1 \rangle \right)^2 \]

**In practice,**

\[ e \rightarrow e_{eff}(E2) = e_{bare} + e_{pol}(E2) \]

For low-energy transitions \( e_{pol}(E2) > 0 \)
Estimate of static E2 polarization charge using ISGQR and IVGQR in a harmonic oscillator model

\[ e_{pol}(E2) = e \left( \frac{Z}{A} - 0.32 \frac{N - Z}{A} + \left( 0.32 - 0.3 \frac{N - Z}{A} \right) \tau_z \right) \]

from ISGQR
n. excess to preserve the local ratio of n & p in IS GQR

from IVGQR

IV coupling field should not act on the total density at any point

Bohr & Mottelson, Vol.II, eq.(6-386b)

\[ \hbar \omega_{\text{ISGQR}} = 58 \text{ A}^{-1/3} \text{ MeV} \]
\[ \hbar \omega_{\text{IVGQR}} = 135 \text{ A}^{-1/3} \text{ MeV} \]

ISGQR increases both \( e_{pol}^n(E2) \) and \( e_{pol}^p(E2) \)

IVGQR increases \( e_{pol}^n(E2) \)
while decreases \( e_{pol}^p(E2) \)

Neutron excess of the core makes both \( e_{pol}^n(E2) \) and \( e_{pol}^p(E2) \) smaller.

For neutrons \( \tau_z = +1 \)
\[ e_{pol}^n(E2) = e \left( \frac{Z}{A} + 0.32 - 0.62 \frac{N - Z}{A} \right) \rightarrow \text{smaller, as (N-Z) becomes larger, for a given } A. \]

For protons \( \tau_z = -1 \)
\[ e_{pol}^p(E2) \approx e \left( \frac{Z}{A} - 0.32 \right) \]

ex.
\[ e_{pol}^p(E2) \approx e \left( \frac{20}{60} - 0.32 \right) \approx 0 \text{ for } ^{40}_{20} \text{Ca}_{40} \]
The value of $e_{pol}(E2)$ depends somewhat on nucleon orbits. In particular, the polarization effect decreases for weakly-bound nucleons, since those nucleons being outside the nuclear surface cannot efficiently polarize the core.

A simple approximate correction is to multiply the standard $e_{pol}(E2)$ in the previous page by

$$\frac{\left(\frac{3}{5}\right)R^2}{\langle j_2 | r^2 | j_1 \rangle}$$

Note $\langle \ell | r^2 | \ell \rangle \rightarrow \infty$ for $\ell=0$ and 1 neutrons, as $\epsilon_{\ell}(<0) \rightarrow 0$.

For neutrons

$$\langle \ell_2 | r^n | \ell_1 \rangle \quad \text{with} \quad \ell_1 + \ell_2 \leq n + 1 \quad \text{diverges as} \quad \epsilon_{\ell_1}, \epsilon_{\ell_2}(<0) \rightarrow 0$$
ex. Derivation of the first term of \[ e_{\text{pol}}(E2) = e \frac{Z}{A} + \ldots... \]

In the **harmonic oscillator model** one can show:

“One particle outside of the closed shell induces a mass quadrupole moment in the closed shell, which is equal to its own mass quadrupole moment.”


Mass quadrupole moment

\[
m(IS, \lambda = 2) = m_{sp}(IS, \lambda = 2) + m_{\text{core-pol}}(IS, \lambda = 2)
\]

Equilibrium shape for a system of a single-particle outside of closed shell

← self-consistency condition of potential and density

Then, in the harmonic oscillator model one obtains

\[
m_{\text{core-pol}}(IS, \lambda = 2) = m_{sp}(IS, \lambda = 2)
\]

∴ For E2 operator (Z : proton number of the core, A : nucleon number of the core)

\[
e_{\text{pol}}(E2) = e \frac{Z}{A} \quad \text{for both protons and neutrons}
\]

Note: this **harmonic oscillator model** produces the frequency of ISGQR

\[
\hbar \omega_{ISGQR} = \sqrt{2} \hbar \omega_0 = 58 A^{-1/3} \quad \text{MeV}
\]

which is consistent with the observed systematics.
3) Magnetic dipole moment of a single nucleon

\[ \vec{\mu} = g_\ell \vec{\ell} + g_s \vec{s} \]
\[ g_\ell = \begin{cases} 1 & \text{for proton} \\ 0 & \text{for neutron} \end{cases} \]
\[ g_s = \begin{cases} 5.58 & \text{for proton} \\ -3.82 & \text{for neutron} \end{cases} \]

\[ \mu = \langle j, m = j | g_\ell \ell_z + g_s s_z | j, m = j \rangle = j \left\{ g_\ell \pm (g_s - g_\ell) \frac{1}{2\ell + 1} \right\} \quad \text{for} \quad j = \ell \pm 1/2 \]

M1 transition operator

\[ M(M1, \mu) = \sqrt{\frac{3}{4\pi}} \frac{e\hbar}{2Mc} \mu^\mu \]

In practice,

\[ g_s \to g_s^{\text{eff}} \quad \text{and} \quad g_\ell \to g_\ell^{\text{eff}} \]

For low-energy transitions

\[ (g_s^{\text{eff}} / g_s) < 1 \]

since the relevant \((\tau\sigma)(\tau\sigma)\) type interaction is repulsive.

Empirical values in medium-heavy nuclei are

\[ (g_s^{\text{eff}} / g_s) = 0.6 \sim 0.7 \quad \text{for both protons and neutrons}, \]

while those in lighter nuclei are somewhat closer to unity.

The spin-saturated core (i.e. \(l\)-\(s\) closed nuclei such as \(^{16}\text{O}\) and \(^{40}\text{Ca}\)) cannot spin-polarize in the lowest order

\[ \to (g_s^{\text{eff}} / g_s^{\text{free}}) \approx 1 \quad \text{for one-particles outside the spin-saturated core.} \]
Writing
\[ g_{\ell}^{\text{eff}}(p) = 1 + \delta g_{\ell}(p) \quad \text{and} \quad \delta g_{\ell}^{\text{eff}}(n) = \delta g_{\ell}(n) \]

Empirical values are \( \delta g_{\ell}(p) \approx +0.1 \) and \( \delta g_{\ell}(n) \approx -0.05 \)


Those \( \delta g_{\ell} \) values are compatible with the effect of the meson-exchange current, while they are also consistent with the modification in the current implied by the velocity-dependent effective interaction.

