# Thermodynamic fluctuations in the relativistic grand-canonical ensemble

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# *We already know a lot…*

**A very good reference:**
**Kovtun (2012), ''Lectures on hydrodynamic fluctuations in relativistic theories''**

**However, our understanding of hydrodynamics has evolved considerably!**

- **Mathematics side (causality, stability, well-posedness…);**
- **Physics side (stable hydrodynamics frames, long-lived non-hydrodynamic modes, analytic results on gradient expansion, exact results in kinetic theory…);**

**New challenging questions, e.g.:**

- **«Is fluctuating BDNK unstable?»** $\langle(\delta A)^2\rangle < 0$ !?!
- **«How do we make a fluctuating theory causal?»**

**Often, the best way forward is to go back to the basics (i.e. to let Landau guide us).**

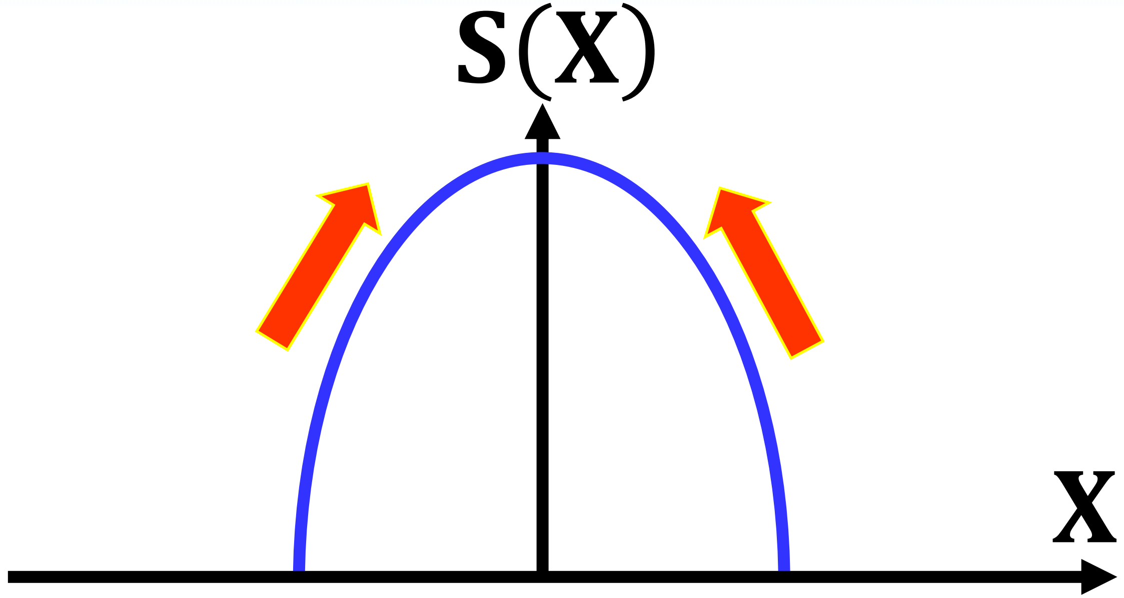
# *Spoiler alert!*

If the fluctuating theory is constructed very carefully (details later 😉), the following facts hold:

1. All correlators exist and are well-defined distributions;
2. All uncertainties are non-negative definite by construction: $\langle \delta A^2 \rangle \geq 0$;
3. The fluctuation-dissipation theorem (in all its formulations) is recovered;
4. If the noise is covariantly Markovian, the fluctuating theory is Lorentz-covariant;
5. The fluctuating theory is causal, in the sense that the fluctations cannot be used to send information faster than the speed of light;
6. The fluctuating theory is stable, in the sense that the macrostate $\psi = \langle \psi \rangle$ is the most probable state, and the fluctations do not "condense";
7. The disperison relations fulfill all QFT-based microcausality criteria;
8. The Martin-Siggia-Rose effective action is well-defined and well-behaved;
9. There is a KMS-type symmetry for the theory.

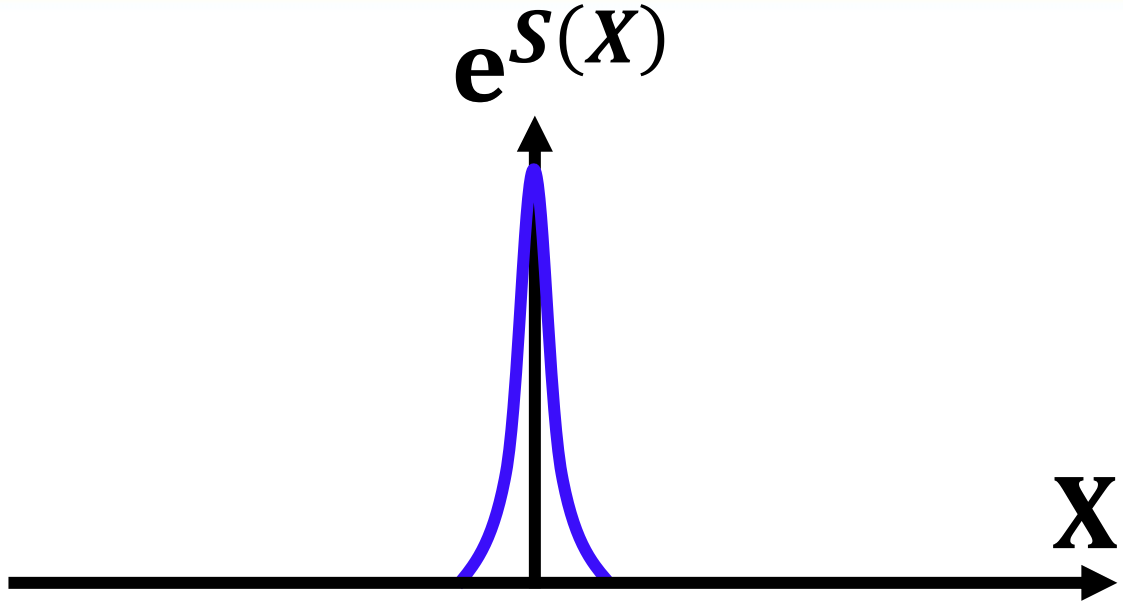
# Quick introduction to fluctuations

# *Two not so consistent pictures*




$$\frac{dS(X)}{dt} \geq 0$$

**Irreversible dynamics:**
**Entropy grows until it is maximised**



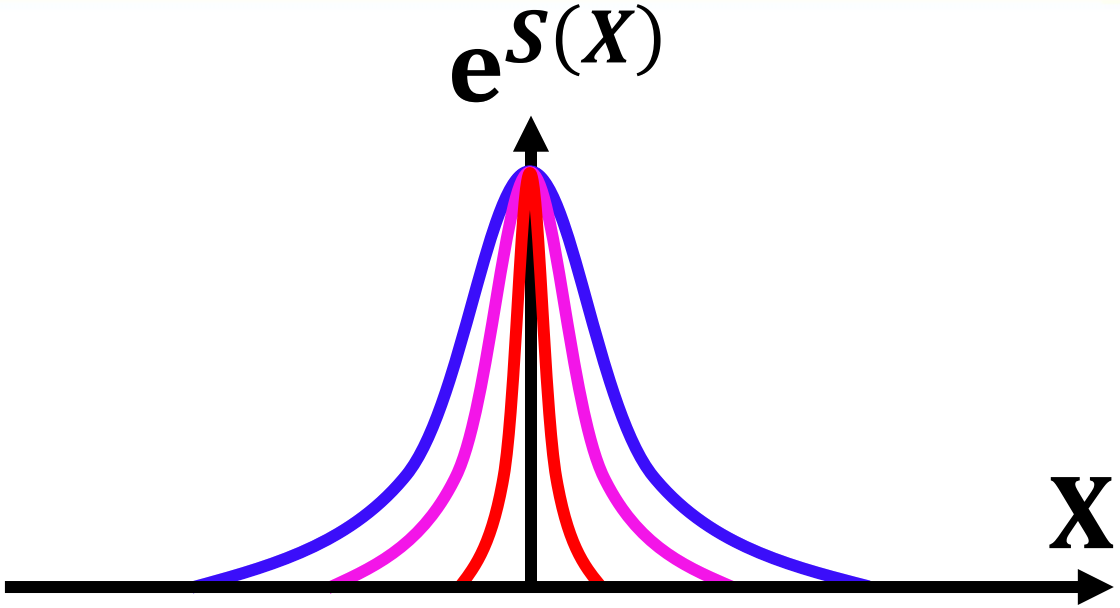

$$Prob(X) \propto e^{S(X)}$$

**Statistical mechanics:**
**All microscopic states have equal probability**

# *They agree only for* $N \to \infty$

$$Prob(X) = \frac{e^{S(X)}}{\int e^{S(X)} dX} = \frac{e^{Ns(X)}}{\int e^{Ns(X)} dX}$$

$$\approx \frac{e^{-\frac{1}{2}Ns''X^2}}{\int e^{-\frac{1}{2}Ns''X^2} dX} = \sqrt{\frac{Ns''}{2\pi}}\, e^{-\frac{1}{2}Ns''X^2}$$




If $N \to \infty$, then $Prob(X) \to \delta(X)$

However, at finite N, there are corrections. In particular, X acquires a finite uncertainty:

$$\langle X^2 \rangle \approx \frac{1}{Ns''}$$

# *Fluctuation-dissipation theorem in a toy model*

**Only two macrostates:**

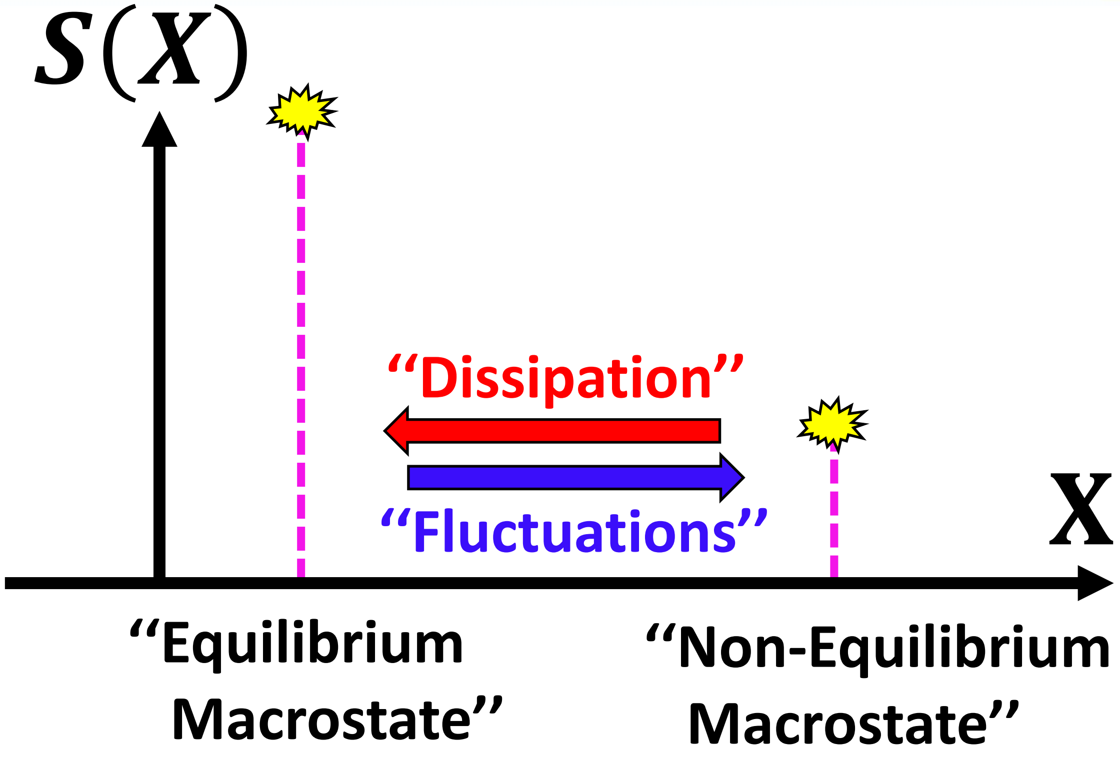
- "Equilibrium" (i.e. high entropy);
- "Non-Equilibrium" (i.e. low entropy).

