Current Status and Future Perspectives of Lattice Gauge Theories

Yoshinobu Kuramashi
University of Tsukuba

September 25, 2010, Wako
Green functions in path integral formulation on 4-dim. space-time lattice

\[ \langle \mathcal{O}[U_\mu, q, \bar{q}] \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \mathcal{D}q \mathcal{D}\bar{q} \; \mathcal{O}[U_\mu, q, \bar{q}] \exp \left\{ - \int d^4x \mathcal{L}[U_\mu, q, \bar{q}] \right\} \]

Numerical integration with Monte Carlo method

\[ \langle \mathcal{O}[U_\mu, q, \bar{q}] \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}[U_\mu^{(i)}, q^{(i)}, \bar{q}^{(i)}] + \mathcal{O} \left( \frac{1}{\sqrt{N}} \right) \]

\[ U^{(i)}_\mu, q^{(i)}, \bar{q}^{(i)} : i\text{-th configuration} \]

statistical error \( \propto 1/\sqrt{N} \)
Simulation Parameters

Few parameters

- **4-dim. volume**: $V = NX \cdot NY \cdot NZ \cdot NT$
- **lattice spacing**: $a$ (as function of bare coupling $g$)
- **quark masses**: $m_q (q=\text{u,d,s,c,b,t})$
Major Systematic Errors

- Finite volume effects
  \[ \Rightarrow \text{larger } V = N_X \cdot N_Y \cdot N_Z \cdot N_T \]

- Finite lattice spacing effects
  \[ \Lambda_{\text{QCD}} \ll 1/a, \text{ currently } m_b > 1/a \]
  \[ \Rightarrow \text{smaller } a \]

- Quench approximation (neglect quark vacuum polarization)
  \[ \Rightarrow 2+1 \ (m_u = m_d \neq m_s) \text{ flavor simulation} \]

- Chiral extrapolation
  \[ \Rightarrow \text{simulations at physical quark masses (physical point)} \]

Need heavier computational cost to diminish the systematic errors

\[ \text{cost } \propto (\text{physical vol.})^{1.25} \cdot (\text{lattice spacing})^{-6\sim-7} \cdot (\text{quark mass})^{-2\sim-3} \]
## Quark Actions

<table>
<thead>
<tr>
<th>action</th>
<th>group</th>
<th>features</th>
<th>lat11 (expected)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wilson tree-imp</td>
<td>BMW</td>
<td>O((\alpha \cdot a)) error brute force</td>
<td>(6 fm)(^3), a(\sim)0.06fm physical point</td>
</tr>
<tr>
<td>KS</td>
<td>MILC</td>
<td>nonlocal</td>
<td>(6 fm)(^3), a(\sim)0.1fm physical point</td>
</tr>
<tr>
<td>Twisted Mass</td>
<td>ETM</td>
<td>Wilson-type</td>
<td>(3.8 fm)(^3), a(\sim)0.06fm m(_\pi)&lt;250MeV</td>
</tr>
<tr>
<td>Wilson NP-imp</td>
<td>PACS-CS</td>
<td>O(a(^2)) error</td>
<td>(6 fm)(^3), a(\sim)0.09fm physical point</td>
</tr>
<tr>
<td>DWF</td>
<td>RBC/UKQCD</td>
<td>(almost) GW</td>
<td>(4.5 fm)(^3), a(\sim)0.14fm m(_\pi)&lt;180MeV</td>
</tr>
<tr>
<td>Overlap</td>
<td>JLQCD</td>
<td>GW Q fixed</td>
<td>(2.6 fm)(^3), a(\sim)0.11fm m(_\pi)~300MeV</td>
</tr>
</tbody>
</table>
What Await First Principle Calculations? (1)

- Hadron spectrum as a fundamental quantity
  - easy to understand current status of lattice QCD calculation

- Unstable particles
  - new quark composite states (tetraquark etc.)
  - $\pi\pi \rightarrow \rho$ resonance is a simplest case

- Nuclei based on QCD
  - nuclear force $\Rightarrow$ Hatsuda
  - direct construction in lattice QCD
What Await First Principle Calculations? (2)

• Hadron matrix elements of phenomenological importance
  – nonperturbative renormalization
  – use of GW for 4-fermi operators ⇒Christ