(Bohr & Mottelson, Vol. II, p.484)

Core polarization effect may not simply be described in terms of a renormalization of bare one-particle operators. Thus, effective magnetic moment operator may have, for example, a term like

\[ (\delta \mu)_{\nu} = f(r)(Y_{2s})_{\lambda=1,\nu} \]

radial distribution of the polarizing particle
4) E1 transition operator, which should be orthogonal to the center of mass motion that must not create an excitation,

\[ M(E1, \mu = 0) = \sqrt{\frac{3}{4\pi}} e^\sum_i z_i \]

\[ e^\sum_i z_i \to e^\sum_i \left( z_i - \frac{1}{A} \left( \sum_j z_j + \sum_k z_k \right) \right) = e^\sum_i z_i - \frac{e}{A} Z \left( \sum_j z_j + \sum_k z_k \right) \]

\[ = \frac{N}{A} e^\sum_i z_i - Z \frac{e}{A} \sum_k z_k \]

In practice, in stable nuclei

\[ \left| e_{eff}^p (E1) \right| < \frac{N}{A} e \quad \text{and} \quad \left| e_{eff}^n (E1) \right| < \frac{Z}{A} e \]

due to the polarization effect associated with IVGDR (Iso Vector Giant Dipole Resonance).

\[ \begin{cases} e_{eff}^p (E1) = e(1 + \chi) \frac{N}{A} \\ e_{eff}^n (E1) = -e(1 + \chi) \frac{Z}{A} \end{cases} \]

where \( \chi \approx -0.7 \) (estimate in B&M VoL.II).

ex. Empirical values obtained in the Pb region are

\[ \left| e_{eff}^p (E1) \right|^2 \sim (0.10)e^2 > \left| e_{eff}^n (E1) \right|^2 \]

(from the analysis of E1 decays of octupole multiplet members in \(^{209}\text{Bi}\) and \(^{207}\text{Pb}\).)

In very light halo nuclei such as $^{11}$Be, one may expect

$$|e_{\text{eff}}^p (E1)| \approx \frac{N}{A} e \quad \text{and} \quad |e_{\text{eff}}^n (E1)| \approx \frac{Z}{A} e$$

\begin{align*}
\{ & \text{weakly-bound orbits} \rightarrow \text{a change of shell structure and wave-functions} \\
& \text{halo particles} \rightarrow \text{difficult to polarize the core} \}
\end{align*}
Observed low-energy E1 transitions in stable spherical nuclei are usually very much hindered;

In medium-heavy nuclei $B(E1) < (10^{-5}) B_W(E1)$

\[ e_{\text{eff}}(E1) \] values, due to the nuclear shell-structure there is no close-lying one-particle configurations that can be connected by E1 operators in either light or medium-heavy nuclei;

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<thead>
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<tr>
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<td>2s_{1/2}, 1d_{3/2}, 1d_{5/2}</td>
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<tr>
<td>3s_{1/2}, 2d_{3/2}, 2d_{5/2}, 1g_{7/2}, 1h_{11/2}</td>
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The strong hindrance of low-energy E1 transitions makes it almost impossible to obtain any nuclear structure information from the $B(E1)$ values.
6.2. From the $Y_{20}$ deformed intrinsic system to laboratory system

The intrinsic wave functions are not eigenstates of angular momentum, while the states observed in the laboratory system are the eigenstates. Thus, one has to construct the total wave functions using respective intrinsic wave functions.

Angular momentum projection from a deformed intrinsic wave function is one way of getting back an eigenstate of angular momentum. However, the projection includes no possible rotational perturbation of intrinsic states.

Particle-rotor model with particles (or some intrinsic degrees of freedom) referred to the body-fixed system is another model, in which angular momentum is a good quantum number.

In the following the simplest and practical (though approximate) way of getting back total angular momentum (Bohr & Mottelson, Vol.II), which is generally expected to work better in heavier nuclei.

In 6.2. a general form of the total wave function for a given intrinsic wave function with $Y_{20}$ deformed intrinsic shape (i.e. axially symmetric and R-invariant shape) is derived. The formulas can be used not only for intrinsic one-particle configurations but also for more complicated intrinsic configurations.

In 6.3. energies with $Y_{20}$ deformed intrinsic shape are described.

In 6.4. electromagnetic properties of the system with $Y_{20}$ deformed intrinsic shape are described.
From now on:

\[(1, 2, 3) : \text{body-fixed system} \]
\[(x, y, z) : \text{laboratory system} \]

\[
\begin{align*}
\mu : & \text{components referred to the laboratory system} \\
\nu : & \text{components referred to the body-fixed system}
\end{align*}
\]

\[
\begin{align*}
\vec{K} & = \Omega \\
\vec{I} & = \vec{R} + \vec{J}
\end{align*}
\]

Total angular momentum \( \vec{I} \)

axially-sym shape \( \rightarrow \ \vec{K} (\leftarrow I_3) = \Omega (\leftarrow J_3) \)

No collective rotation about symmetry axis ; \( R_3 = 0 \)

(OBS. No collective rotation in spherically-symmetric nuclei)
Total (= intrinsic x rotational) wave functions 
and consequences of symmetry

If the intrinsic and rotational parts of the Hamiltonian are separated, the eigenstates of the Hamiltonian are the product form

$$\Psi_{\alpha,I} = \Phi_\alpha(q) \varphi_{\alpha,I}(\omega)$$

where $\alpha$ : quantum number specifying intrinsic states, 
$q$ : intrinsic variable, 
$\omega$ : angular variables specifying the orientation of the deformed body with respect to the laboratory system, 
$I$ : angular-momentum quantum-numbers.

