Color Confinement and Random Matrices

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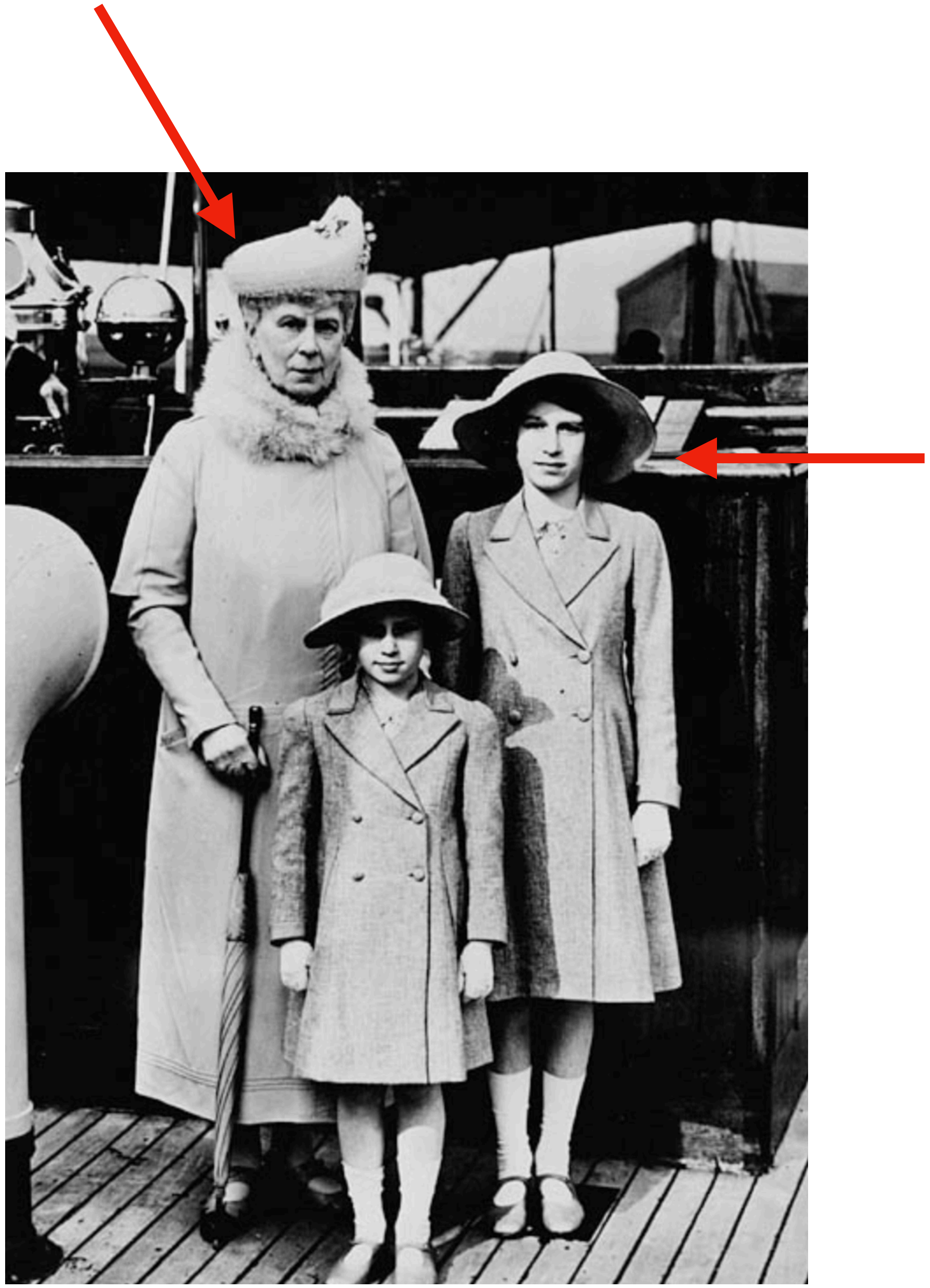


Queen Mary

Queen Elizabeth II

(from Wikipedia)

A puzzle

- QCD does not have center symmetry. Why is Polyakov loop a good 'order parameter'?

Maybe 'approximate' center symmetry?

A puzzle

- QCD does not have center symmetry. Why is Polyakov loop a good 'order parameter'?

~~Maybe 'approximate' center symmetry?~~

No

A puzzle

- QCD does not have center symmetry. Why is Polyakov loop a good 'order parameter'?

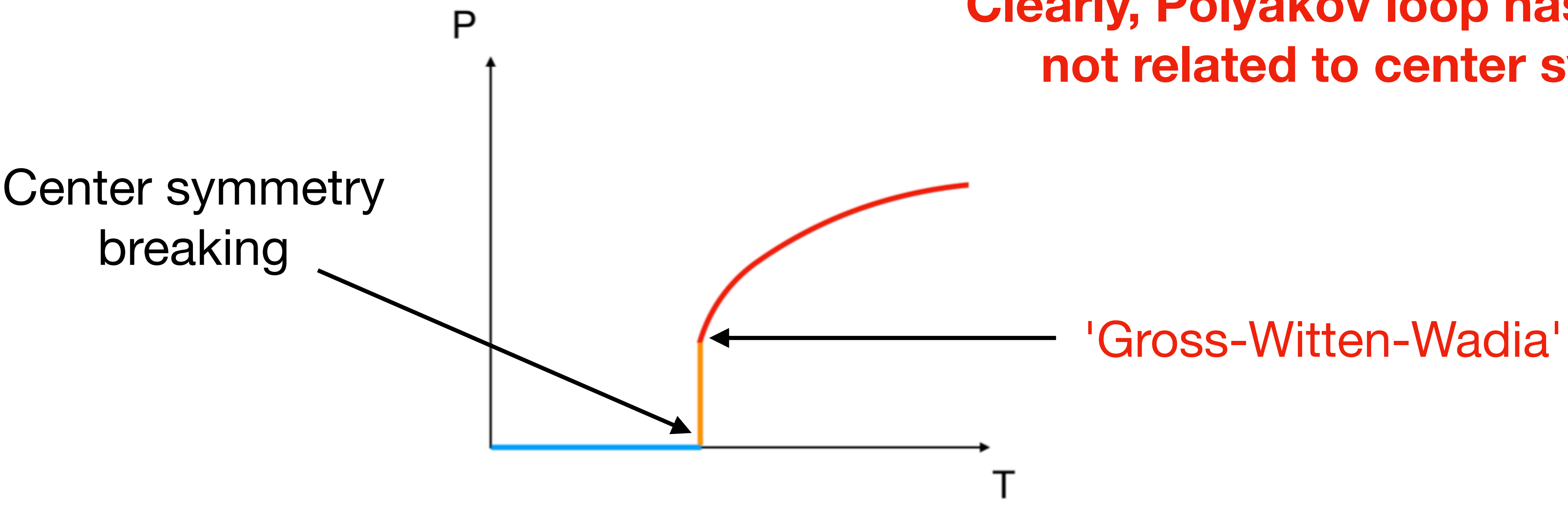
~~Maybe 'approximate' center symmetry?~~