• Chiral symmetry breaking
  – GW case ⇒Fukaya
  – Wilson case Giusti-Lüscher 09

• Finite temperature and density ⇒Ejiri

• Application to BSM ⇒Yamada, Suzuki
Plan of Talk

§1. Introduction
§2. Recent Algorithmic Improvement
§3. Hadron Spectrum
§4. $\pi\pi \rightarrow \rho$ Resonance
§5. Nuclei in Lattice QCD
§6. Nonperturbative Renormalization
§7. Summary
§2. Recent Algorithmic Improvement

time consuming part in HMC = integration of Hamilton eq. in MD

\[
\frac{d}{d\tau} U_\mu(x) = P_\mu(x) U_\mu(x) \\
\frac{d}{d\tau} P_\mu(x) = - F_g(x, \mu) - F_q(x, \mu)
\]

\[D^{-1}[U_\mu,m_q]\] is required for evaluation of \(F_q\)
Expected Simulation Cost as of 2001

- $N_f=2+1$
- 100 configs
- $a=0.1\text{fm}$
- $L=3\text{fm}$

$\Rightarrow$ need algorithmic improvement
Multiple Time Scale Integration

1. separation of UV and IR contributions: \( F_q \rightarrow F_q^{UV} + F_q^{IR} \)

   domain decomposition / mass preconditioning
   \( \det[M] = \det[M'] \cdot \det[M/M'] \)

2. adaptive choice of step size

   \( \delta \tau_g \|F_g\| \approx \delta \tau_q^{UV} \|F_q^{UV}\| \approx \delta \tau_q^{IR} \|F_q^{IR}\| \)

reduce the number of \( D^{-1}[U_{\mu,m_q}] \) calculation

ex. \( \|F_g\| : \|F_q^{UV}\| : \|F_q^{IR}\| = 16 : 4 : 1 \)
Effectiveness of the improvement

\[ N_f = 2 + 1 \]
\[ 100 \text{ configs} \]
\[ a = 0.1 \text{ fm} \]
\[ L = 3 \text{ fm} \]

\[ \Rightarrow \text{physical point simulation is possible} \]
Why Physical Point Simulation?

• difficult to trace chiral logs for chiral extrapolation

• ChPT is not always a good guiding principle

• direct treatment of resonances based on phase shift

• simulations with different up and down quark masses
Toward ExaScale Computing

• difficulties
  - hierarchical structure of parallelism
  - diminishing B/F
  - floating-point performance >> network performance
  ⇒ need algorithmic improvements/development
    for weak scaling

• positive points
  - remaining systematic errors are finite volume and finite a
§3. Hadron Spectrum

Fundamental quantities both in physical and technical senses

physical side
- physical input ⇒ m_u, m_d, m_s, … ⇒ reproduce hadron spectrum?
  (ex. m_π, m_K, m_Ω)
- validity of QCD / determination of m_q

technical side
- hadron correlators in terms of quark fields

\[ \langle \mathcal{O}_h(t) \mathcal{O}_h^\dagger(0) \rangle \sim \exp(-m_h t) \] ⇒ extract m_h by fit

quark diagrams
- from Wick contractions
  - meson
  - baryon
History of Hadron Spectrum Calculation

1981  First calculation of hadron masses in quenched approx.
      Hamber-Parisi
      Demonstrate the possibility of first principle calculations

1996～2000  Precision measurement in quenched approx.
            CP-PACS
            Clear deviation from the experiment

2000～  Initiate 2+1 flavor QCD simulations
        CP-PACS/JLQCD, MILC, …
        Incorporate u,d,s vacuum polarization effects
        Reduce ud quark mass toward physical value
Hadron Spectrum in Quenched QCD

physical input $m_\pi$, $m_K$ or $m_\phi$, $m_\rho \Rightarrow m_u = m_d$, $m_s$, $a$

~10% deviation from experimental values
Chiral Behavior (1)

reduction of ud quark mass ⇒ check chiral properties as a by-product

Amorós-Bijnens-Talavera 01

logarithmic curvature expected from ChPT
Chiral Behavior (2)

PACS-CS 10

nontrivial curvature observed

ChPT fit gives reasonable LECs
Hadron Spectrum in 2+1 Flavor QCD

physical input $m_\pi, m_K, m_\Omega \Rightarrow m_u=m_d, m_s, a$

consistent within 2~3% error bars
similar results are obtained by other groups

PACS-CS 09
What Next?