Rotational wave functions ;

(1) In 2-dimensional rotation (a rotation about a fixed axis)

$$\varphi_{\alpha,I}(\omega) \sim \exp \left( iM\theta \right) \quad \omega \rightarrow \theta \quad I \rightarrow M$$

(2) In 3-dimensional rotation

$$\varphi_{\alpha,I}(\omega) \sim D_{MK}^I(\omega) \quad \omega \rightarrow 3 \text{ Euler angles } (\Phi,\theta,\psi), \text{ to specify the orientation of the body.}$$

$I \rightarrow 3 \text{ quantum numbers:}$

$$\left( \vec{I} \right)^2, \ M (\leftarrow l_z), \ K (\leftarrow l_3)$$
\[ \vec{I}^2 D_{MK}^l = I(I+1)D_{MK}^l \]
\[ I_x D_{MK}^l = MD_{MK}^l \]
\[ I_z D_{MK}^l = KD_{MK}^l \]

\[ \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi D_{MM'}^{l'}(\omega)^* D_{M'M_1}^{l_1} \cdot (\omega) = \frac{8\pi^2}{2I+1} \delta(I_1, I') \delta(M, M_1) \delta(M', M'_1) \]

\[ [I_x, I_y] = iI_z \]
\[ [I_1, I_2] = -iI_3 \]

\[ \langle I, M | I_x \pm iI_y | I, M \mp 1 \rangle = \left( I(I+1) - M(M \mp 1) \right)^{1/2} \]
\[ \langle I, K | I_1 \pm iI_2 | I, K \pm 1 \rangle = \left( I(I+1) - K(K \pm 1) \right)^{1/2} \]

\[ I_x, I_y, I_z \] : referred to the lab.system
\[ I_1, I_2, I_3 \] : referred to the body-fixed system

\[ I_x, I_y, I_z \] : give the change in the state vector when the lab system is rotated about one of its own axes.
\[ I_1, I_2, I_3 \] : describe the change in the state vector when the lab system is rotated about an axis of the body-fixed system.
Rotational degrees of freedom \(\leftrightarrow\) restricted by the symmetry of deformation

- **ex. Spherically symmetric nuclei** \(\rightarrow\) No collective rotation
- **ex. Axially-symmetric deformed nuclei** \(\rightarrow\) No collective rotation about the symmetry axis
- **ex. \(R\)-invariant axially-symmetric deformation**
  \(\rightarrow\) rotation \(R_\perp(\pi)\) (\(\equiv\) rotation \(\pi\) about the axis \(\perp\) symmetry axis)
  must not be included in the rotational degrees of freedom

Correspondingly,
the form of total wave function (in general, a sum of products of intrinsic and rotational wave-functions) is governed by the symmetry of deformation.
Total wave function for $Y_{20}$ deformed intrinsic shape

(a) axially-symmetric shape $\rightarrow$ no collective rotation about the sym axis (=3-axis)

$$K (\leftarrow I_3) = \Omega (\leftarrow J_3)$$

$$\Psi_{KIM} = \Phi_K(q)D^I_{MK}(\phi, \theta, \psi)\sqrt{2I+1\over 8\pi^2}$$

$$= \Phi_K(q)D^I_{MK}$$

$R_\perp(\pi) \rightarrow R_2(\pi) \equiv \text{rotation } \pi \text{ about the 2-axis}$

(b) $R$-invariant shape, in addition to axial symmetry (taking $K > 0$)

$$\Psi_{KIM} = \sqrt{2I+1\over 16\pi^2}\left\{\Phi_K(q)D^I_{MK}(\phi, \theta, \psi) + (-1)^{I+K}\Phi_{-K}(q)D^I_{M,-K}(\phi, \theta, \psi)\right\}$$ ($\dagger$)

Rotation by $R_2(\pi)$ does not belong to collective rotation (quantum effect!).

i.e. from the two intrinsic states with $K$ and $-K$, only a single rotational state can be formed for a given $I$.

Note for a given $I$.

Obs. The 1st and 2nd term in ($\dagger$) can be connected by the operator with $\Delta K = 2K$.

$$\rightarrow (-1)^I \text{ dependent term in observables}$$

ex. For $K=1/2$ bands $\rightarrow$ the term $(\propto (-1)^I)$ in the energy

For a Hamiltonian with a coupling between intrinsic and rotational motion, a set of wave functions ($\dagger$) can be used as a basis for diagonalization. ex. particle-rotor model (Bohr & Mottelson, vol.II, Chap. 4A).
**R-invariance**: deformation is invariant under $R_2(\pi)$ (≡ rotation $\pi$ about the 2-axis)

Then, $R_2(\pi)$ is not included in collective rotational degrees of freedom.

$R \equiv R_2(\pi)$ can be expressed as
- $R_e \equiv R_2(\pi)$, rotation $\pi$ of the lab system ($x$, $y$, $z$) about the 2-axis
- $R_i \equiv R_2(\pi)$, rotation $\pi$ of the body about the 2-axis

$\Psi$: total wave-function

\[ R_e \Psi = R_i \Psi \]

is determined by

\[ (1 + R_i^{-1}R_e)\Phi_K(q)D_{MK}^I(\phi, \theta, \psi) = \Phi_K(q)D_{MK}^I(\phi, \theta, \psi) + (-1)^{l+K} \Phi_{-K}(q)D_{M-K}^I(\phi, \theta, \psi) \rightarrow (\$) \]

\[ R_e D_{MK}^I(\phi, \theta, \psi) = e^{-i\pi l_2}D_{MK}^I(\phi, \theta, \psi) = (-1)^{l+K} D_{M-K}^I(\phi, \theta, \psi) \]

$\Phi_{-K}(q) \equiv R_i^{-1}\Phi_K(q)$ : Intrinsic state with $-K$, which is degenerate with $\Phi_K(q)$

In fact, $\Phi_{-K}(q) = T \Phi_K(q)$ where $T$: time reversal operator

\[ R|K\rangle \propto |-K\rangle \] since $R_i$ inverts the direction of the 3-axis.

**R-inv** $\rightarrow$ Total wave function is a definite combination of two degenerate states with $K$ and $-K$. 
\[
\Psi_{KIM} = \sqrt{\frac{2I+1}{16\pi^2}} \left\{ \Phi_K(q)D_{MK}^I(\phi, \theta, \psi) + (-1)^{I+K} \Phi_{-K}(q)D_{MK}^I(-\phi, -\theta, -\psi) \right\}
\]

Euler angles: \( \omega \equiv (\phi, \theta, \psi) \)

**R-inv shape →**

the cross term of the first and second terms in the above \{ ... \} can produce;

**ex.1** \(( -1)^I \) dependent term in the expectation value of the operator \( j_{\pm}I_{\mp} \) (sim Coriolis coupling)

\[
\propto (-1)^{I+K} \left\langle \Phi_K(q)D_{MK}^I(\omega) \left| j_{\pm}I_{\mp} \right| \Phi_K(q)D_{MK}^I(\omega) \right\rangle
\]

\[
\propto (-1)^{I+K} \left\langle \Phi_K(q) \left| j_{\pm} \right| \Phi_K(q) \right\rangle \int d\omega D_{MK}^I(\omega)D_{MK}^I(\omega)
\]

that is non-zero only for \( K=1/2 \).