No

Pure YM on small S^3

Aharony et al 2003, Sundborg 1999

Clearly, Polyakov loop has a meaning not related to center symmetry

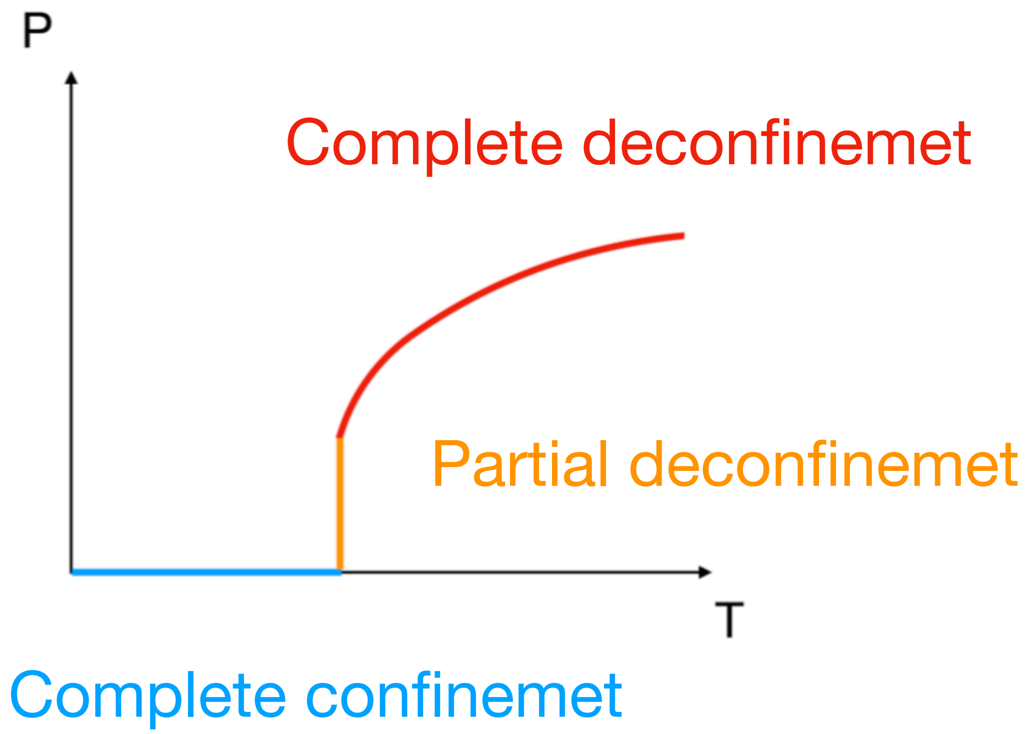


Large N

- Polyakov loop is related to gauge symmetry

- Confinement ~ Bose-Einstein Condensation

MH-Shimada-Wintergerst 2020



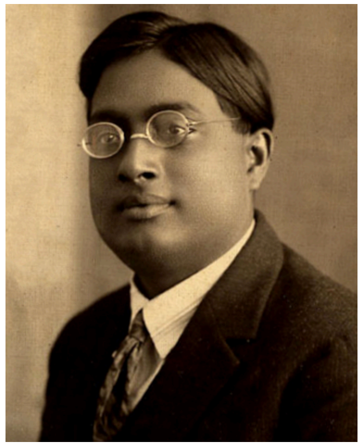
Finite N

- Polyakov loop is related to gauge symmetry

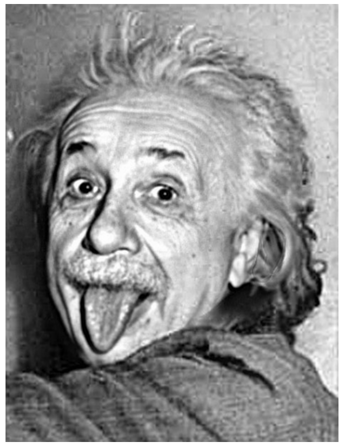
- Gross-Witten-Wadia has finite-N counterpart

MH-Watanabe 2023, MH-Ohata-Shimada-Watanabe 2023

Historically the first example of
non-Abelian gauge theory
in the large-N limit



Bose



Einstein

N indistinguishable bosons

N bosons in 3d harmonic trap

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^{\,2}}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^{\,2} \right)$$

$$\hat{\vec{x}}_i = (\hat{x}_i, \hat{y}_i, \hat{z}_i)$$

$$\hat{\vec{p}}_i = (\hat{p}_{x,i}, \hat{p}_{y,i}, \hat{p}_{z,i})$$

Fock states $|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{3} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$

States related by S_N permutation are identical.

➡ S_N permutation is gauged.

Summation over singlet states $Z(T)=\mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}(e^{-\hat{H}/T})$

Summation over all states & projection to singlet states

$$Z(T)=\frac{1}{\mathrm{vol}(G)}\int_G dg\mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}(\hat{g}e^{-\hat{H}/T})$$

$G=\mathrm{SU}(N)$ + adjoint fields → Yang-Mills, Matrix Model

$G=\mathrm{S}_N$ + fundamental fields → N indistinguishable bosons

For Yang-Mills and Matrix Model:

$$Z(T) = \int [dA_t][dX]e^{-S[A_t,X]}$$

$$\Updownarrow$$

Feynman's method

$$Z(T) = \frac{1}{\mathrm{vol}G}\int_G dg\,\mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}/T}\right)$$

$$\Updownarrow$$

$$Z(T) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}}(e^{-\hat{H}/T})$$

Non-interacting bosons × N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S$_N$ gauge symmetry

Non-interacting bosons × 2N^2

$$\hat{H}_{\text{Gaussian}} = \sum_{\alpha=1}^{N^2} \left(\frac{1}{2} \hat{P}_{I,\alpha}^2 + \frac{1}{2} \hat{X}_{I,\alpha}^2 \right)$$

I =1,2

SU(N) gauge symmetry

Non-interacting bosons × N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Bose-Einstein Condensation

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle$$

Non-interacting bosons × $2N^2$

$$\hat{H}_{\text{Gaussian}} = \sum_{\alpha=1}^{N^2} \left(\frac{1}{2} \hat{P}_{I,\alpha}^2 + \frac{1}{2} \hat{X}_{I,\alpha}^2 \right)$$

$I = 1,2$

SU(N) gauge symmetry

N bosons in 3d harmonic trap

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^{\,2}}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^{\,2} \right)$$

$$\begin{aligned} \hat{\vec{x}}_i &= (\hat{x}_i, \hat{y}_i, \hat{z}_i) \\ \hat{\vec{p}}_i &= (\hat{p}_{x,i}, \hat{p}_{y,i}, \hat{p}_{z,i}) \end{aligned}$$

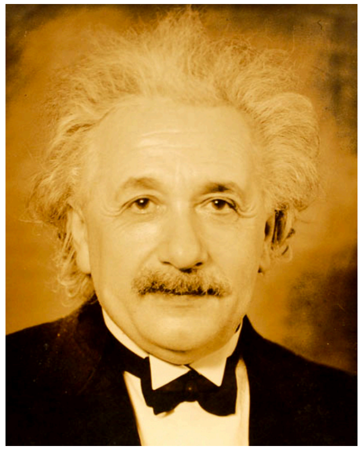
Fock states $|\vec{n}_1, \vec{n}_2, \cdots, \vec{n}_N\rangle \equiv \prod_{i=1}^{3} \frac{\hat{a}_{i1}^{\dagger n_{i1}}}{\sqrt{n_{i1}!}} \frac{\hat{a}_{i2}^{\dagger n_{i2}}}{\sqrt{n_{i2}!}} \cdots \frac{\hat{a}_{iN}^{\dagger n_{iN}}}{\sqrt{n_{iN}!}} |0\rangle$

$$\begin{aligned} Z(T) &= \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \sum_{\vec{n}_1, \cdots, \vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{\sigma} e^{-\hat{H}/T} | \vec{n}_1, \cdots, \vec{n}_N \rangle \\ &= \frac{1}{N!} \sum_{\vec{n}_1, \cdots, \vec{n}_N} e^{-(E_{\vec{n}_1} + \cdots + E_{\vec{n}_N})/T} \underbrace{\left(\sum_{\sigma \in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \vec{n}_{\sigma(1)}, \cdots, \vec{n}_{\sigma(N)} \rangle \right)}_{\text{measures the amount of redundancy}} \end{aligned}$$