**Markovian evolution equation:**

$$\frac{d}{dt}\begin{bmatrix} P_E \\ P_N \end{bmatrix} = \begin{pmatrix} -R_{E\to N} & R_{N\to E} \\ R_{E\to N} & -R_{N\to E} \end{pmatrix}\begin{bmatrix} P_E \\ P_N \end{bmatrix}$$

**Stationary state:**

$$\begin{bmatrix} P_E \\ P_N \end{bmatrix} = \begin{bmatrix} e^{S_E} \\ e^{S_N} \end{bmatrix}$$




FDT: $R_{E\to N} = e^{S_N - S_E} R_{N\to E}$

# *Why should this be relevant for hydrodynamics?*

Hydrodynamics focuses on average values: $\partial_\mu \langle J^\mu \rangle = 0$

Fluctuating theories compute uncertaintees: $C^{\mu\nu} = \langle J^\mu J^\nu \rangle - \langle J^\mu \rangle \langle J^\nu \rangle$

Why should hydrodynamics care about fluctations?

Two reasons:

1. Hydrodynamics is non-linear;
2. Hydrodynamics studies evolution over long times.

Example: $\dot{X} = X^2 \;\; => \langle \dot{X} \rangle = \langle X^2 \rangle = \langle X \rangle^2 + \Delta X^2$

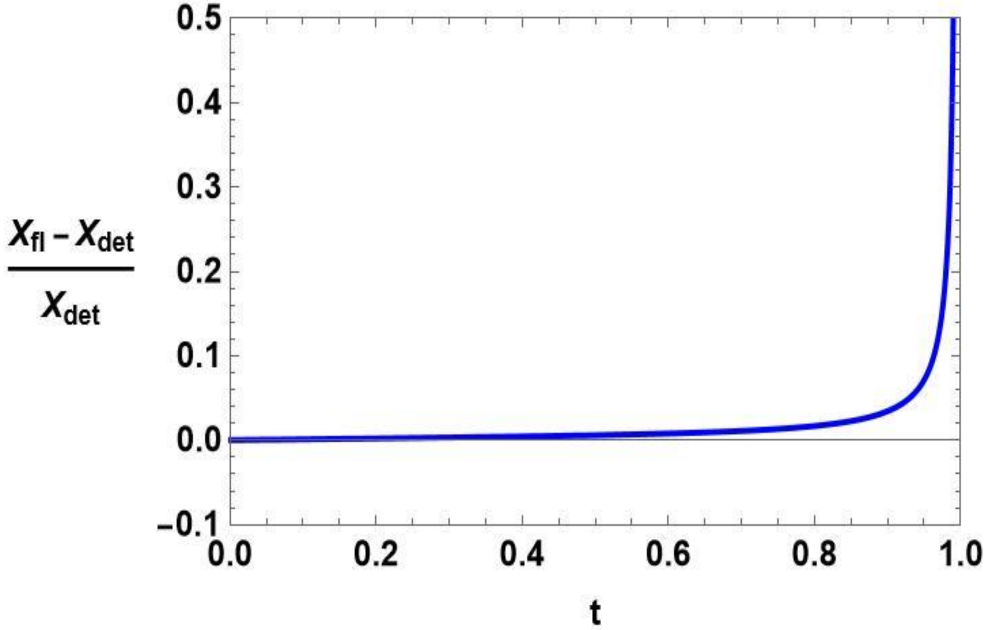
The evolution of $\langle X \rangle$ is not a function of $\langle X \rangle$ alone.

You need to know also $\Delta X$.

Suppose that $\Delta X$ is does not depend on time.

Then: $X(t) = \Delta X \tan\left[\arctan\left(\frac{X(0)}{\Delta X}\right) + t\, \Delta X\right]$

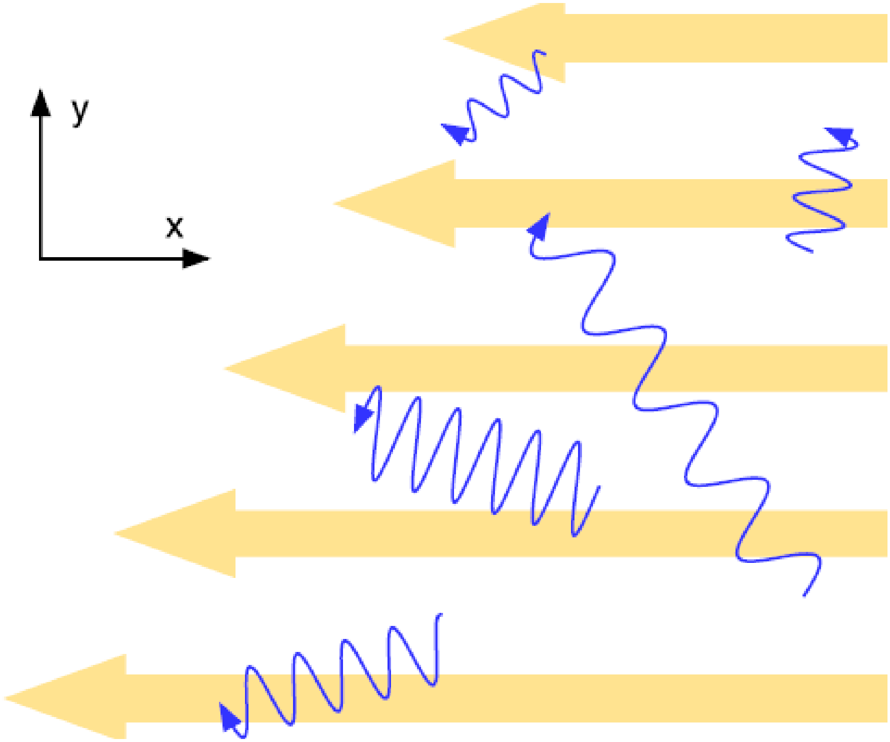
"Secular effect": The impact of $\Delta X$ cumulates in time.



# *A famous example: The "stickiness of sound"*

Fluctuation-induced sound waves diffuse just like particles, and lead to a renormalization of the shear viscosity!

$$\eta_{ren} \approx \eta_{bare} + \frac{\text{coefficient}}{\eta_{bare}^2}$$

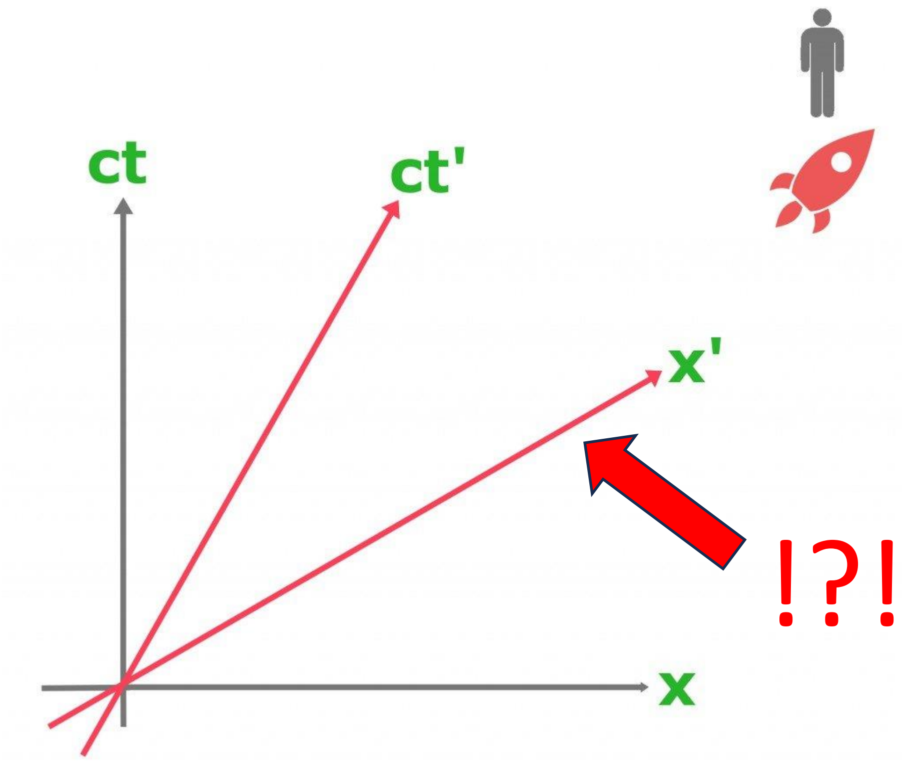



Credit: Kovtun (2012), "Lectures on hydrodynamic fluctuations in relativistic theories"

Kovtun, Moore, Romatschke: PRD (2021)