• 2+1 \((m_u=m_d\neq m_s)\) flavor simulation at the physical point
  \(PACS-CS\ 10\)
  avoid chiral extrap. with 200\(\sim\)300\(\text{MeV}\) < \(m_\pi\) < 500\(\sim\)600\(\text{MeV}\)

• 1+1+1 \((m_u\neq m_d\neq m_s)\) flavor simulation at the physical point
  – electromagnetic interactions
  quenched study: Eichten et al. 96, Blum et al. 07,10
  MILC@lat10, BMW@lat10
  – u-d quark mass difference

• Direct treatment of resonances in lattice QCD
  decay width of resonance states

\[
\begin{array}{c|c|c}
K^0(d\bar{s}) & 497.6\text{MeV} \\
\hline
K^+(u\bar{s}) & 493.7\text{MeV}
\end{array}
\]
§3. \( \pi \pi \to \rho \) Resonance

\( \rho \) resonance in \( e^+e^- \to \pi^+\pi^- \) cross section

Difficulties in lattice QCD:
- 2 or 2+1 flavor QCD simulations at \( 2m_\pi < m_\rho \)
- how to treat resonances or scattering in a finite box?
Basic Idea

Finite size formula: Lüscher 91,86
energy shift due to interaction in a finite box \Rightarrow phase shift

ex. 1-dim. Schrödinger wave function \( \psi(x,y)=f(x-y) \) with periodic BC

\[
\text{free} \quad p_n = \frac{2\pi}{L} n \quad n \in \mathbb{Z}
\]

\[
\text{interacting} \quad \exp(2i\delta(k)) \exp(ikL) = 1
\]

\[k(\neq p_n) \Leftrightarrow \delta(k)\]
Strategy

1). Choose ππ kinematics such that \( \sqrt{s} \sim m_\rho \)

2). Calculate correlation matrix \( \rightarrow \) extract energy eigen values

\[
\begin{pmatrix}
\langle \mathcal{O}_{\pi\pi}(t)\mathcal{O}_{\pi\pi}^\dagger(0) \rangle & \langle \mathcal{O}_{\pi\pi}(t)\mathcal{O}_{\rho}^\dagger(0) \rangle \\
\langle \mathcal{O}_{\rho}(t)\mathcal{O}_{\pi\pi}^\dagger(0) \rangle & \langle \mathcal{O}_{\rho}(t)\mathcal{O}_{\rho}^\dagger(0) \rangle \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
C_1 \exp(-W_1(k_1)t) & 0 \\
0 & C_2 \exp(-W_2(k_2)t) \\
\end{pmatrix}
\]

diagonalize

3). Momentum \( k_i \rightarrow \) phase shift \( \delta(k_i) \) by Lüscher’s formula
Correlation Matrix with Quark Diagram

\[ \langle O_{\pi\pi}(t) O_{\pi\pi}^\dagger(0) \rangle \]

\[ \langle O_\rho(t) O_\rho^\dagger(0) \rangle \]

\[ \langle O_{\pi\pi}(t) O_\rho^\dagger(0) \rangle \]

\[ \langle O_\rho(t) O_{\pi\pi}^\dagger(0) \rangle \]

more complicated, more difficult
Energy Eigen Values

\[ \rho_3(0,0,2\pi/L) \& \pi(0,0,0)\pi(0,0,2\pi/L) \]

\[ \Delta E_i = W_i - \{E_{\pi}(0,0,0) + E_{\pi}(0,0,2\pi/L)\} \]

attractive \[ \Delta E_1 < 0 \]

repulsive \[ \Delta E_2 > 0 \]

finite size formula gives:
\[ \tan \delta(k_1) = +0.0942(47) \]
\[ \tan \delta(k_2) = -0.165(50) \]
Effective $\rho \rightarrow \pi \pi$ Coupling Constant $g_{\rho\pi\pi}$

Simple phenomenological description of $\rho \rightarrow \pi \pi$ decay by

\[ \mathcal{L}_{\text{int}} = g_{\rho\pi\pi} \epsilon_{abc} \rho_{\mu}^a \pi^b \partial^\mu \pi^c \]