京都　三十三間堂

Sanjusangendo, Kyoto

N=1001



(Einstein visited Kyoto in 1922)

$$Z(T) = \frac{1}{N!} \sum_{\sigma \in \mathrm{S}_N} \sum_{\vec{n}_1,\cdots,\vec{n}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \hat{\sigma} e^{-\hat{H}/T} | \vec{n}_1, \cdots, \vec{n}_N \rangle$$

$$= \frac{1}{N!} \sum_{\vec{n}_1,\cdots,\vec{n}_N} e^{-(E_{\vec{n}_1}+\cdots+E_{\vec{n}_N})/T} \underline{\left(\sum_{\sigma \in \mathrm{S}_N} \langle \vec{n}_1, \cdots, \vec{n}_N | \vec{n}_{\sigma(1)}, \cdots, \vec{n}_{\sigma(N)} \rangle \right)}$$

$$|\vec{0}, \vec{0}, \cdots, \vec{0}\rangle \qquad N!$$

$$|\vec{n}_1, \cdots, \vec{n}_N\rangle \qquad 1$$

(all of them are different)

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle \qquad (N-M)!$$

This enhancement triggers BEC.

Einstein, 1924

$$Z(T) = \frac{1}{\mathrm{vol}(G)} \int_G dg \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}(\hat{g} e^{-\hat{H}/T})$$

$G = \mathrm{S}_N$ + fundamental fields → N indistinguishable bosons

'genuine' gauge invariance leads to

the enhancement factor $N! = \mathrm{vol}(\mathrm{S}_N)$

This enhancement triggers BEC.

(Einstein, 1924)

$G = \mathrm{SU}(N)$ + adjoint fields → Yang-Mills, Matrix Model

'genuine' gauge invariance leads to

the enhancement factor $\mathrm{vol}(\mathrm{SU}(N)) \sim e^{N^2}$

This enhancement triggers color confinement.

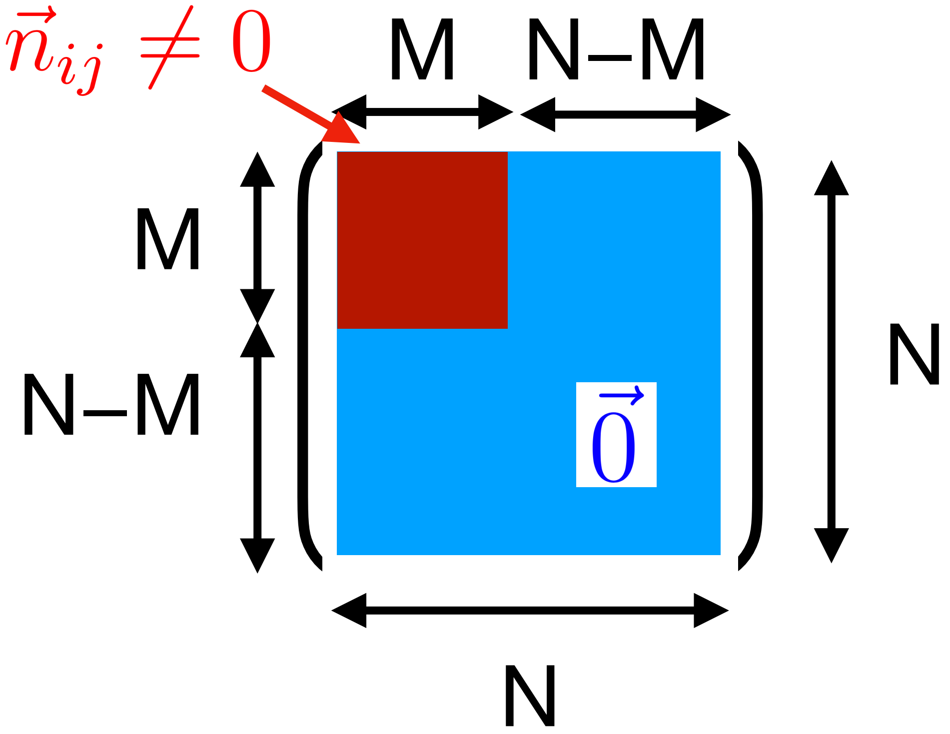
(MH-Shimada-Wintergerst, 2020)

Partially-BEC state

$$|\vec{n}_1,\cdots,\vec{n}_M,\vec{0},\cdots,\vec{0}\rangle \quad (N-M)!$$

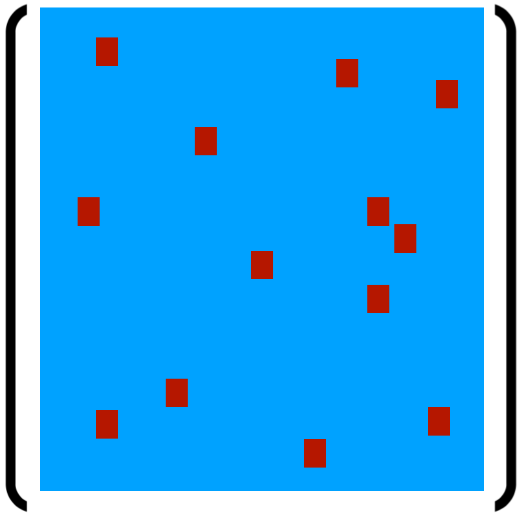
Partially-confined state

(MH-Maltz, 2016; Berenstein, 2018;
MH-Ishiki-Watanabe, 2018;
MH-Jevicki-Peng-Wintergerst, 2019;
Watanabe et al, 2020)



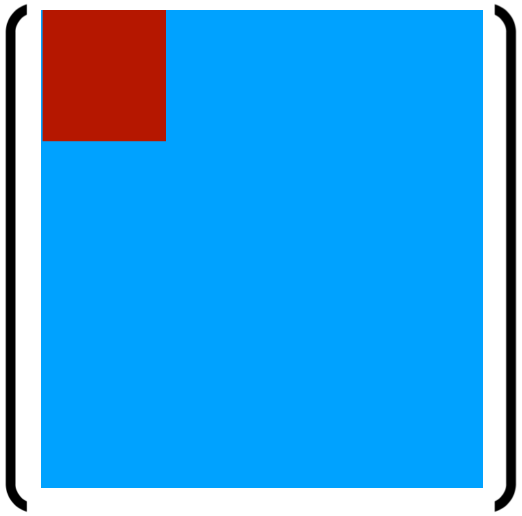
$$\mathrm{vol}(\mathrm{SU}(N-M)) \sim e^{(N-M)^2}$$

👎



no symmetry

👍



Larger enhancement factor

$$\mathrm{vol}(\mathrm{SU}(N-M)) \sim e^{(N-M)^2}$$

MH-Shimada-Wintergerst, 2020

Non-interacting bosons × N

$$\hat{H} = \sum_{i=1}^{N} \left(\frac{\hat{\vec{p}}_i^2}{2m} + \frac{m\omega^2}{2} \hat{\vec{x}}_i^2 \right)$$