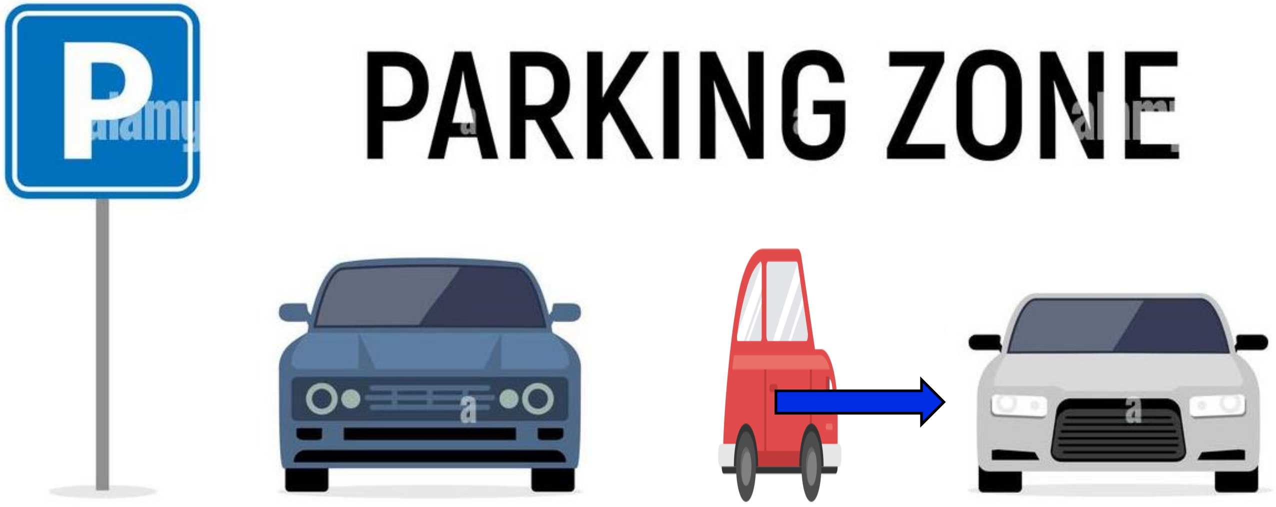
# Relativity introduces subtleties

# *Relativity of Simultaneity*




$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \end{cases}$$

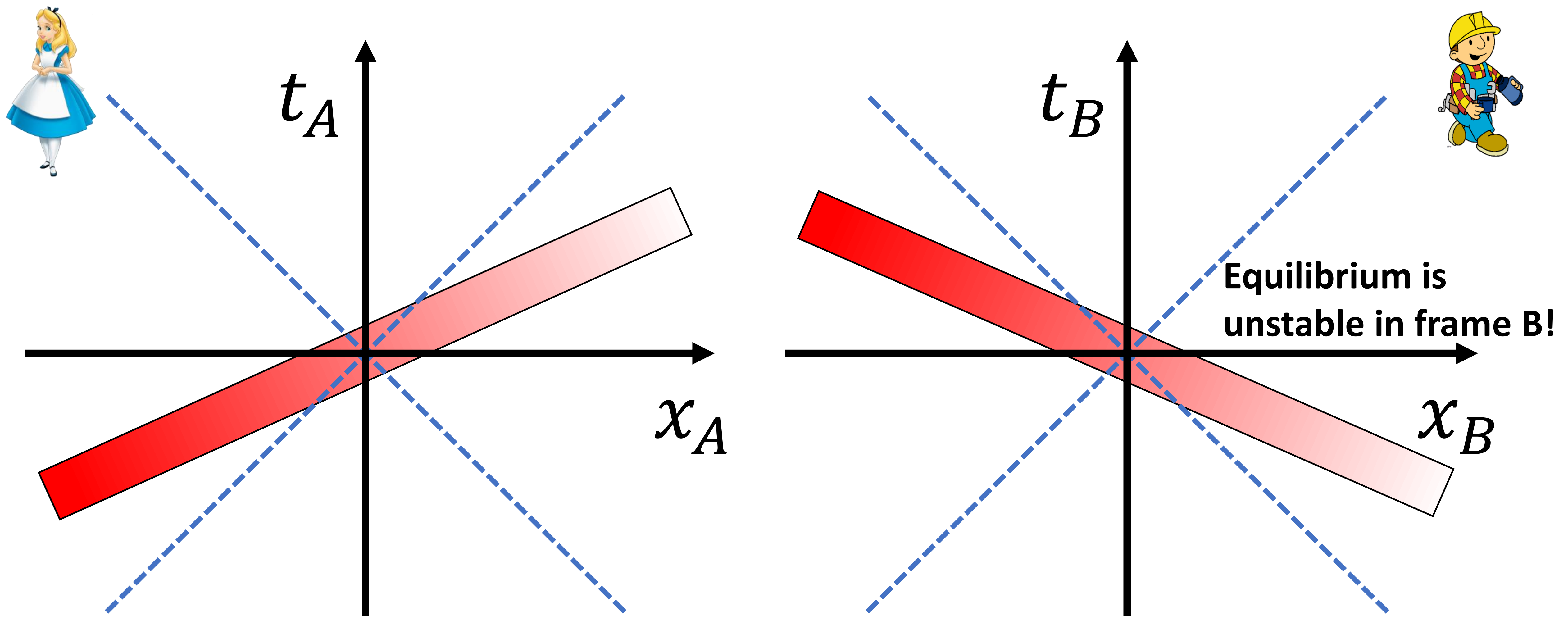
# *Some intuitive concepts are not invariant (e.g. "fitting in")*



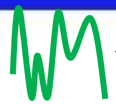

**Parking at rest, moving car**


**Car at rest, moving parking**

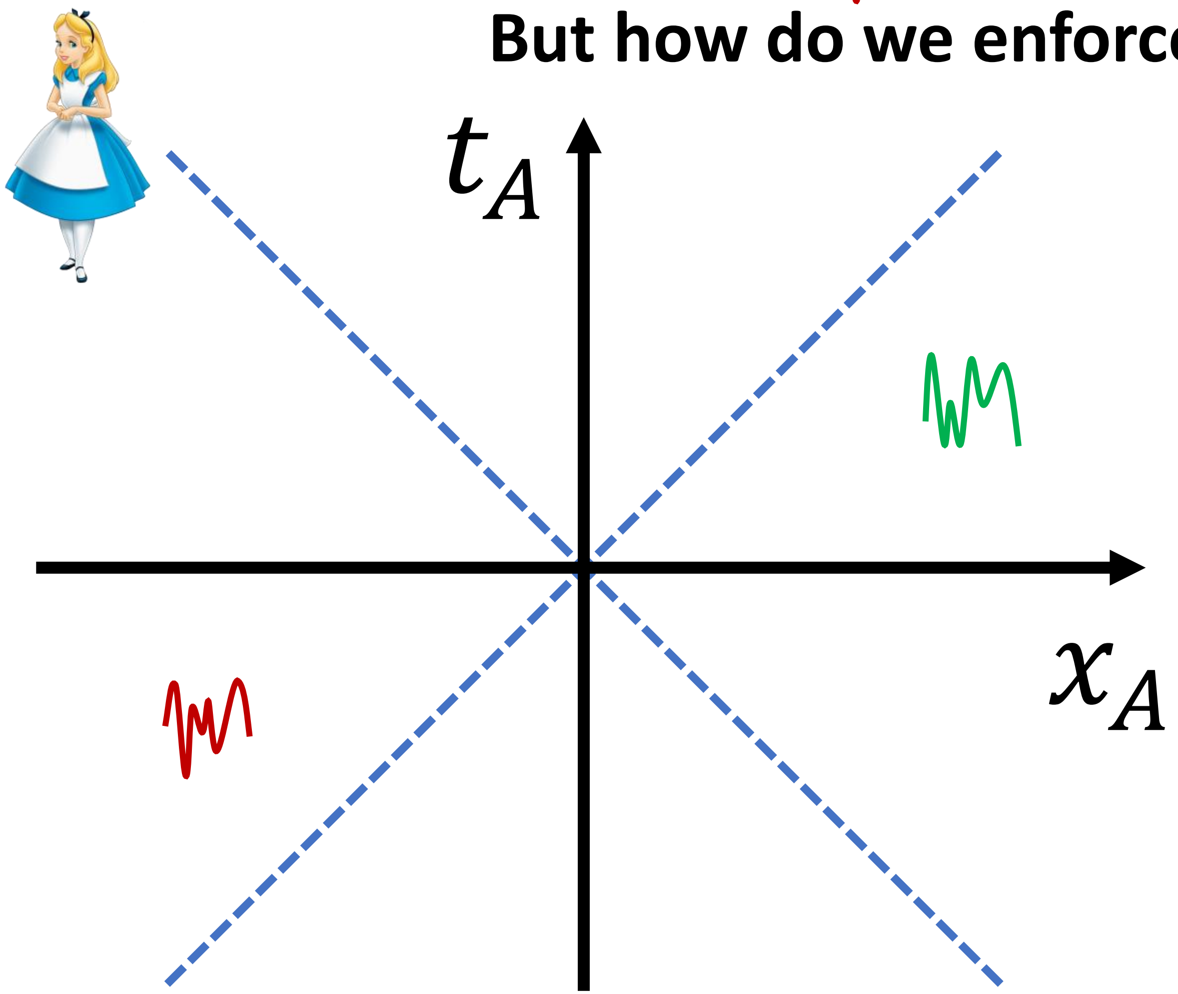
# *Without causality, "stability" is not invariant!*