\( \therefore \) \( j_{\pm} \) and \( I_{\mp} \) change \( K \)-value only by \( \pm 1 \).

\( \rightarrow \) \(( -1)^I \) dependent term in the rotational energy of \( K=1/2 \) bands.

**ex.2** \(( -1)^I \) dependent part of matrix elements of the operator \( T_{\mu}^\lambda = \sum_\nu T_{\nu}^\lambda D_{\mu\nu}^\lambda(\omega) \)

\[
\propto (-1)^{I+K} \left\langle \Phi_K(q)D_{MK}^I(\omega) \left| \sum_\nu T_{\nu}^\lambda D_{\mu\nu}^\lambda(\omega) \right| \Phi_K(q)D_{MK}^I(\omega) \right\rangle
\]

\[
\propto (-1)^{I+K} \sum_\nu \left\langle \Phi_K(q) \left| T_{\nu}^\lambda \right| \Phi_K(q) \right\rangle \int d\omega D_{MK}^I(\omega)D_{MK}^I(\omega)
\]

can be non-zero for \( \nu = 2K \).

For example, in \( B(M1) \) within a given \( K=1/2 \) band, and

in \( B(E2) \) within a given \( K=1 \) band, but

not in \( B(E2) \) within a given \( K=1/2 \) band.

\( \lambda = 1 \) and \( |\nu| \leq 1 \) for \( M1 \)

\( \lambda = 2 \) and \( |\nu| \leq 2 \) for \( E2 \)
\textbf{\(K=0\) band}

\[\Psi_{K=0,IM} = D^I_{M,K=0}(\phi, \theta, \psi)\Phi_{K=0}(q) = \sqrt{\frac{4\pi}{2I+1}}Y_{IM}(\theta, \phi)\Phi_{K=0}(q)\]

\[R_eY_{IM}(\theta, \phi) = Y_{IM}(\pi - \theta, \phi + \pi) = (-1)^I Y_{IM}(\theta, \phi)\]

\textcolor{blue}{\textbf{inverts the direction of the sym axis (=3-axis)}}

\[R_i\Phi_{K=0} = r\Phi_{K=0}\]

\[R_e\Psi = R_i\Psi \quad \Rightarrow \quad (-1)^I = r\]

\(l = \text{even for } r = +1\)

\(l = \text{odd for } r = -1\)

The ground state of even-even nuclei has \(K=0\) and \(r = +1\)

(Pairwise-occupied (\(\pm \Omega\)) nucleon states have \(r = +1\).)

\[\Phi(1,2) = \frac{1}{\sqrt{2}}(\phi_{\Omega}(1)\phi_{\Omega}(2) - \phi_{\overline{\Omega}}(1)\phi_{\overline{\Omega}}(2))\]

\(\Phi(1,2) = \frac{1}{\sqrt{2}}(\phi_{\overline{\Omega}}(1)\phi_{\Omega}(2) + \phi_{\Omega}(1)\phi_{\overline{\Omega}}(2)) = \Phi(1,2)\)

This explains: the ground-band of even-even nuclei has only \(l^\pi = 0^+, 2^+, 4^+\), ….
one-particle states in the many-body system

In spherical case

[ closed-shell core with J=0 ] → spherical potential
{ one-particle + closed-shell core (J=0) } : one-particle states

In $Y_{20}$ deformed case

[ pairwise-occupied even-even core with K=0 ] → $Y_{20}$ deformed potential
{ one-particle + even-even core (K=0) } : one-particle states

For a moderate deformation,

the values of $e_{pol}(E\lambda)$ and $g_{pol}(M\lambda)$ in one-particle operators due to the virtual excitations of Giant Resonances of the core remain nearly the same as in spherical case.

However, $e_{pol}(E\lambda,|\nu|)$ and $g_{pol}(M\lambda,|\nu|)$ are expected, since the properties of GR in $Y_{20}$ deformed nuclei depend on the tensor components $|\nu|$ in the intrinsic system.
6.3. Energies with $Y_{20}$ deformed intrinsic shape

If the deformation and rotation degrees of freedom can be approximately separated, one expects a rotational band associated with each intrinsic configuration. In other words, to observe rotational spectra is a simple way to find that the nucleus is deformed.

One-particle energies obtained in a deformed potential correspond to the energies of band-head states with the intrinsic one-particle configurations.

In the present section we describe the properties of the states close to band-head states, without taking into account Coriolis perturbation of the intrinsic structure.

Rotational energy associated with a given one-particle configuration (where $K = \Omega$),

$$E_{\text{rot}}(K, I) \approx A \left\{ I(I + 1) + a(-1)^{I + \frac{1}{2}}(I + \frac{1}{2})\delta(K, \frac{1}{2}) \right\}$$

where \( a \equiv -\langle \Omega | j_+ | \bar{\Omega} \rangle \)

decoupling parameter \( a \approx -\langle [Nn_3 \Lambda \Omega] | j_+ | [Nn_3 \Lambda \Omega] \rangle = \delta(\Omega,1/2) \delta(\Lambda,0) (-1)^N \) for normal-parity orbits

$$a = (-1)^{j-1/2} \left( j + \frac{1}{2} \right)$$ for a single-j configuration

Thus, for normal-parity orbits the band-head state with $\Omega=1/2$ is almost always $I=1/2$, though the rotational spectra may deviate from $I(I+1)$.
ex. The $N=13$ th neutron orbit is seen in low-lying excitations in $^{25}\text{Mg}_{13}$

Note (a) $I \geq K$ (→ $I_z$)
(b) the bandhead state has $I=K$.
Exception may occur for $K=1/2$ bands.
(c) some irregular rotational spectra are observed for $K=1/2$ bands.