S_N gauge symmetry

Bose-Einstein Condensation

$$|\vec{n}_1, \cdots, \vec{n}_M, \vec{0}, \cdots, \vec{0}\rangle$$

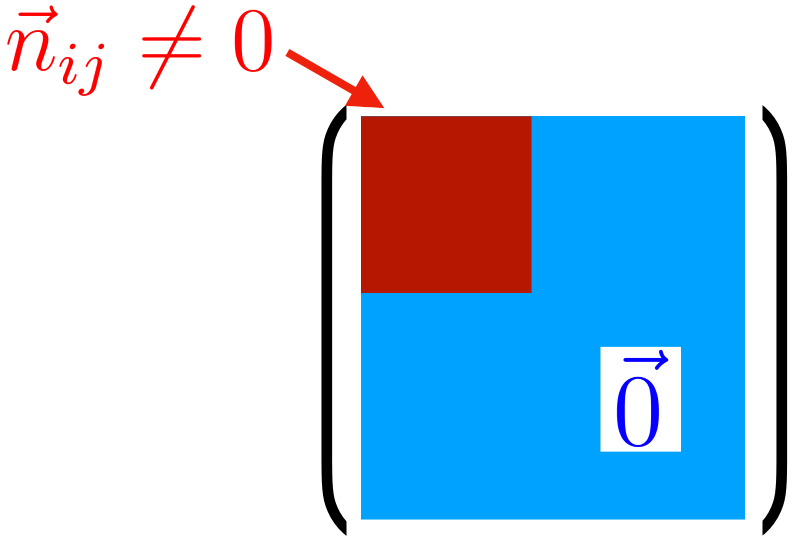
Non-interacting bosons × 2N²

$$\hat{H}_{\text{Gaussian}} = \sum_{\alpha=1}^{N^2} \left(\frac{1}{2} \hat{P}_{I,\alpha}^2 + \frac{1}{2} \hat{X}_{I,\alpha}^2 \right)$$

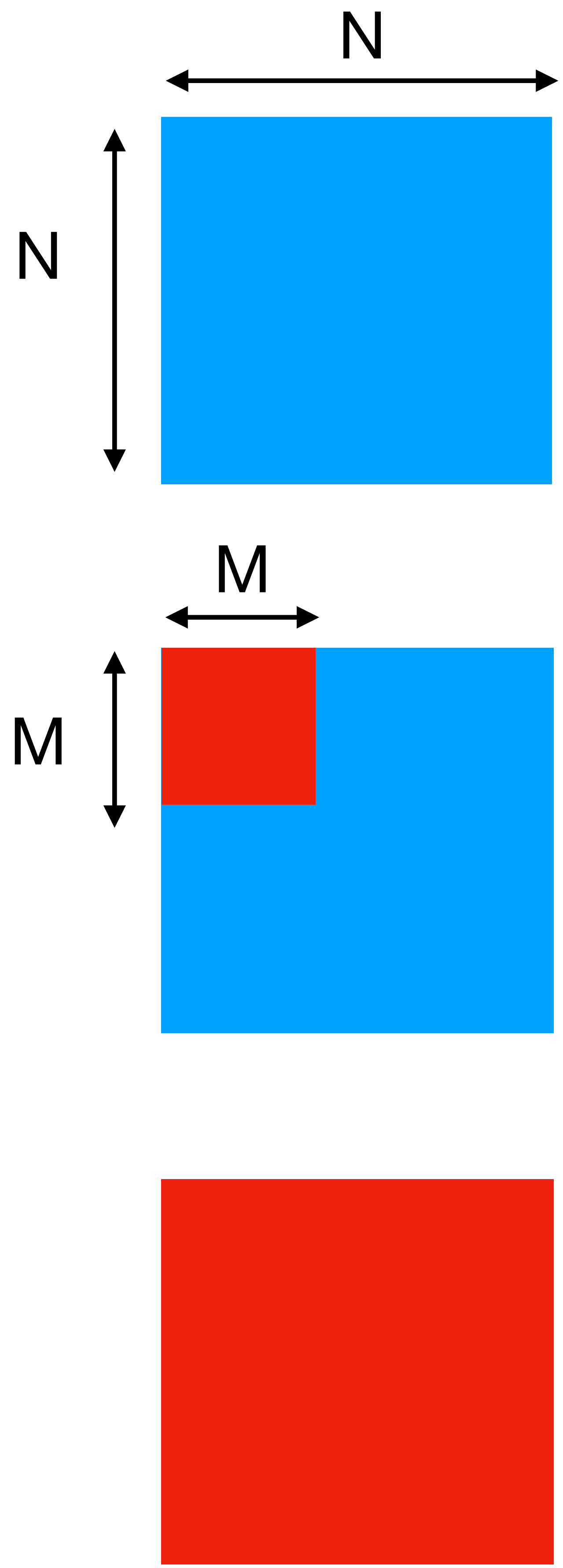
$I = 1,2$

SU(N) gauge symmetry

Partial confinement



Generalization to finite coupling.
(MH, 2021)



Completely Confined

Partially Confined
(= Partially Deconfined)

MH-Maltz, 2016 (JHEP)
MH-Ishiki-Watanabe, 2018 (JHEP)
MH-Jevicki-Peng-Wintergerst, 2019 (JHEP)
MH-Shimada-Wintergerst, 2020 (JHEP)