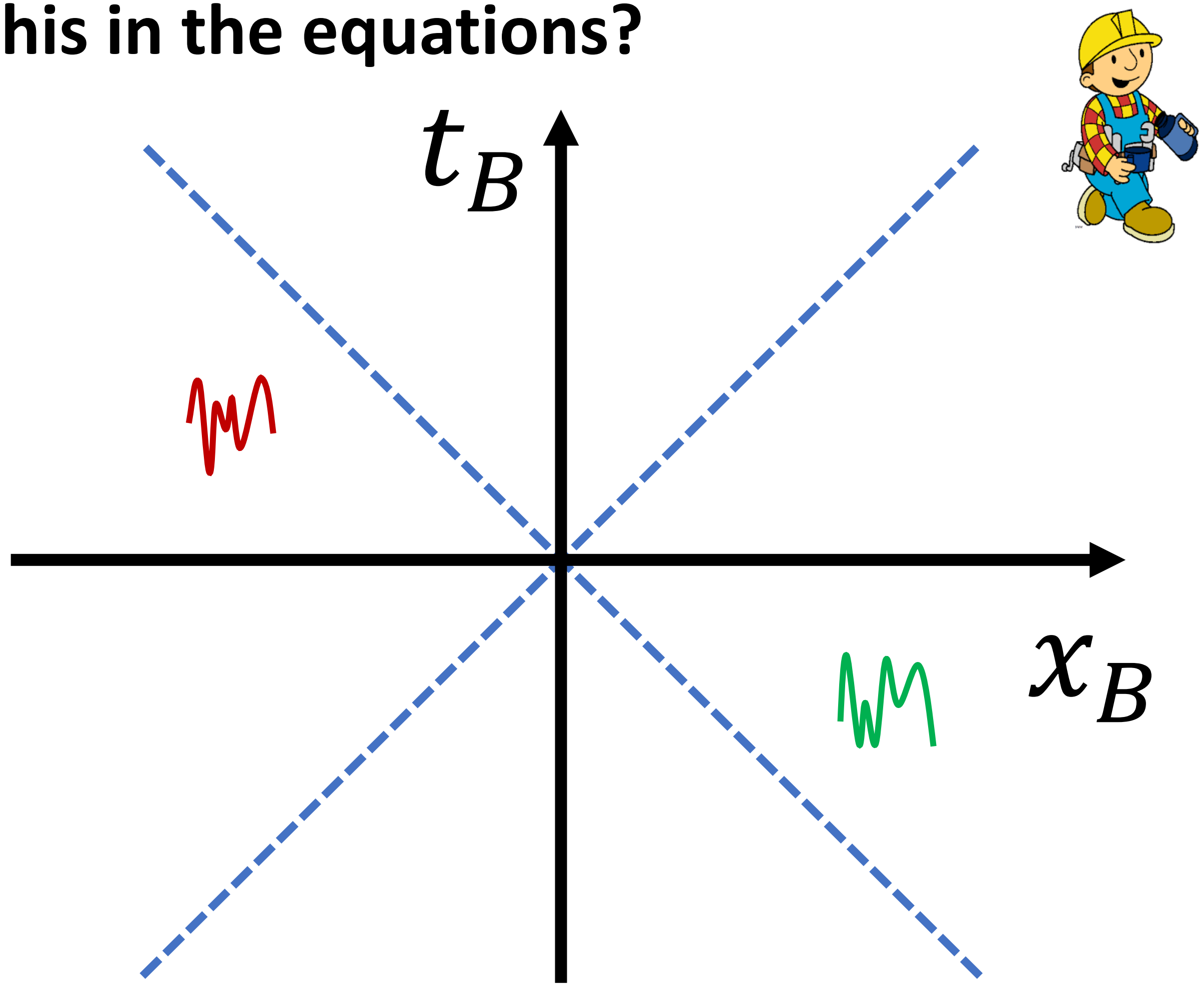


# *What makes a fluctuating theory causal?*

**"" cannot cause ""**

**But how do we enforce this in the equations?**



# Relativistic grand-canonical probabilities: Formal derivation

# *An old trick*

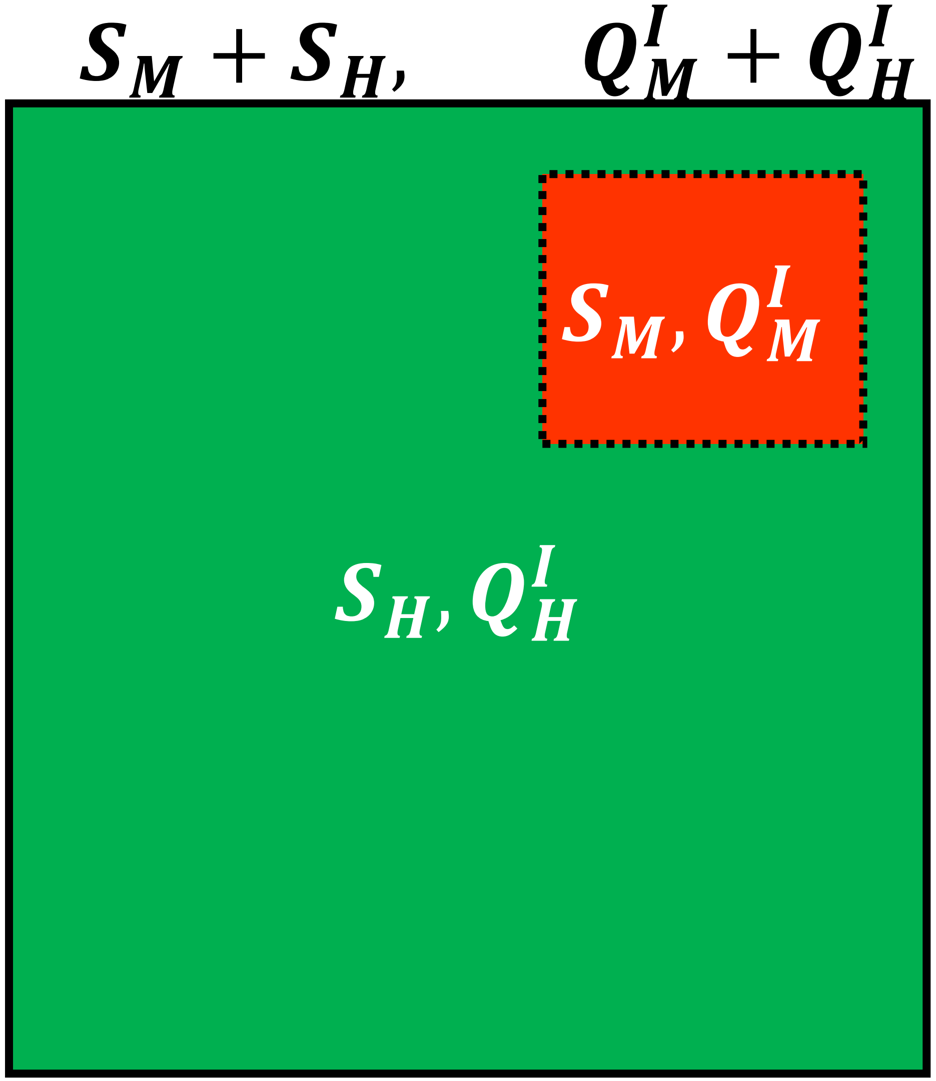
**We split an isolated system in two parts: Medium ($M$) and Heat-bath ($H$); $H \gg M$.**

**Noether Charges:** $Q^I = \{N_b, Q_e, p^\mu, J^{\mu\nu}, \dots\}$

**Ideal heat-bath:** $S_H(Q_H^I) = -\alpha_I^\star Q_H^I$ **+constant**
**($\alpha_I^\star$ are constants)**

**Second law of thermodynamics:**

$$0 \le \Delta S_{tot} = \Delta S_M + \Delta\left(-\alpha_I^\star Q_H^I\right) = \Delta S_M - \alpha_I^\star \Delta Q_H^I$$
$$= \Delta S_M + \alpha_I^\star \Delta Q_M^I = \Delta\left(S_M + \alpha_I^\star Q_M^I\right) = \Delta\Phi$$



# *Relativistic extremum principle*

**The equilibrium state of a Medium ($M$) in contact with a bath having intensive parameters $\alpha_I^\star$ is the state that maximizes $\Phi = S_M + \alpha_I^\star Q_M^I$ for unconstrained variations.**

**Let us use this principle to find the Grand-Canonical density matrix $\widehat{\rho}_{GC}$ :**

$$S_M = -Tr(\widehat{\rho}\, ln\widehat{\rho}\,)$$

$$Q_M^I = Tr(\widehat{\rho}\, \widehat{Q}^I\,)$$

$$\boxed{\widehat{\rho}_{GC} = \frac{e^{\alpha_I^\star \widehat{Q}^I}}{Z}}$$

**Generalization of**

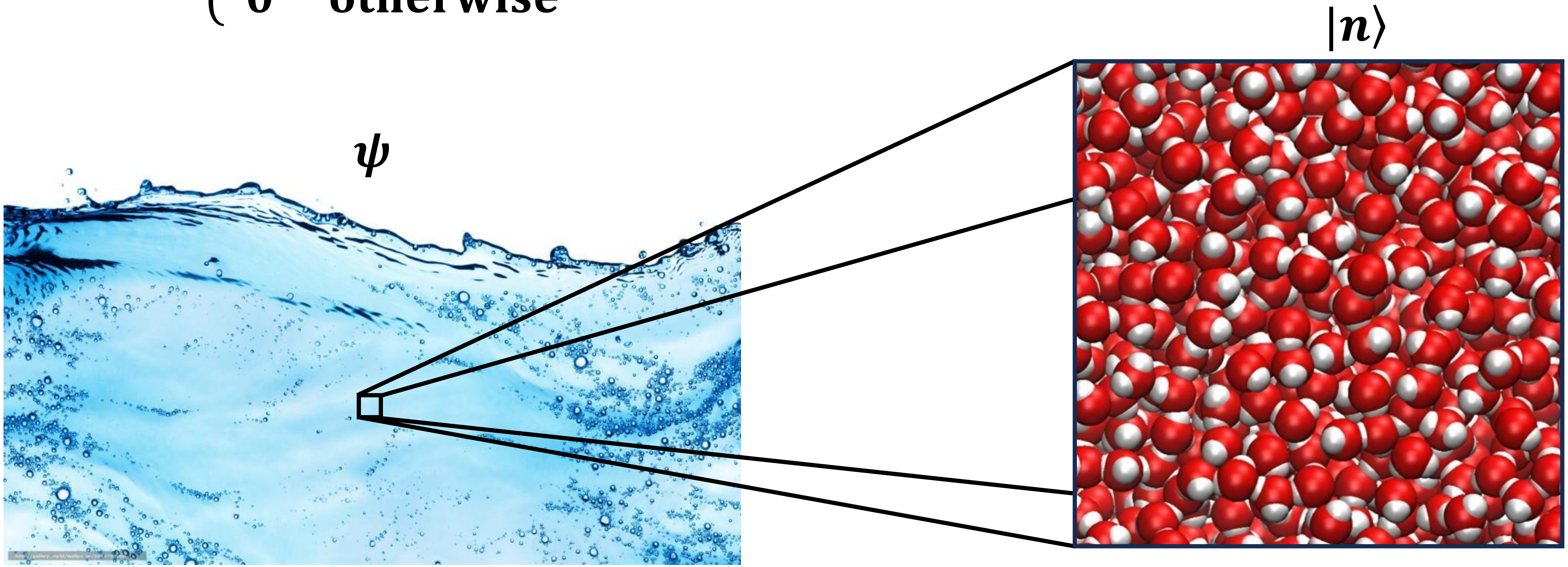
$$\widehat{\rho}_{GC} = \frac{e^{-\beta^\star(\widehat{H} - \mu^\star \widehat{N})}}{Z}$$

**with arbitrary Noether charges.**

# *Probability of macrostates*

**Each macroscopic state $\psi$ has an associated projector $\widehat{P}(\psi)$:**

$$\widehat{P}(\psi)|n\rangle = \begin{cases} |n\rangle & \text{if } |n\rangle \text{ is a microscopic realization of } \psi \\ 0 & \text{otherwise} \end{cases}$$



# *Probability of macrostates*

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**The corresponding probability of occupation is**