1) Leading-order E2 and M1 intensity relation works pretty well

$\rightarrow Q_0 \approx +50 \text{ fm}^2 \rightarrow \delta \approx 0.4$

$(g_K - g_R) \approx 1.4$ for [202 5/2] etc.
Rotational spectra unique in the intrinsic configuration with $\Omega = 1/2$

$$E_{\text{rot}}\left(K = \Omega = \frac{1}{2}, I\right) = \frac{\hbar^2}{2\lambda} \left\{ I(I+1) + a(-1)^{\frac{1}{2}}(I + \frac{1}{2}) \right\}$$

For one-particle in a single j-shell ($\approx$ high-j shell)

\[
a = (-1)^{j-1/2}\left(j + \frac{1}{2}\right)
\]

\[
\begin{align*}
  &+1 \quad \text{for } j=1/2 & 1/2 \\
  &-2 \quad \text{for } j=3/2 & 3/2 \\
  &+3 \quad \text{for } j=5/2 & 1/2 \\
  &-4 \quad \text{for } j=7/2 & 3/2 \\
  &+5 \quad \text{for } j=9/2 & 5/2 \\
  &-6 \quad \text{for } j=11/2 & 3/2 \text{ and } 7/2 \\
  &+7 \quad \text{for } j=13/2 & 5/2
\end{align*}
\]

In rotational bands with high-j configuration $[ I = j \text{ mod } 2 ]$ levels are pushed down relative to $[ I = j-1 \text{ mod } 2 ]$ levels, also after including the full Coriolis coupling.


\[
a = -\langle j, m = 1/2 | j, m = 1/2 \rangle = (-1)^{j-1/2} \langle j, m = 1/2 | j, m = -1/2 \rangle = (-1)^{j-1/2}\left(j + \frac{1}{2}\right)
\]
6.4. Electromagnetic properties of the system with $Y_{20}$ deformed intrinsic shape

Writing $|KIM\rangle$ for the state with the wave function $\Psi_{KIM}$ in ($\$\$),

$$\langle K_2 I_2 M_2 | T_{\lambda \mu} | K_1 I_1 M_1 \rangle = \frac{1}{\sqrt{2I_2 + 1}} C(I_1, \lambda I_2; M_1, \mu M_2) \langle K_2 I_2 \| T_\lambda \| K_1 I_1 \rangle$$

the reduced transition probability is written as

$$B(\lambda; I_1 \rightarrow I_2) = \frac{1}{2I_1 + 1} \left| \langle K_2 I_2 \| T_\lambda \| K_1 I_1 \rangle \right|^2$$

Using Bohr and Mottelson, Vol.II, eqs.(4-91) and (4-92) for the expressions of $\langle K_2 I_2 \| T_\lambda \| K_1 I_1 \rangle$

$$B(\lambda; K_1 I_1 \rightarrow K_2 I_2) = \left\{ C(I_1, \lambda I_2; K_1, K_2 - K_1, K_2) \langle K_2 \| T_{\lambda, K_2 - K_1} \| K_1 \rangle ight. + (-1)^{I_1 + K_1} \langle C(I_1, \lambda I_2; -K_1, K_1 + K_2, K_2) \langle K_2 \| T_{\lambda, K_1 + K_2} \| K_1 \rangle \right\}^2$$

for $(K_1 \neq 0$ and $K_2 \neq 0$)

For matrix elements within a band, the second term inside $\{ \}$ vanishes for

$$c(-1)^{2K} = +1$$

where $c = -1$ ($+1$) for electric (magnetic) transitions

If $K_1 = 0$,

$$B(\lambda, K_1 = 0, I_1 \rightarrow K_2 I_2) = C(I_1, \lambda I_2; 0K_2 K_2) \langle K_2 \| T_{\lambda, K_2} \| K_1 = 0 \rangle^2$$

$$\begin{cases} 2 & \text{for } K_2 \neq 0 \\ 1 & \text{for } K_2 = 0 \end{cases}$$

For matrix elements within a K=0 band, $\langle K = 0 \| T_{\lambda, 0} \| K = 0 \rangle = 0$, for magnetic operators.
For reference,

If the intrinsic moments $T_{\lambda \mu}$ does not depend on $I_\pm$, the matrix element between the two states with the form of the wave function, ($\Psi$), is given by

$$\langle K_2 I_2 \| T_\lambda \| K_1 I_1 \rangle = (2I_1 + 1)^{1/2} \left\{ C(I_1 \lambda I_2 ; K_1, K_2 - K_1, K_2) \langle K_2 \| T_{\lambda, \nu = K_2 - K_1} \| K_1 \rangle \right.$$  

$$+ (-1)^{I_1 + K_1} C(I_1 \lambda I_2 ; -K_1, K_1 + K_2, K_2) \langle K_2 \| T_{\lambda, \nu = K_1 + K_2} \| K_1 \rangle \right\}$$  

for $(K_1 \neq 0, K_2 \neq 0)$  

BM, Vol. II, eq.(4-91)

If one of the bands, or both, has $K=0$,

$$\langle K_2 I_2 \| T_\lambda \| K_1 = 0, I_1 \rangle = (2I_1 + 1)^{1/2} C(I_1 \lambda I_2 ; 0 K_2 K_2) \langle K_2 \| T_{\lambda, \nu = K_2} \| K_1 = 0 \rangle \left\{ \begin{array}{ll} \sqrt{2} & K_2 \neq 0 \\ 1 & K_2 = 0 \end{array} \right\}$$  

BM, Vol. II, eq.(4-92)

When the intrinsic states are one-particle configurations, the intrinsic matrix elements of $M_1, E_1$ and $E_2$ operators

$$\langle K_2 \| T_{\lambda, \mu} \| K_1 \rangle \quad \text{and} \quad \langle K_2 \| T_{\lambda, \mu} \| \bar{K}_1 \rangle$$

can be evaluated using Tables 1 and 2 appended in the end of Chap. 4, depending on whether the wave function of the one-particle configuration is approximated by an $[N n_3 \Lambda \Omega]$ representation or a single-j configuration.
Transitions between two bands with intrinsic configurations $\alpha_1, \Omega_1 (= K_1)$ and $\alpha_2, \Omega_2 (= K_2)$

ex. If $(-1)^{I+K}$ term is absent or negligible,

$$B(\lambda; \alpha_1 K_1 I_1 \rightarrow \alpha_2 K_2 I_2) = C(I_1\lambda I_2; K_1, K_2 - K_1, K_2)^2 \langle \alpha_2 K_2 | T_\lambda | \alpha_1 K_1 \rangle^2$$

kinematical factor

intrinsic matrix element, common in all transitions

= 0 for $\ |I_1 - I_2| > \lambda$ or $\ |K_1 - K_2| > \lambda$

The ratio of $B(\lambda)$ values between the members of given two bands is obtained from the Clebsch-Gordan coefficients,$;$

$$C(I_1\lambda I_2; K_1, K_2 - K_1, K_2)^2$$

$B(\lambda) : B(\lambda) : B(\lambda)$

$\approx C(I\lambda I + 1; K_1, K_2 - K_1, K_2)^2 : C(I\lambda I; K_1, K_2 - K_1, K_2)^2 : C(I\lambda I - 1; K_1, K_2 - K_1, K_2)^2$
1) Magnetic dipole (M1) moments and transitions