Completely Deconfined

lower
energy

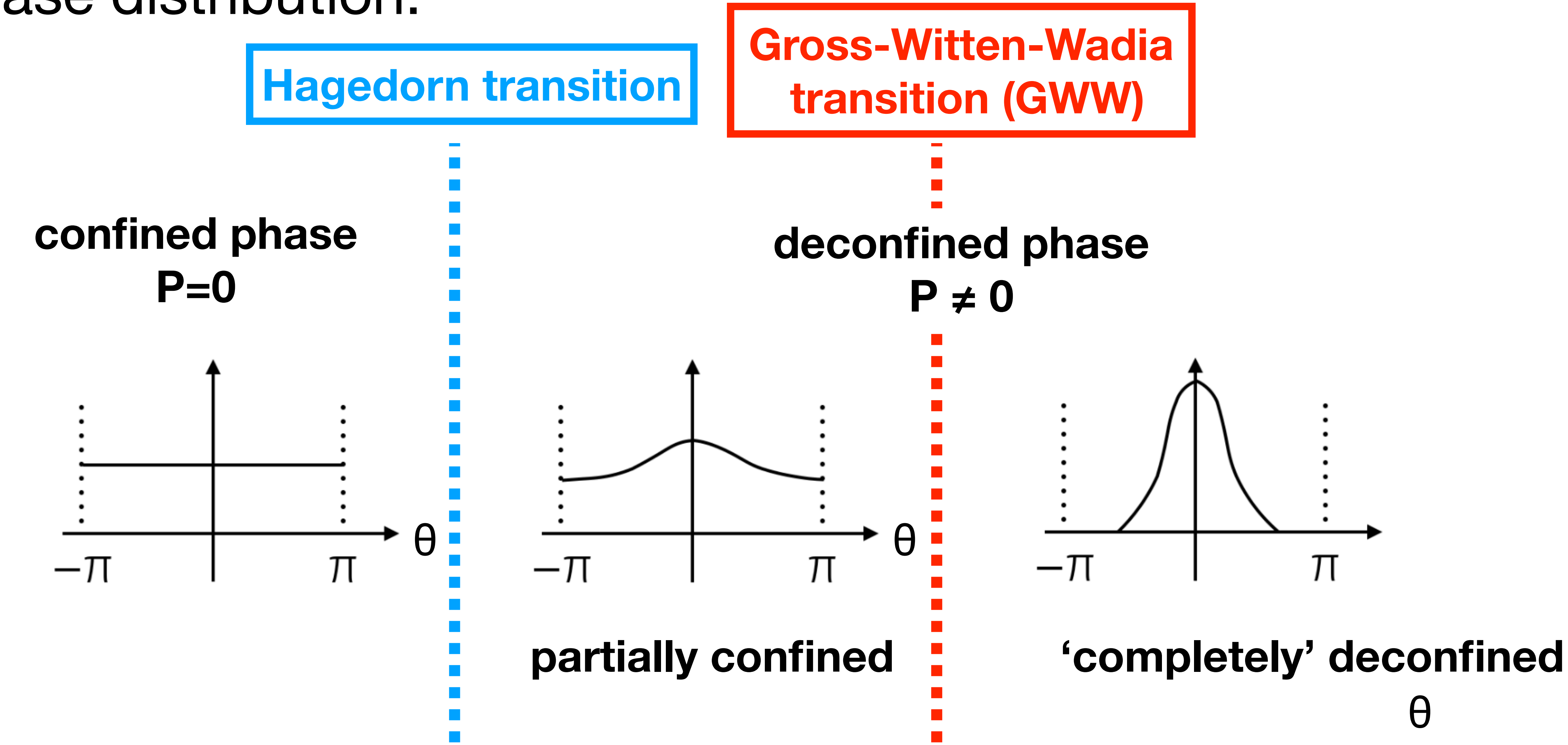
higher
energy

- Polyakov loop

(Wilson loop wrapped on
the temporal circle)

$$P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

- Phase distribution:



Hagedorn transition

Gross-Witten-Wadia transition (GWW)

confined phase
P=0

deconfined phase
P ≠ 0

partially confined

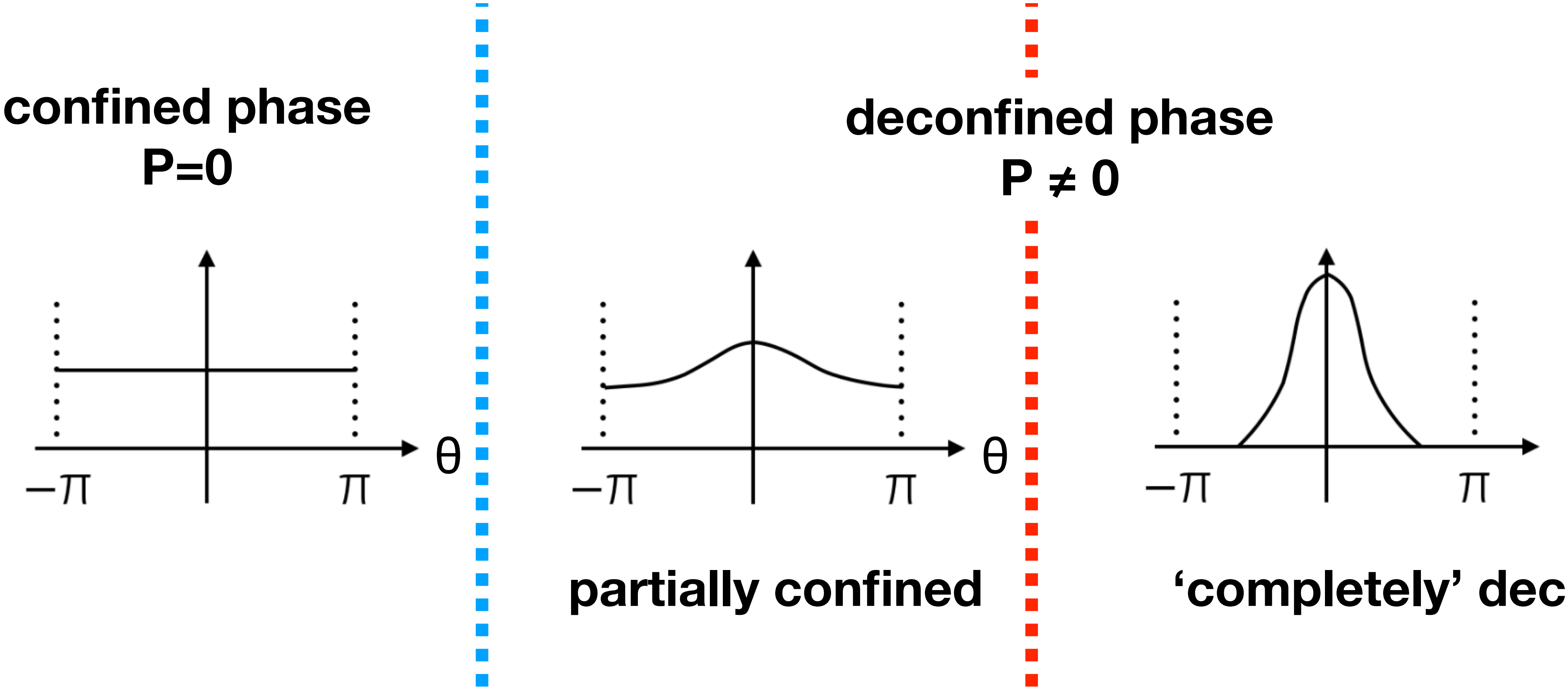
'completely' deconfined

- Polyakov loop

$$P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

(Wilson loop wrapped on
the temporal circle)

- Phase distribution:



Hagedorn transition

Gross-Witten-Wadia transition (GWW)

confined phase
P=0

deconfined phase
P ≠ 0

partially confined

‘completely’ deconfined
θ

Completely BEC **Partially BEC**

Feynman essentially said this in 1953

$$Z(T) = \frac{1}{\mathrm{vol}G} \int_G dg \, \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}} \left(\hat{g} e^{-\hat{H}/T} \right)$$

$$\sim \frac{1}{\mathrm{vol}G} e^{-E_{\mathrm{typical}}/T} \int_G dg \langle \mathrm{typical} | \hat{g} | \mathrm{typical} \rangle$$

Polyakov loop

$$Z(T) = \int [dA_t][dX] e^{-S[A_t, X]}$$

$$\Updownarrow \quad \textbf{Feynman's Ph.D. thesis}$$

$$Z(T) = \frac{1}{\mathrm{vol}G} \int_G dg \, \mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}} \left(\hat{g} e^{-\hat{H}/T} \right)$$