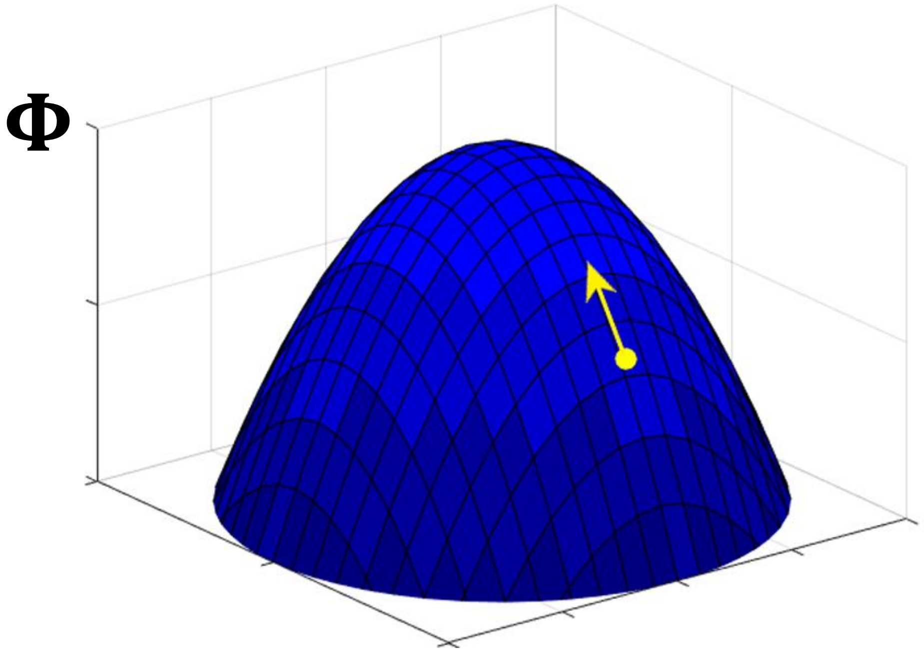
$$P(\psi) = Tr\left[\hat{\rho}_{GC}\hat{P}(\psi)\right]$$

$$= Tr\left[\frac{e^{\alpha_I^\star \hat{Q}^I}}{Z}\hat{P}(\psi)\right]$$

$$\approx Tr\left[\hat{P}(\psi)\right]\frac{e^{\alpha_I^\star Q^I(\psi)}}{Z}$$

$$= \frac{e^{S(\psi)+\alpha_I^\star Q^I(\psi)}}{Z}$$

$$\boxed{P(\psi) = \frac{e^{\Phi(\psi)}}{Z}}$$



# *Quick example: Chemical potential of non-conserved ultrarelativistic particles*

**Gas of non-conserved particles (like photons, but with Boltzmann statistics) at fixed temperature $T$, whose chemical potential $\mu (= \psi)$ fluctuates:**

$$S(\mu) = VaT^3 e^{\mu/T}\,(4 - \mu/T)$$

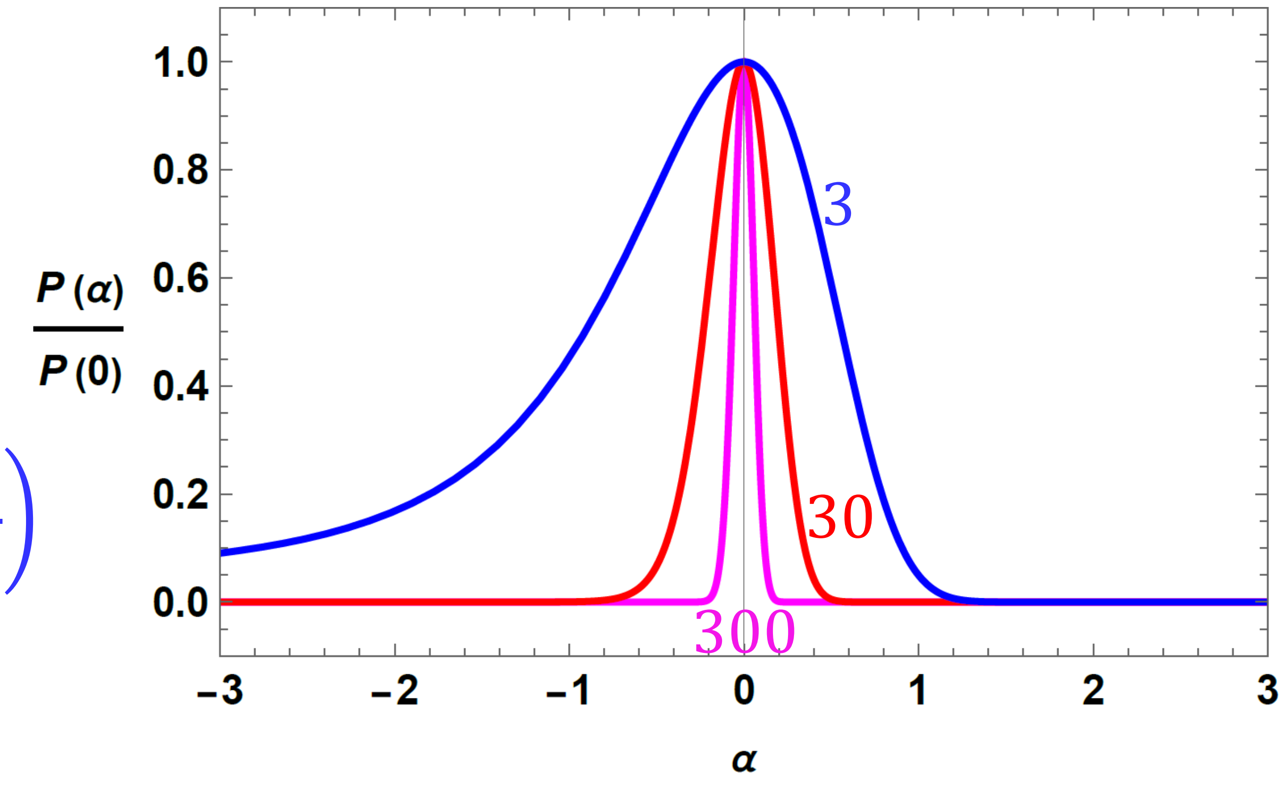
$$U(\mu) = 3VaT^4 e^{\mu/T}$$

$$\Phi(\mu) = VaT^3 e^{\frac{\mu}{T}}\left(1 - \frac{\mu}{T}\right)$$

**Therefore ($\alpha = \mu/T$):**

$$\frac{P(\alpha)}{P(0)} = \frac{exp\left(N_{eq}e^{\alpha}(1-\alpha)\right)}{exp\left(N_{eq}\right)} \approx exp\left(-\frac{N_{eq}\alpha^2}{2}\right)$$

**Typical fluctation:** $\langle\alpha^2\rangle \approx N_{eq}^{-1}$



# *We are interested in applications to continuum mechanics*

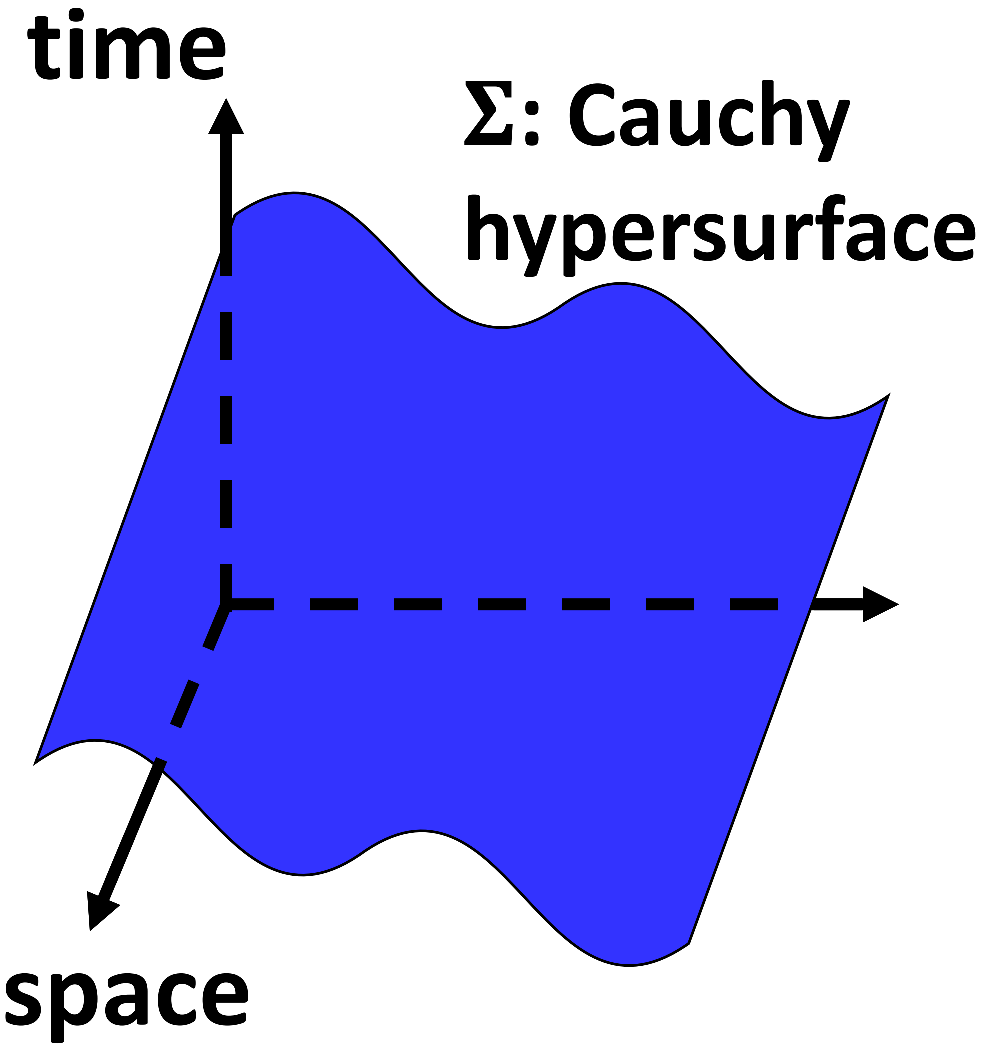
**Recall:** $P[\psi] = \frac{e^{\Phi[\psi]}}{Z} = \frac{e^{S[\psi]+\alpha_I^\star Q^I[\psi]}}{Z}$

**Express extenstensive variables as fluxes of corresponding currents:**

$$S = \int_\Sigma s^\mu d\Sigma_\mu \qquad Q^I = \int_\Sigma J^{I\mu} d\Sigma_\mu$$