Phase shift $\delta(k)$ in terms of $g_{\rho\pi\pi}$

\[ \tan \delta(k) = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{k^3}{\sqrt{s(M_R^2 - s)}} \]

\[ (2k)^2 = s - (2m_\pi)^2 \]

$g_{\rho\pi\pi}$, $M_R$ are fit parameters

“physical value” of $g_{\rho\pi\pi}$

\[ \Gamma_{\text{ph}} = \frac{(g_{\rho\pi\pi}^{\text{ph}})^2 (k_{\text{ph}})^3}{6\pi (m_{\rho}^{\text{ph}})^2} = 149.1(8) \text{ MeV} \Rightarrow g_{\rho\pi\pi}^{\text{ph}} = 5.98(2) \]
Results for Decay Width

\[ \frac{k^3}{\tan \delta(k) / \sqrt{s}} = \frac{6\pi}{g_{\rho\pi\pi}} (M_R^2 - s) \]

\[ g_{\rho\pi\pi} = 5.24(51) \Rightarrow \Gamma = 113(22)\text{MeV} \]

\[ M_R = 0.4064(46) \]

\[ g_{\rho\pi\pi}^{ph} = 5.98(2) \]

\[ \Gamma^{ph} = 149.1(8)\text{MeV} \]
Current Status

Only recently 2 or 2+1 flavor QCD simulations at $2m_\pi < m_\rho$ make the investigation possible

\[ g_{\rho\pi\pi}^{ph} = 5.98(2) \]

<table>
<thead>
<tr>
<th>group</th>
<th>year</th>
<th>#flavor</th>
<th>$m_\pi$ [MeV]</th>
<th>$g_{\rho\pi\pi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP-PACS</td>
<td>07</td>
<td>2</td>
<td>330</td>
<td>6.25(67)</td>
</tr>
<tr>
<td>QCDSF</td>
<td>lat08</td>
<td>2</td>
<td>240〜810</td>
<td>5.3(+2.1)(-1.5)</td>
</tr>
<tr>
<td>ETMC</td>
<td>10,lat10</td>
<td>2</td>
<td>290〜480</td>
<td>6〜7</td>
</tr>
<tr>
<td>PACS-CS</td>
<td>lat10</td>
<td>2+1</td>
<td>410</td>
<td>5.24(51)</td>
</tr>
<tr>
<td>BMW</td>
<td>lat10</td>
<td>2+1</td>
<td>200,340</td>
<td>5.5(2.9),6.6(3.4)</td>
</tr>
</tbody>
</table>
§5. Nuclei from Lattice QCD

Previous studies of multi-nucleon system

• ΛΛ system ⇒ H dibaryon: unbound
  Mackenzie-Thacker 85, Iwasaki et al. 88,
  Negele et al.@lat98, Wetzorke et al.@lat99,
  Wetzorke-Karsch@lat02

• NN system ⇒ Deuteron: unbound
  Fukugita et al. 95, NPLQCD 06

• NNN system ⇒ Triton: unbound
  NPLQCD 09

No one has succeeded in detecting binding energies
Direct Construction of Nuclei on the Lattice

Exploratory study for $^4$He and $^3$He nuclei

- Large binding energy $\Delta E_{^4\text{He}} = 28.3$ MeV
- $^4$He has double magic numbers ($Z=2, N=2$)

Difficulties in multi-nucleon systems

1. No. of Wick contractions
2. How to distinguish bound state from scattering state?
Wick Contractions

He nucleus correlator in terms of quark fields

\[ \langle O_{4\text{He}}(t)O_{4\text{He}}^\dagger(0) \rangle_{t \to 0} \propto C \exp(-m_{4\text{He}}t) \quad \Delta E_{4\text{He}} = E_{4\text{He}} - 4E_N \]

\[ ^4\text{He} \; \text{operator consists of two protons (udu) and two neutrons (dud)} \]