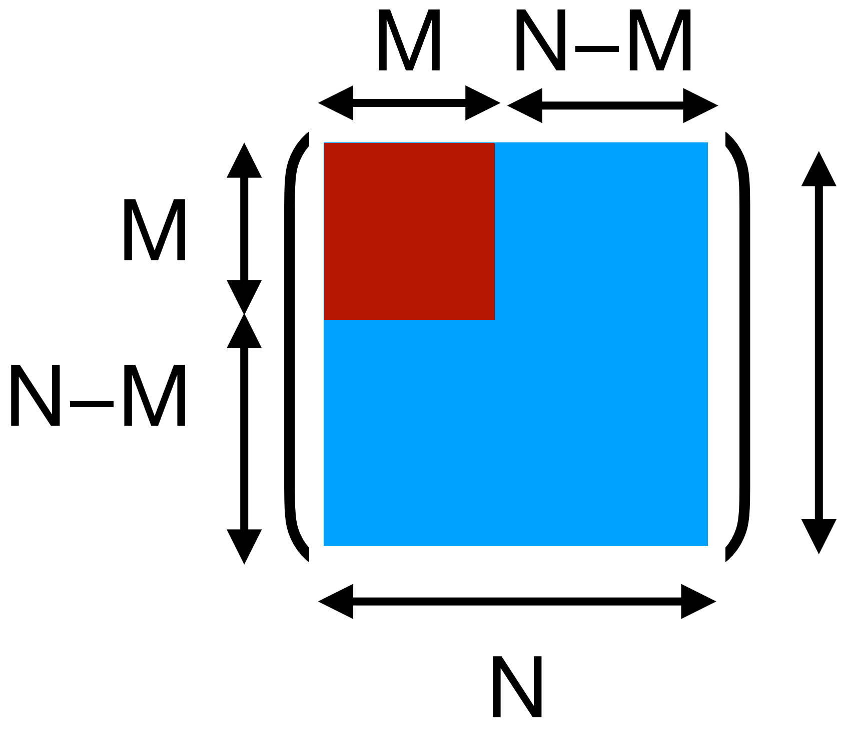
$$\Updownarrow$$

$$Z(T) = \mathrm{Tr}_{\mathcal{H}_{\mathrm{inv}}} (e^{-\hat{H}/T})$$

$$
\begin{aligned}
Z(T) &= \frac{1}{\mathrm{vol}G}\int_G dg\,\mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}/T}\right)\\
&\sim \frac{1}{\mathrm{vol}G}e^{-E_{\mathrm{typical}}/T}\int_G dg\langle\mathrm{typical}|\hat{g}|\mathrm{typical}\rangle
\end{aligned}
$$

Polyakov loop

Typical $\hat{g}$'s which leave $|\text{typical}\rangle$ unchanged dominate the phase distribution

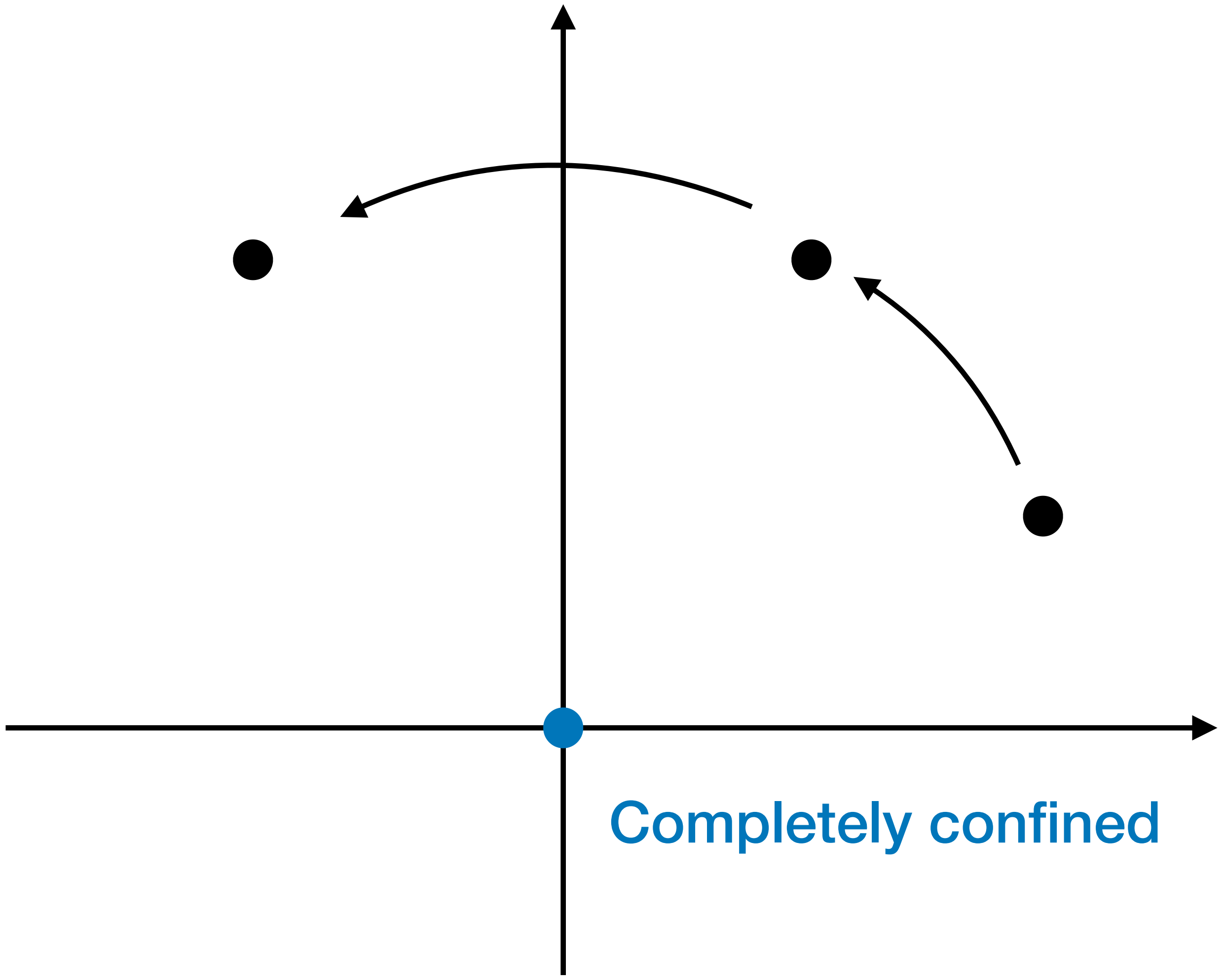


$$
g \sim \begin{pmatrix} 1 & \\ & \Omega \end{pmatrix}
$$

Haar random

$$
|\vec{n}_1,\cdots,\vec{n}_M,\vec{0},\cdots,\vec{0}\rangle
$$

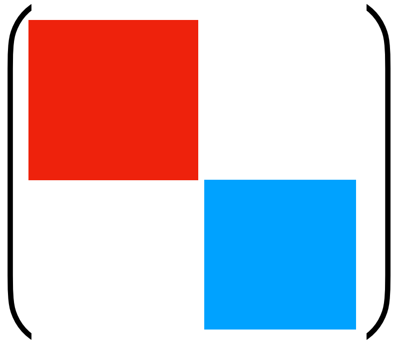
- Extended Hilbert space ⊃ Singlet Hilbert space
- Gauge orbit ⟷ singlet