**Result:**

$$P[\psi] = \frac{e^{\int_\Sigma \left(s^\mu[\psi]+\alpha_I^\star J^{I\mu}[\psi]\right) d\Sigma_\mu}}{Z}$$

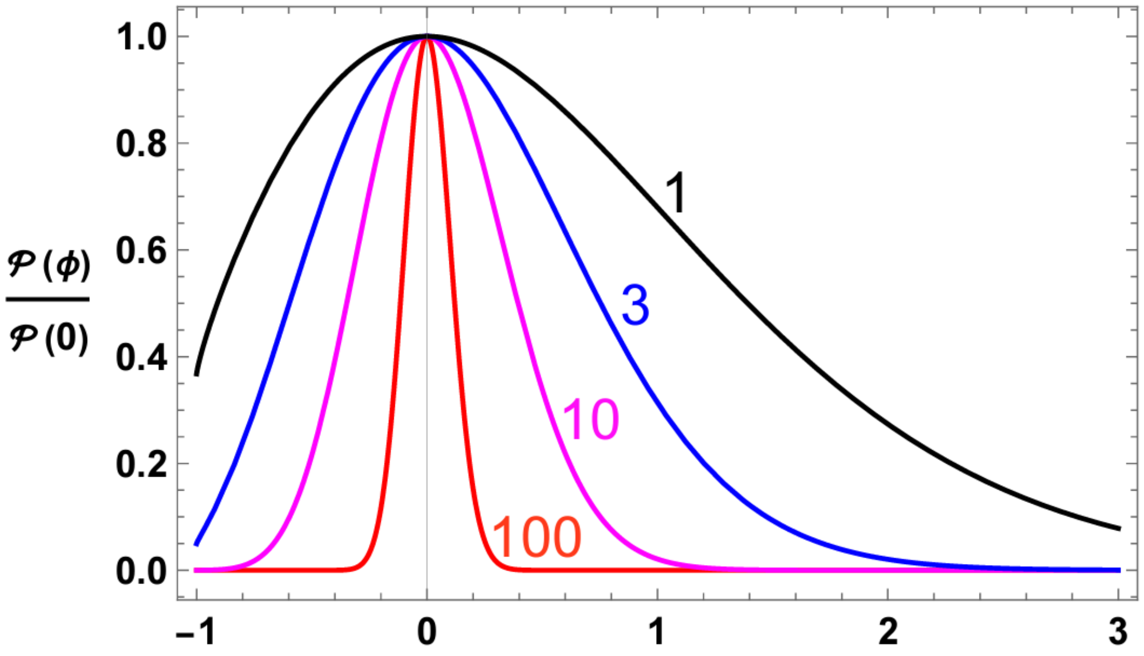


# *Example: Relativistic kinetic theory*

**Kinetic distribution function:** $\psi = f\left(x^{\mu}, q^{I}\right)$

**Equilibirum distribution (MB statistics):** $f_{eq} = e^{\alpha_I^{\star} q^I}$

**Relative non-equilibrium deviation:** $f = f_{eq}(1+\phi)$




$$P[\phi] \propto exp \int [\phi - (1+\phi) ln(1+\phi)]\, dN_{eq}$$

$$\approx exp \int -\frac{\phi^2}{2}\, dN_{eq}$$

**Typical fluctation:** $\left\langle \phi^2 \right\rangle_{N_{eq}} \approx N_{eq}^{-1}$

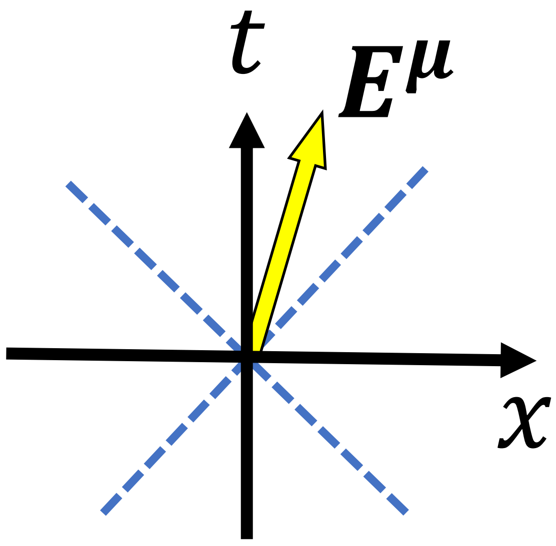
# Gaussian approximation

# *Expand the exponent to second order*

Small fluctations: $\psi = \psi_{eq} + \delta\psi$

Information current

$$\frac{P[\delta\psi]}{P[0]} = \frac{e^{\int_\Sigma \left(s^\mu[\psi_{eq}+\delta\psi]+\alpha_I^\star J^{I\mu}[\psi_{eq}+\delta\psi]\right)d\Sigma_\mu}}{e^{\int_\Sigma \left(s^\mu[\psi_{eq}]+\alpha_I^\star J^{I\mu}[\psi_{eq}]\right)d\Sigma_\mu}} \approx e^{-\int_\Sigma E^\mu[\delta\psi]d\Sigma_\mu + O(\delta\psi^3)}$$




**Relevant properties:**

1. It is a quadratic form in the fields: $E^\mu = \frac{1}{2}\delta\psi^T K^\mu \delta\psi$;
2. It is timelike future directed, with $K^0$ positive definite;
3. In the absence of fluctations, one has $\partial_\mu E^\mu \leq 0$;
4. When the above facts hold, the linearised non-fluctuating theory is causal, stable, symmetric hyperbolic and thermodynamically consistent.
5. Onsager symmetry can be derived from the above facts.

# *Gaussian equal-time correlation functions*

**All equal-time correlators are Gaussian functional integrals:**

$$\langle \psi(x)\psi^T(y)\rangle = \frac{\int D\psi\ e^{-\int E^0 d^3x}\psi(x)\psi^T(y)}{\int D\psi\ e^{-\int E^0 d^3x}}$$

# Example 1: Kinetic theory (massless MB)

**Notation:** $\langle \mathrm{F(x)G(y)} \rangle = \overline{\mathrm{FG}}\, \delta^3(\mathrm{x}-\mathrm{y})$

$$E^\mu = \frac{1}{2}\int \phi^2 f_{eq} p^\mu \frac{d^3p}{(2\pi)^3 p^0}$$

$$\overline{\phi_p \phi_q} = \frac{(2\pi)^3}{f_{eq}(p)} \delta^3(p-q)$$

$$\overline{J^\mu J^\nu} = \frac{1}{3} n_{eq} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{J^0 T^{\mu\nu}} = P_{eq} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{T^{00} T^{\mu\nu}} = 4TP_{eq} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{T^{01} T^{\mu\nu}} = 4TP_{eq} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Example 2: Israel-Stewart fluid (massless MB)

$$E^0 = \frac{1}{2}\left(\frac{(\delta n)^2}{n} + \frac{3n}{T^2}(\delta T^2) + 4n\delta u^j \delta u_j + b_1 \delta v^j \delta v_j + b_2 \delta \pi^{jk} \delta \pi_{jk}\right)$$

$$\overline{J^\mu J^\nu} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & \frac{n}{4} + \frac{1}{b_1} & 0 & 0 \\ 0 & 0 & \frac{n}{4} + \frac{1}{b_1} & 0 \\ 0 & 0 & 0 & \frac{n}{4} + \frac{1}{b_1} \end{bmatrix}$$

$$\overline{J^0 T^{\mu\nu}} = P \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{T^{00} T^{\mu\nu}} = 4TP_{eq} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overline{T^{01} T^{\mu\nu}} = 4TP_{eq} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# *Example 3: Relativistic conductor*

$$E^0 = \frac{1}{2}\left(\rho^2 + \frac{\tau}{D} J^k J_k + \frac{\Sigma}{D}\mathcal{E}^k\mathcal{E}_k + \frac{\Sigma}{D}\mathcal{B}^k\mathcal{B}_k\right)$$

**With constraints:** $\partial_k\,\mathcal{E}^k = \rho \quad \partial_k\,\mathcal{B}^k = 0$

$$\langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle = \delta^3(x-y) - \frac{\Sigma e^{-\sqrt{\frac{\Sigma}{D}}|x-y|}}{4\pi D|x-y|}$$

**(Debye screening)**

$$\left\langle\mathcal{B}_j(\mathbf{x})\mathcal{B}_k(\mathbf{0})m^k\right\rangle = \frac{\Sigma}{4\pi D}\left[\frac{3x_j(x\cdot m)-|x|^2 m_j}{|x|^5} + \frac{8\pi}{3}m_j\delta^3(x)\right]$$

**(Magnetic dipole)**

# Example 4: Elastic medium

$$E^0 = \frac{1}{2}\left(\partial_t \xi_j \partial_t \xi^j + 2\mu\, \partial_{(j}\xi_{k)}\partial^{(j}\xi^{k)} + \lambda\left(\partial_j \xi^j\right)^2\right)$$

$$\langle \xi_j(\mathrm{x})\xi_k(0)\rangle = \left(\frac{1}{\mu} + \frac{1}{2\mu+\lambda}\right)\frac{\delta_{jk}}{8\pi|x|} + \left(\frac{1}{\mu} - \frac{1}{2\mu+\lambda}\right)\frac{x_j x_k}{8\pi|x|^3}$$

# Correlations at non-equal times

# *Add some noise*

$\mathcal{L}(\partial_\mu)\psi = \xi \quad \Longrightarrow \quad \psi(x) = \int \mathcal{G}(x - x')\xi(x')d^4x'$

**Multiply the first evaluated at $x$ and the second at 0:**

$$\mathcal{L}(\partial_\mu)\psi(x)\psi^T(0) = \xi(x)\int \xi^T(x')\mathcal{G}^T(-x')d^4x'$$

**Average:**

$$\mathcal{L}(\partial_\mu)\langle\psi(x)\psi^T(0)\rangle = \int\langle\xi(x)\xi^T(x')\rangle\mathcal{G}^T(-x')d^4x'$$