(One-particle) M1 operator in the intrinsic (= body-fixed) system

\[ \vec{M}_1 \propto g_R \vec{R} + g_\ell \vec{\ell} + g_s \vec{s} \]

\[ (M1)_v = \sqrt{\frac{3}{4\pi}} \frac{e\hbar}{2Mc} (g_R R_v + g_\ell \ell_v + g_s s_v) \]

\[ \vec{I} = \vec{R} + \vec{\ell} + \vec{s} \]

rotational angular momentum of the even-even core

magnetic moment \hspace{1cm} M1 transition

\[ \sqrt{\frac{3}{4\pi}} \frac{e\hbar}{2Mc} (g_R I_v + (g_\ell - g_R) \ell_v + (g_s - g_R) s_v) \]

\[ \therefore \] The operator \( I_v \) does not make any transitions.
\( g_R = \frac{Z}{A} \): a uniform rotation of a charged body

\( g_R \) values obtained from observed magnetic moments of \( 2_1^+ \) states of even-even nuclei using \( \mu = g_R I \) are somewhat smaller than \( Z/A \).

\[ g_R \approx \frac{\Im_p}{\Im_p + \Im_n} \]

where \( \Im \) (= moments of inertia) \( \rightarrow \) larger for \( \Delta \rightarrow \) smaller

ex. In even-even rare-earth nuclei the pairing gap \( \Delta_p > \Delta_n \rightarrow g_R < \frac{Z}{A} \)

In odd-A nuclei one may expect

\[
\begin{aligned}
\begin{cases}
  g_R > \frac{Z}{A} & \text{for odd-Z nuclei where } \Delta_p \rightarrow \text{smaller} \text{ and } \Im_p \rightarrow \text{larger} \\
  g_R < \frac{Z}{A} & \text{for odd-N nuclei where } \Delta_n \rightarrow \text{smaller} \text{ and } \Im_n \rightarrow \text{larger}
\end{cases}
\end{aligned}
\]

Indeed, one observes \( (g_R)_{\text{odd}-Z} > (g_R)_{\text{odd}-N} \)

In practice,

\[ g_s \rightarrow g_s^{\text{eff}} \quad \text{and} \quad g_\ell \rightarrow g_\ell^{\text{eff}} \]

Furthermore, in axially-symmetric deformed nuclei one generally expects

\[ g_{s_3} \neq g_{s_1} = g_{s_2} \]
For one-particle configuration with \( \Omega \) in \( Y_{20} \) deformed shape potential, we have \( K = \Omega \), and static magnetic dipole moments and M1 transition probabilities within a given one-particle configuration (i.e. within a given band) can be written

\[
\mu = g_R I + (g_K - g_R) \frac{K^2}{I + 1} + \delta(K,1/2) \frac{g_K - g_R}{4(I + 1)} (2I + 1)(-1)^{I + 1/2} b
\]

\[
B(M1;K,I_1 \to K,I_2 = I_1 \pm 1) = \begin{cases} 
\frac{3}{4\pi} \left( \frac{e\hbar}{2Mc} \right)^2 (g_K - g_R)^2 K^2 (C(I_1 I_2; K0K))^2 & \text{for } K > \frac{1}{2} \\
\frac{3}{16\pi} \left( \frac{e\hbar}{2Mc} \right)^2 (g_K - g_R)^2 \left\{ 1 + (-1)^{I - I_1 + 1/2} b \right\}^2 (C(I_1 I_2; 1/2,0,1/2))^2 & \text{for } K = \frac{1}{2}
\end{cases}
\]

where \( l_\ge \) denotes the greater of \( l_1 \) and \( l_2 \),

\[
g_K K = \left\langle \Omega | g_\ell \ell_3 + g_s s_3 | \Omega \right\rangle
\]

and \( b \) ( = magnetic decoupling parameter) is defined by

\[
(g_K - g_R) b = \left\langle \Omega = 1/2 | (g_\ell - g_R) \ell_+ + (g_s - g_R)s_+ | \Omega = 1/2 \right\rangle
\]

which can be rewritten

\[
(g_K - g_R) b = -(g_\ell - g_R) a - \frac{1}{2} (-1)^\ell (g_s + g_K - 2g_\ell)
\]

\[
j_+ = \ell_+ + s_+
\]
Observed $g_R$ factors from the 2+ first rotational states of even-even nuclei

$g_R$ and $g_K$ factors in odd-Z and odd-N nuclei obtained by combining a measured magnetic moment with a measured B(M1) value

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Orbit</th>
<th>$g_R$</th>
<th>$(g_R)_{obs.}$</th>
<th>$(g_R)_{calc.}$</th>
<th>$(g_R)<em>{eff}/(g_R)</em>{free}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{153}\text{Eu}$</td>
<td>413 5/2</td>
<td>0.47</td>
<td>0.67</td>
<td>0.30</td>
<td>0.57</td>
</tr>
<tr>
<td>$^{159}\text{Yb}$</td>
<td>411 3/2</td>
<td>0.42</td>
<td>1.83</td>
<td>2.28</td>
<td>0.71</td>
</tr>
<tr>
<td>$^{165}\text{Ho}$</td>
<td>523 7/2</td>
<td>0.43</td>
<td>1.35</td>
<td>1.53</td>
<td>0.72</td>
</tr>
<tr>
<td>$^{167}\text{Tm}$</td>
<td>411 1/2</td>
<td>0.41</td>
<td>-1.57</td>
<td>-2.44</td>
<td>0.79</td>
</tr>
<tr>
<td>$^{175}\text{Lu}$</td>
<td>404 7/2</td>
<td>0.31</td>
<td>0.73</td>
<td>0.41</td>
<td>0.55</td>
</tr>
<tr>
<td>$^{181}\text{Ta}$</td>
<td>404 7/2</td>
<td>0.29</td>
<td>0.78</td>
<td>0.41</td>
<td>0.48</td>
</tr>
<tr>
<td>$^{185}\text{Re}$</td>
<td>402 5/2</td>
<td>0.42</td>
<td>1.61</td>
<td>1.90</td>
<td>0.74</td>
</tr>
<tr>
<td>$^{187}\text{Re}$</td>
<td>402 5/2</td>
<td>0.41</td>
<td>1.63</td>
<td>1.90</td>
<td>0.76</td>
</tr>
</tbody>
</table>