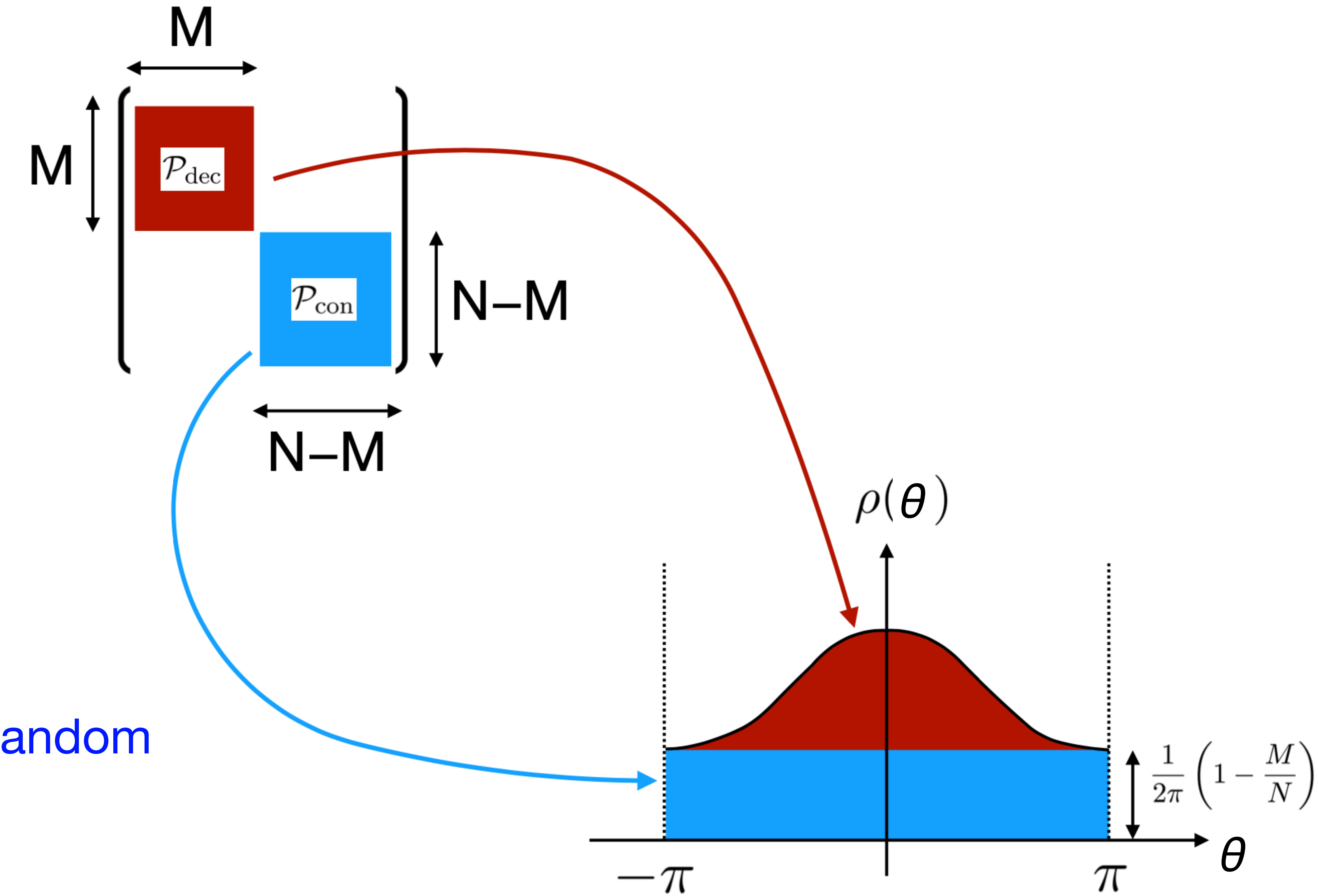


Orbit = SU(N) / stabilizer

Polyakov loop ~ stabilizer

Stabilizer = SU(N-M)
for partially-deconfined states

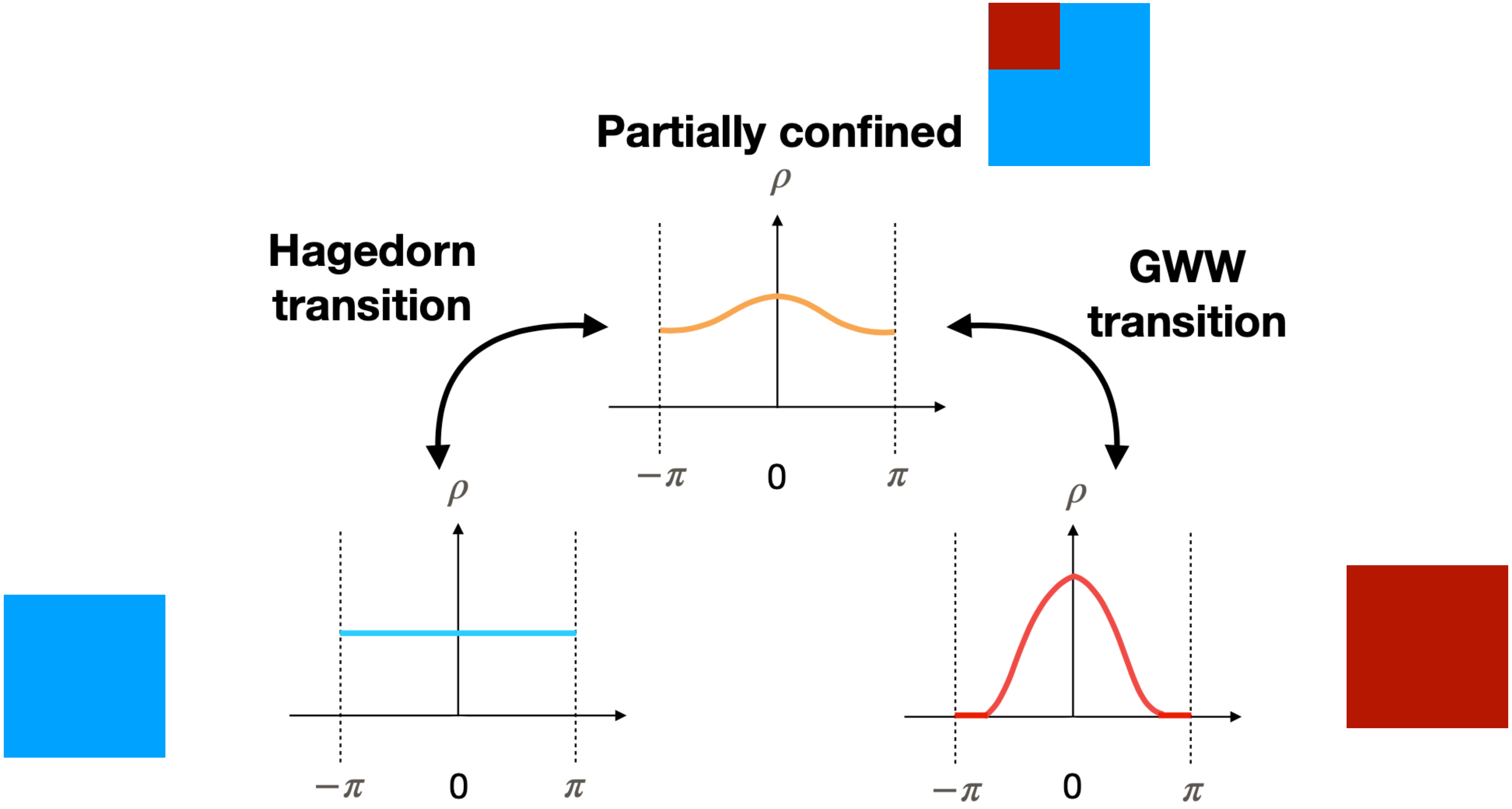




$\mathcal{P}_{\text{con}}$

Haar random

Polyakov Loop $$P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$



Partially confined

Hagedorn transition