**Assume covariant Markovianity of the noise:**

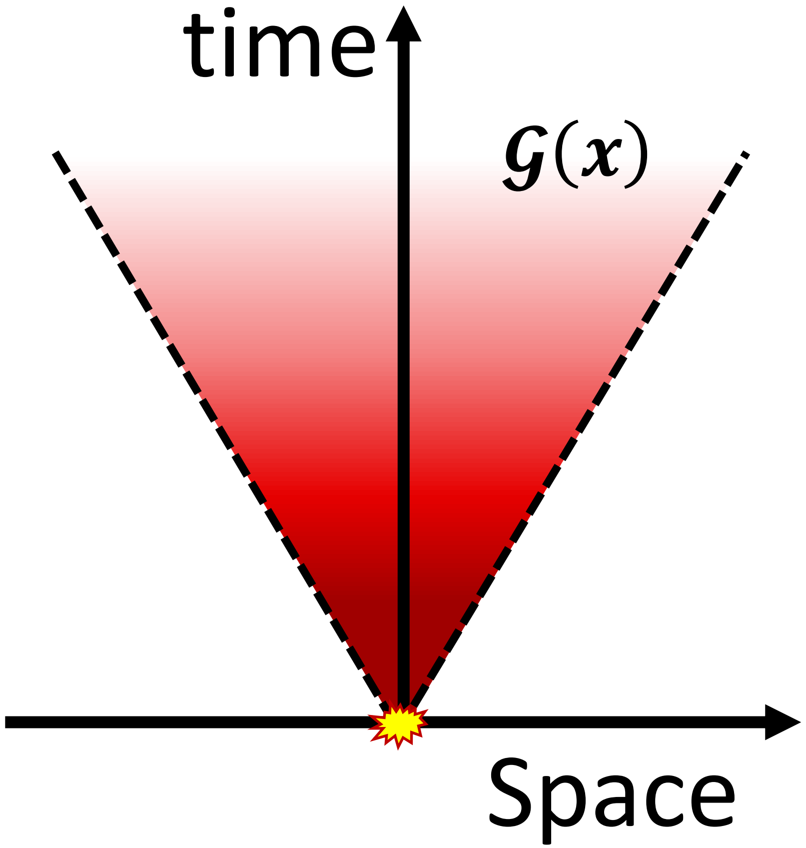
$$\langle\xi(x)\xi^T(x')\rangle = Q\delta^4(x - x')$$

**Result:**

$$\mathcal{L}(\partial_\mu)\langle\psi(x)\psi^T(0)\rangle = Q\mathcal{G}^T(-x)$$

**If $x$ is outside the past lightcone:**

$$\mathcal{L}(\partial_\mu)\langle\psi(x)\psi^T(0)\rangle = 0$$

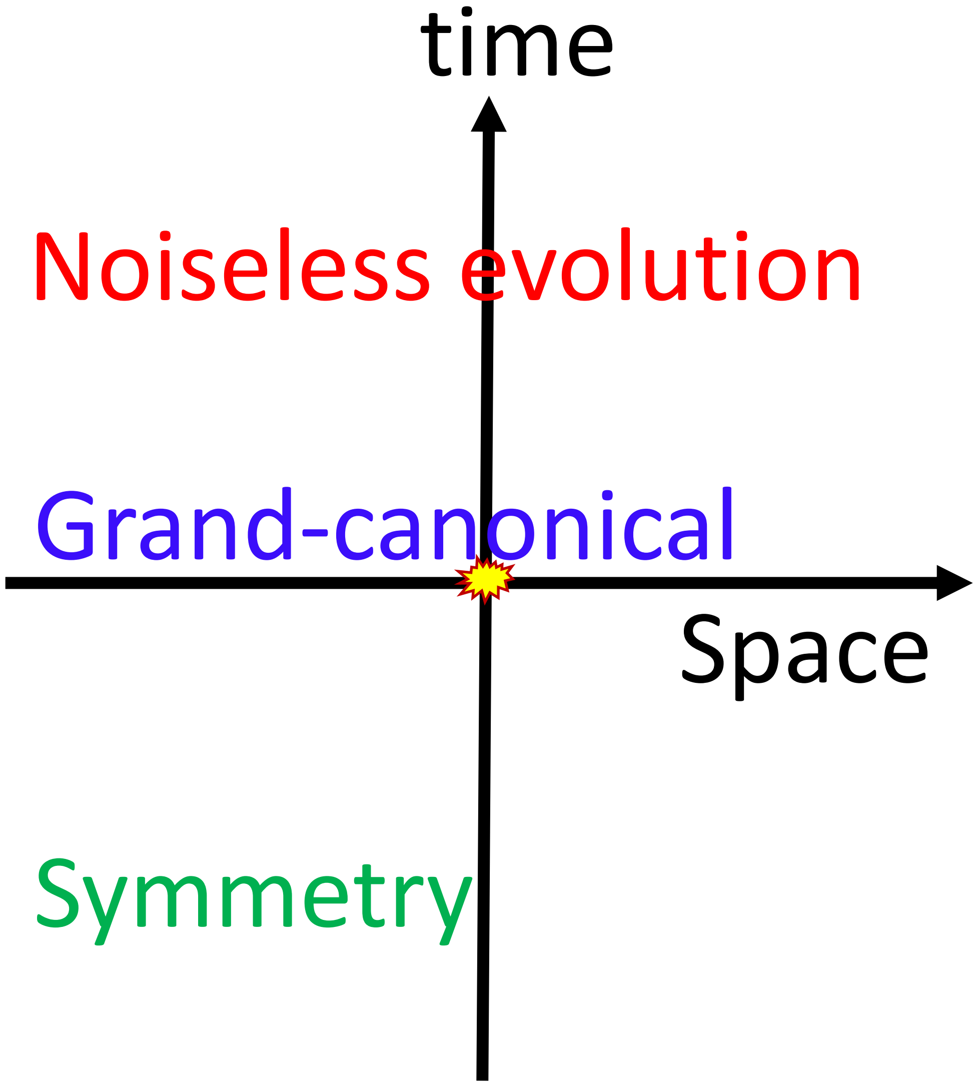


# *Diagram of the correlator* $\langle \psi(x)\psi^T(0)\rangle$

$t > 0$: $\mathcal{L}(\partial_\mu)\langle \psi(x)\psi^T(0)\rangle = 0$

$t = 0$: $\langle \psi(x)\psi^T(0)\rangle = \dfrac{\int D\psi\, e^{-\int E^0 d^3x}\psi(x)\psi^T(0)}{\int D\psi\, e^{-\int E^0 d^3x}}$

$t < 0$: $\langle \psi(x)\psi^T(0)\rangle = \langle \psi(-x)\psi^T(0)\rangle^T$

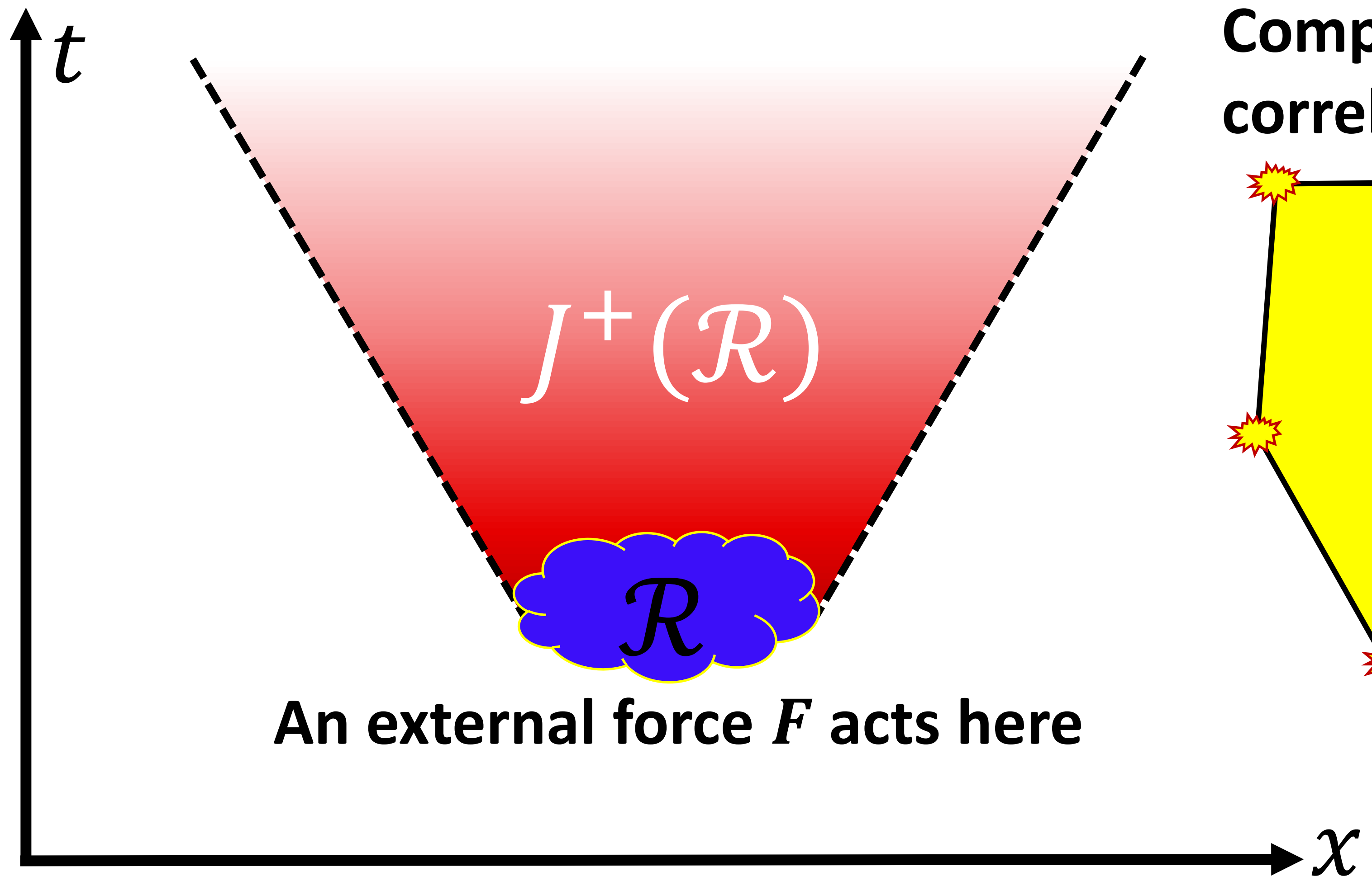


# *Main result*

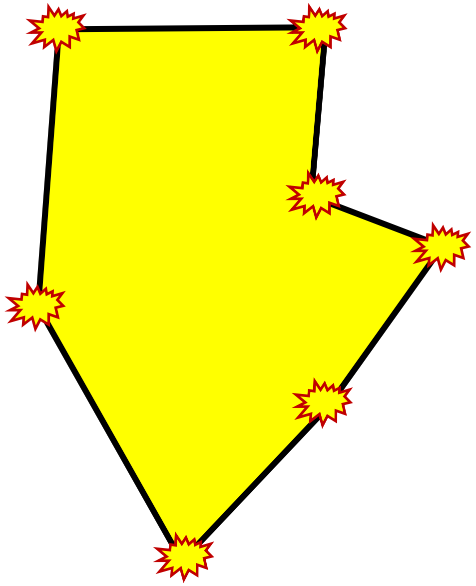
**If you build linear fluctuations using the foregoing procedure, the following facts hold automatically:**