⇒ No. of Wick contraction: \( N_u! \times N_d! = (2N_p + N_n)! \times (2N_n + N_p)! \)

cf. N-N: 3! \times 3! = 36

\[ ^{12}\text{C} : 18! \times 18! \sim 4 \times 10^{31} \]

\[ ^4\text{He}: \; 6! \times 6! = 518400 \]

\[ ^3\text{He}: \; 5! \times 4! = 2880 \]

independent quark diagrams are reduced by imposing \( m_u = m_d \)
Identification of Bound State in a Finite Box

\[ \Delta E < 0 \] both for bound state and attractive scattering state

\[ \Delta E = \text{const} \]

\[ \Delta E \propto \frac{1}{L^3} \]

mandatory to check volume dependence of \( \Delta E \)
Volume Dependence of $\Delta E_{4\text{He}}$

Exploratory study with $m_N=1.6$ GeV in quenched QCD

Yamazaki-YK-Ukawa 10

same order to experimental values
§6. Nonperturbative Renormalization

Two major schemes

- **MOM scheme**  
  Martinelli et al. 95
  copy what is done in perturbative renormalization
  in a nonperturbative way
  - window problem: $\Lambda_{\text{QCD}} \ll \mu \ll 1/a$
  - Gribov copy problem in Landau gauge fixing

- **Schrödinger Functional (SF) scheme**  
  Jansen et al. 96
  a finite-volume renormalization technique:
  massless QCD in an $L^4$ box $\Rightarrow$ renormalization scale $\mu=1/L$
  - step scaling function for scale evolution
    $\sigma_Z(L) = Z(2L)/Z(L)$
  - now available for Ginsparg-Wilson fermions $\Rightarrow$ Takeda
    Taniguchi 05,06, Sint@lat10, Lüscher 06
MOM Scheme

definition of renormalization factor: \( \mathcal{O}(\mu) = Z_\mathcal{O}(\mu a, g(a))\mathcal{O}(a) \)

renormalization condition:

\[
Z_\mathcal{O}(\mu a, g(a))Z^{-1}_\psi(\mu a, g(a))\Gamma_\mathcal{O}(pa)|_{p^2 = \mu^2} = 1
\]

vertex function:

\[
\Lambda_\mathcal{O}(pa) = S^{-1}(pa)G_\mathcal{O}(pa)S^{-1}(pa) \quad \text{off-shell quark mom } p^2
\]

\[
\Gamma_\mathcal{O}(pa) = \frac{1}{12} \text{Tr} \left( \Lambda_\mathcal{O}(pa) \hat{P}_\mathcal{O} \right) \quad \text{gauge fixed}
\]

field renormalization:

\[
Z_\psi(\mu a, g(a)) = \frac{1}{48} \text{Tr} \left( \Lambda_{\mathcal{V}_\mu C}(pa)\gamma_\mu \right)|_{p^2 = \mu^2}
\]

window problem \( (1/L \ll \Lambda_{\text{QCD}} \ll \mu \ll 1/a) \) & Gribov copies
Schrödinger Functional (SF) Scheme

step scaling function

\[ \bar{g}(L), Z(L) \]

\[ \bar{g}(2L), Z(2L) \]

\[ \sigma_g(L) = \frac{\bar{g}(2L)}{\bar{g}(L)} \]

\[ \sigma_Z(L) = \frac{Z(2L)}{Z(L)} \]

scale evolution: \( L \rightarrow 2L \rightarrow 2^2L \rightarrow \cdots \rightarrow 2^nL \rightarrow \cdots \)
Running Quark Mass in SF Scheme

Non-perturbative running mass in SF scheme

\[ m_{SF}(\mu)/M \]

\[ \mu/\Lambda_{SF} \]

RGI quark mass

\[
M = \bar{m}(\mu) \left(2b_0 \tilde{g}^2(\mu)\right)^{-\frac{d_0}{2b_0}} \exp \left(-\int_0^{\tilde{g}(\mu)} \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0g} \right)
\]
§5. Summary

- 2+1 ($m_u=m_d\neq m_s$) flavor simulation near the physical point
  \[\Rightarrow 1+1+1 (m_u\neq m_d\neq m_s)\] flavor simulation at the physical point

- Proper treatment of $\pi\pi\rightarrow\rho$ resonances
  \[\Rightarrow \text{investigate new quark composite states (tetraquark etc.)}\]

- Direct construction of helium nuclei in lattice QCD
  \[\Rightarrow \text{nuclei with larger atomic number}\]

- Nonperturbative renormalization with MOM or SF schemes
  \[\Rightarrow \text{SF schemes for GW actions}\]