Odd-proton configurations

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Orbit</th>
<th>$g_R$</th>
<th>$(g_R)_{obs.}$</th>
<th>$(g_R)_{calc.}$</th>
<th>$(g_R)<em>{eff}/(g_R)</em>{free}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{152}\text{Gd}$</td>
<td>521 3/2</td>
<td>0.32</td>
<td>-0.48</td>
<td>-0.61</td>
<td>0.79</td>
</tr>
<tr>
<td>$^{157}\text{Gd}$</td>
<td>521 3/2</td>
<td>0.26</td>
<td>-0.53</td>
<td>-0.61</td>
<td>0.87</td>
</tr>
<tr>
<td>$^{161}\text{Dy}$</td>
<td>642 5/2</td>
<td>0.21</td>
<td>-0.34</td>
<td>-0.45</td>
<td>0.76</td>
</tr>
<tr>
<td>$^{161}\text{Dy}$</td>
<td>523 3/2</td>
<td>0.32</td>
<td>0.17</td>
<td>0.39</td>
<td>0.44</td>
</tr>
<tr>
<td>$^{169}\text{Dy}$</td>
<td>523 5/2</td>
<td>0.27</td>
<td>0.25</td>
<td>0.39</td>
<td>0.64</td>
</tr>
<tr>
<td>$^{167}\text{Er}$</td>
<td>633 7/2</td>
<td>0.18</td>
<td>-0.26</td>
<td>-0.39</td>
<td>0.67</td>
</tr>
<tr>
<td>$^{171}\text{Yb}$</td>
<td>521 1/2</td>
<td>0.28</td>
<td>1.43</td>
<td>1.75</td>
<td>0.82</td>
</tr>
<tr>
<td>$^{173}\text{Yb}$</td>
<td>512 5/2</td>
<td>0.28</td>
<td>-0.48*</td>
<td>-0.79*</td>
<td>0.71*</td>
</tr>
<tr>
<td>$^{177}\text{Hf}$</td>
<td>514 7/2</td>
<td>0.26</td>
<td>0.21</td>
<td>0.40</td>
<td>0.52</td>
</tr>
<tr>
<td>$^{179}\text{Hf}$</td>
<td>624 9/2</td>
<td>0.22</td>
<td>-0.22</td>
<td>-0.35</td>
<td>0.63</td>
</tr>
</tbody>
</table>

Table 5-14 Magnetic $g$ factors for odd-$A$ nuclei ($150 < A < 190$). The experimental data are

Figure 4-6 $g$ factors for first excited 2+ states in even-even nuclei. The figure is based on
Can the measured magnetic moment of the ground state with $I\pi=1/2^+$ in $^{11}\text{Be}$ or $^{15}\text{C}$ tell whether the nucleus is spherical or deformed?

\[
\mu_{\text{obs}} = -1.6816(8) \quad \mu_N \quad \text{in } ^{11}\text{Be}_7 \quad \text{(W.Geithner et al.,PRL,1999)}
\]

\[
|\mu_{\text{obs}}| = 1.720(9) \quad \mu_N \quad \text{in } ^{15}\text{C}_9 \quad \text{(K.Asahi et al.)}
\]

The answer is “no”.  


For a spherical shape the relevant one-particle orbit must be $s_{1/2}$. Then, \( \mu = (0.5) g_s \text{eff} \) in \( \mu_N \).

For a prolately deformed shape the one-particle orbit must be the \([220 1/2]\) orbit.

\[
\begin{align*}
g_\ell &= 0 \quad \text{because of neutron,} \\
g_K &= \langle \Omega | g_\ell \ell_3 + g_s s_3 | \Omega \rangle / K = g_s \\
(g_K - g_R) b &= -(g_\ell - g_R) a - \frac{1}{2} (-1)^\ell (g_s + g_K - 2g_\ell) = g_R - \frac{1}{2} (g_s + g_K)
\end{align*}
\]

\[
\mu = g_R I + (g_K - g_R) \frac{K^2}{I+1} + \delta(K,1/2) \frac{g_K - g_R}{4(I+1)} (2I+1)(-1)^{I+1/2} b = (0.5) g_s \text{eff} \quad \text{in } \mu_N. 
\]

(independent of \( g_R \))
2) Electric quadrupole (E2) transitions

With quadrupole deformed intrinsic shape all nucleons collectively contribute to E2 moments.

Intrinsic quadrupole moment with an axially symmetric quadrupole deformation

\[ eQ_0 \equiv \langle K | e \sum_p r_p^2 (3\cos^2 \theta_p - 1) | K \rangle = \left( \frac{16\pi}{5} \right)^{1/2} \langle K | M(E2, \nu = 0) | K \rangle \]

where \( M(E2, \nu) \) denotes the components referred to the body-fixed system.

The E2 moments referring to the lab. system

\[ M(E2, \mu) = \sum_{\nu} M(E2, \nu) D_{\mu\nu}^2 (\omega) \Rightarrow M(E2, \nu = 0) D_{\mu,0}^2 (\omega) \]

\( \omega = (\phi, \theta, \psi) \) : Euler angles

The collective E2 moment above connects states belonging to the same rotational band.

\[ B(E2; KI_1 \rightarrow KI_2) = \frac{5}{16\pi} e^2 Q_0^2 C(I_1, 2I_2; K0K)^2 \]

where for \( I \gg K \),

\[ C(I_1, 2I_2; K0K) \approx \begin{cases} \left( \frac{3}{8} \right)^{1/2} & \text{for } I_2 = I_1 \pm 2 \\ \pm \left( \frac{3}{2} \right)^{1/2} \frac{K}{I} & \text{for } I_2 = I_1 \pm 1 \\ -\frac{1}{2} & \text{for } I_2 = I_1 \end{cases} \]

ex. In well-deformed rare-earth nuclei,

\[ B(E2; K=0, l=2 \rightarrow K=0, l=0) \approx 200 B_{W}(E2) \]
The static quadrupole moment in the lab system

\[ Q = C(I2I; K0K)C(I2I; I0I)Q_0 = \frac{3K^2 - I(I + 1)}{(I + 1)(2I + 3)}Q_0 \]

\[ \begin{align*}
  & I \to \infty \quad \text{keeping a fixed} \quad K \quad ; \\
  & Q \to -\frac{Q_0}{2}
\end{align*} \]