1. **All correlators exist and are well-defined distributions;**
2. **All uncertaintees are non-negative definite by construction: $\langle \delta A^2 \rangle \geq 0$;**
3. **The fluctuation-dissipation theorem (in all its formulations) is recovered;**
4. **If the noise is covariantly Markovian, the fluctuating theory is Lorentz-covariant;**
5. **The fluctuating theory is causal, in the sense that the fluctations cannot be used to send information faster than the speed of light;**
6. **The fluctuating theory is stable, in the sense that the macrostate $\delta\psi = 0$ is the most probable state, and the fluctations do not "condense";**
7. **The disperison relations fulfill all QFT-based microcausality criteria;**
8. **The Martin-Siggia-Rose effective action is well-defined and well-behaved;**
9. **There is a KMS-type symmetry for the theory.**

# *Causality of fluctations*




**Compute n-point correlator here:** $\langle \psi\psi \dots \psi \rangle$



**The value of the correlator is not affected by the external force.**

# *Application 1: BDNK and IS (single-charge diffusion)*

**Equation of motion of average:**

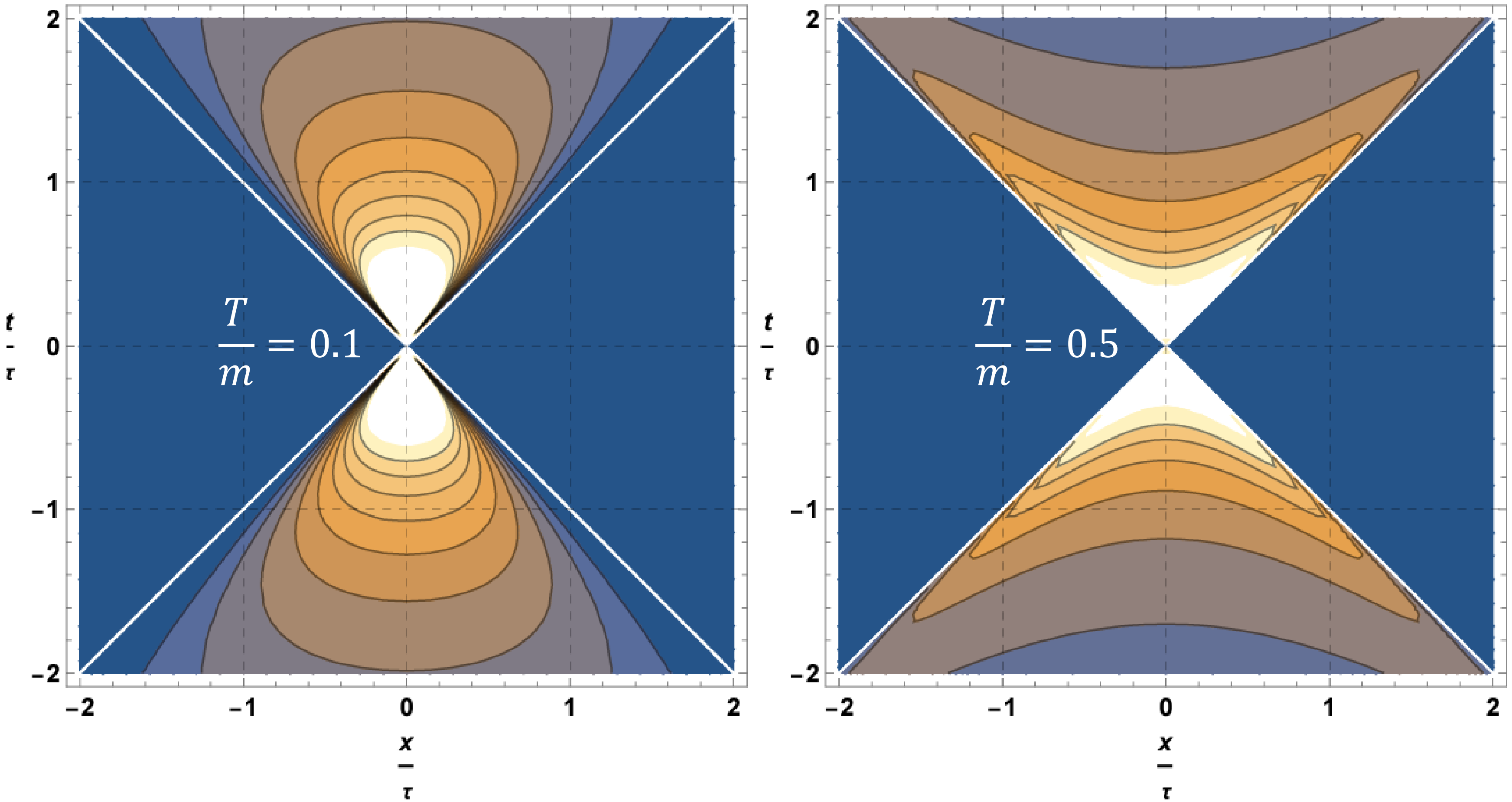
$$\tau\partial_t^2\langle\delta n\rangle + \partial_t\langle\delta n\rangle = D\partial_x^2\langle\delta n\rangle$$

**Two-point correlators:**

$$\langle\delta n(x)\delta n(y)\rangle = T\frac{dn}{d\mu}\int\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\frac{2Dk^jk_j}{\omega^2+\left(Dk^jk_j-\tau\omega^2\right)^2}$$

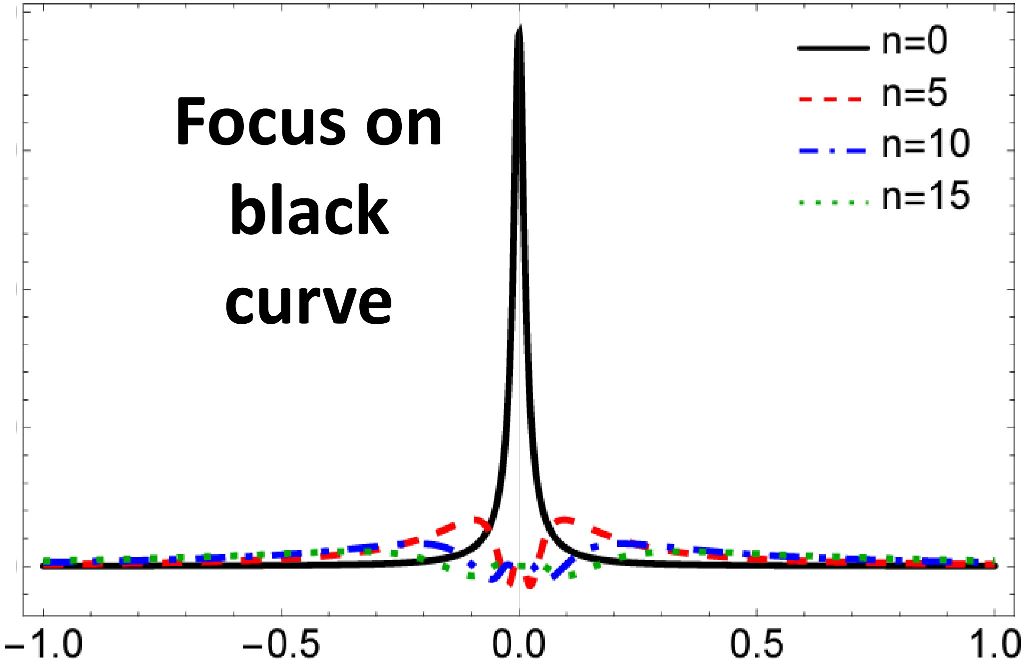
# Application 2: Chemically active diluted solution

$$\langle \delta n(t,x)\delta n(0)\rangle = n_{ur} e^{-|t|/\tau} \frac{t^2\Theta(t^2-x^2)}{8\pi(t^2-x^2)^{5/2}} \frac{m^3}{T^3} exp\left[-\frac{m|t|}{T\sqrt{t^2-x^2}}\right]$$



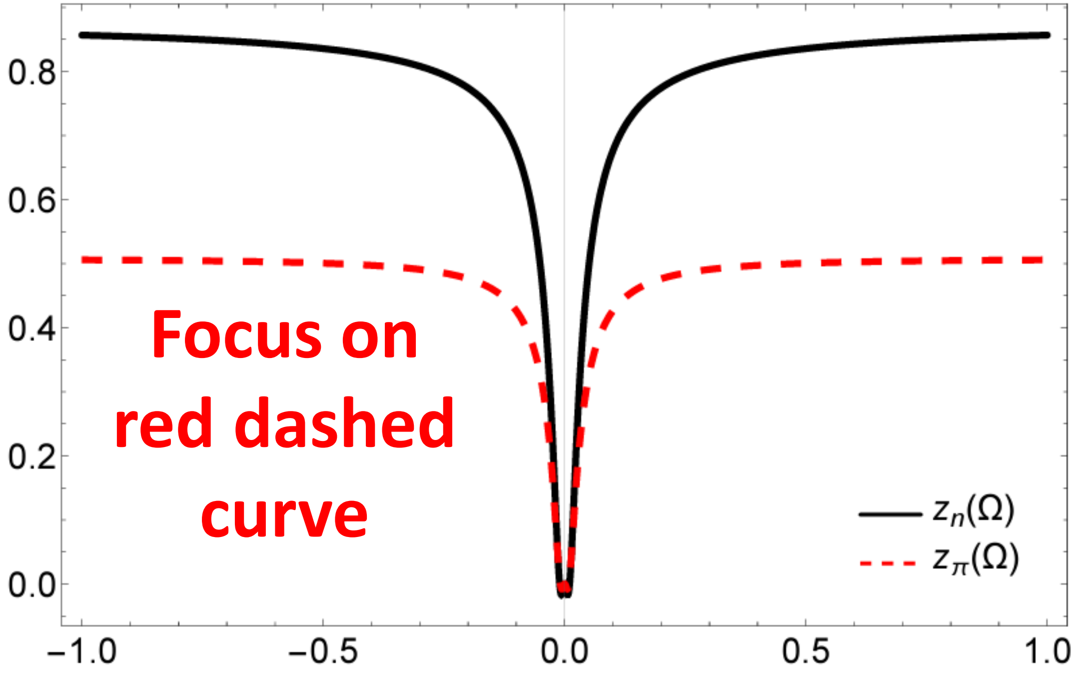
# *Application 3: Non-degenerate massless $\lambda\phi^4$ gas*

**$\langle \pi_{12}\pi_{12} \rangle_\omega$ according to kinetic theory**




$\omega\lambda_{mfp}$

**Relative error of Israel-Stewart compared to kinetic theory**




$\omega\lambda_{mfp}$

# Appendices