For \( I=K \) (i.e. the band head state in most cases)

\[ Q = \frac{I(2I - 1)}{(I + 1)(2I + 3)}Q_0 \]

Note \( I \to \infty \quad \text{keeping} \quad K = I \quad ; \)

\[ Q \to Q_0 \quad ; \quad \text{classical limit} \]
For ellipsoidal shape (or triaxial shape)

$K$ is not a good quantum number, and the collective $E2$ moments depend on two intrinsic quadrupole parameters, $Q_0$ and $Q_2$.

$$M(E2, \mu) = \sum_{\nu} M(E2, \nu) D_{\mu \nu}^2(\omega) \Rightarrow \sqrt{\frac{5}{16\pi}} e^{\{Q_0 D_{\mu 0}^2 + Q_2 (D_{\mu 2}^2 + D_{\mu -2}^2)\}}$$

where

$$Q_0 = \langle \alpha | \sum_p (2x_3^2 - x_1^2 - x_2^2)_p | \alpha \rangle \Rightarrow \left(\frac{4}{5}\right)ZR_0^2 \beta \cos \gamma$$

$$Q_2 = \sqrt{\frac{3}{2}} \langle \alpha | \sum_p (x_1^2 - x_2^2)_p | \alpha \rangle \Rightarrow \left(\frac{4}{5\sqrt{2}}\right)ZR_0^2 \beta \sin \gamma$$

$|\alpha\rangle$ : intrinsic state

$$r^2 Y_{20} = \sqrt{\frac{5}{16\pi}}(2x_3^2 - x_1^2 - x_2^2)$$

$$r^2 Y_{22} = \sqrt{\frac{15}{32\pi}}(x_1 + ix_2)^2$$

$$r^2 Y_{2-2} = \sqrt{\frac{15}{32\pi}}(x_1 - ix_2)^2$$

$$\langle I_2 K_2 \| M(E2) \| I_1 K_1 \rangle = (2I_1 + 1)^{1/2} \left(\frac{5}{16\pi}\right)^{1/2} e^{\{Q_0 C(I_1 2I_2; K_1 0K_2) + Q_2 (C(I_1 2I_2; K_1 2K_2) + C(I_1 2I_2; K_1,-2,K_2))\}}$$
3) Electric dipole (E1) transitions

In $Y_{20}$ deformed nuclei one expects

$$e_{pol}(E1, \nu = 0) \neq e_{pol}(E1, \nu = \pm 1)$$

since GDR (Giant Dipole Resonance) in $Y_{20}$ deformed nuclei splits into 2 peaks with $\nu = 0$ and $\nu = \pm 1$
ex.1. In very light **halo** nuclei such as $^{11}$Be, one may expect

$$|e_{\text{eff}}^p(E1)| \approx \frac{N}{A} e \quad \text{and} \quad |e_{\text{eff}}^n(E1)| \approx \frac{Z}{A} e \quad (\%)$$

$S_n = 504 \text{ keV}$

$^{11}_4\text{Be}_7$

$^{1/2-}_{320} \quad \approx p_{1/2}$

$0 \quad \approx s_{1/2}$

Even if the nucleus is deformed.

a) $\epsilon(s_{1/2})$ is pushed down relative to $\epsilon(p_{1/2})$ due to weakly bound

b) $\{ \text{The } [220] \ 1/2 + \text{ wave function } \sim s_{1/2} \}$ because of halo.

**Observed Strong E1 transition,**

$B(E1; 1/2^+ \rightarrow 1/2^-) = (0.115 \pm 0.01) \ e^2 \text{fm}^2 = 0.36 \ B_w(E1) \ : \text{the largest } B(E1) \text{ so far observed.}$

The observed large $B(E1)$ value can be indeed explained by using the value $(\%)$ together with a deformation $\beta = 0.7\sim 0.8$.\quad (I.H. and S.Shimoura, J.Phys.G:34(2007)2715.)

**Note**

$\frac{1}{2}^- \text{ at } 320 \text{ keV } \sim [101 \ 1/2]$ \quad Asymptotically $<[101 \ 1/2]|E1|[2201/2]> = 0$

The ground $\frac{1}{2}^+ \sim [220 \ 1/2]$ \quad Thus, if it is not a **halo** nucleus, the E1 transitions are much hindered.
ex.2. Both quadrupole- and octupole deformation $\rightarrow$ intrinsic dipole moment.

Relatively large $B(E1) = (10^{-2} \sim 10^{-4})$ values are observed between the yrast positive- and negative-parity bands in the Ra-Th region ($N \sim 136$) and Ba-Sm region ($N \sim 88$), especially for high spins.

Those nuclei are supposed to be quadrupole-soft (or deformed) and octupole-soft (or deformed).

Octupole deformation in addition to quadrupole deformation

$\rightarrow$ a shift between the center of charge and the center of mass

(Electric charge would move toward the surface region with large curvature.)

$\rightarrow$ dipole moment $D$ in the body-fixed frame

In the body-fixed system

$$e \frac{N}{A} \sum_i (p) z_i - e \frac{Z}{A} \sum_k (n) z_k = e \frac{NZ}{A} \left( \frac{1}{Z} \sum_i (p) z_i - \frac{1}{N} \sum_k (n) z_k \right) = e \frac{NZ}{A} (z_{p-c.m.} - z_{n-c.m.})$$

c.m. coordinate for neutrons

c.m. coordinate for protons

Assuming an axially-symmetric shape

$$D_{\nu=0} \propto (\beta_2 \beta_3)_{1-,\nu=0}$$
Octupole softness (or deformation) can be seen from observed very low-lying negative-parity levels in even-even nuclei.

Ex. in $^{224}_{88}Ra_{136}$ the lowest 1- state is known only at 216 keV!

If octupole soft in $Y_{20}$ deformation

$K = 0^-$ band:

$I = 1, 3, 5, \ldots, \ldots,$ all with $\pi = -$.

$K = 1^-$ band:

$I = 1, 2, 3, 4, 5, \ldots,$ all with $\pi = -$. 
ex.3.

Measured $B(E1) \sim 10^{-5} B_{w}(E1)$ values in many deformed rare-earth nuclei, which are supposed not to be octupole soft, are difficult to be explained, especially those in odd-A nuclei.