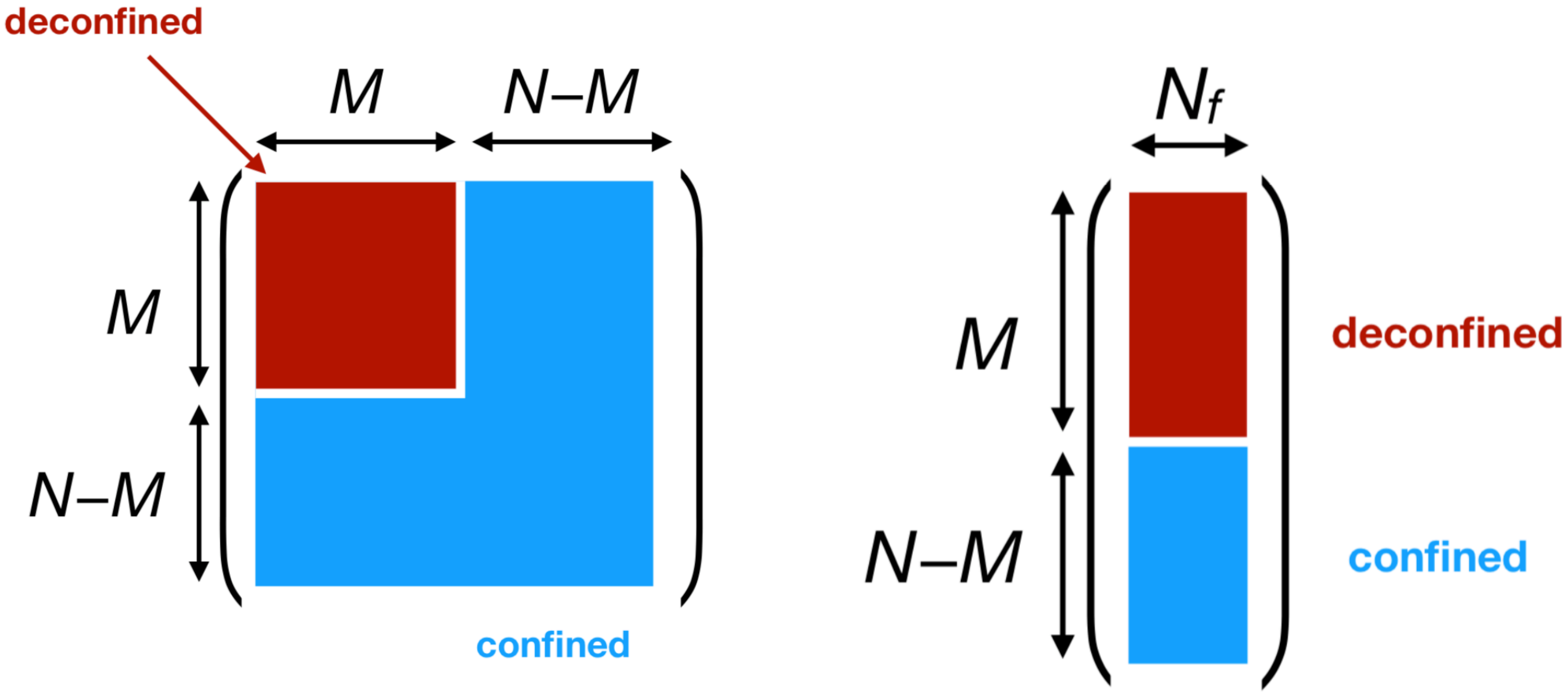
GWW transition

Completely confined

Completely deconfined

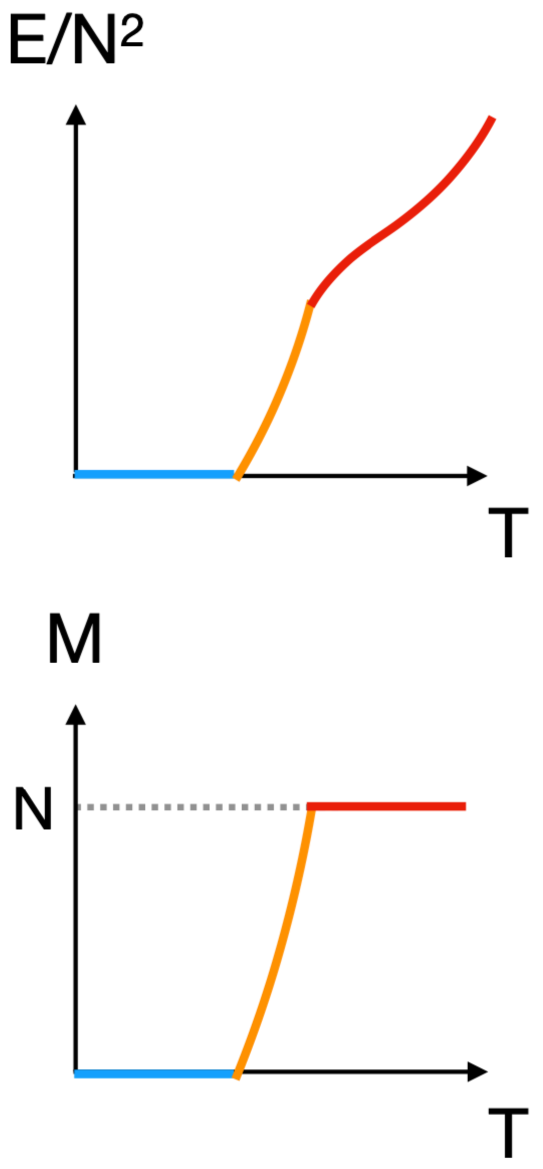
MH-Maltz, 2016 (JHEP)
MH-Ishiki-Watanabe, 2018 (JHEP)
MH-Shimada-Wintergerst, 2020 (JHEP)

QCD phase transition

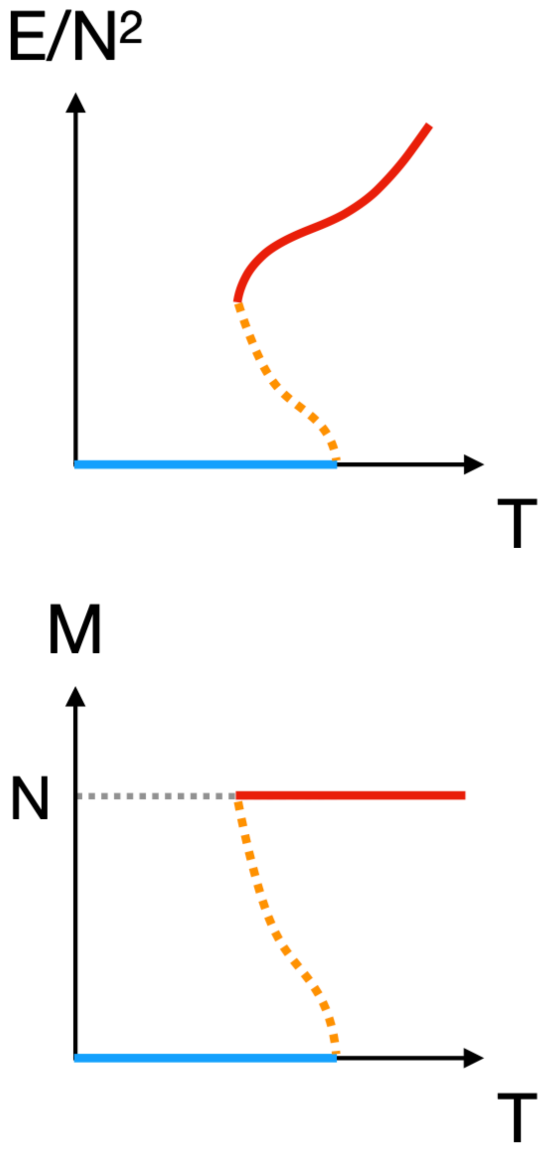


Weak-coupling analysis: MH-Robinson 2019

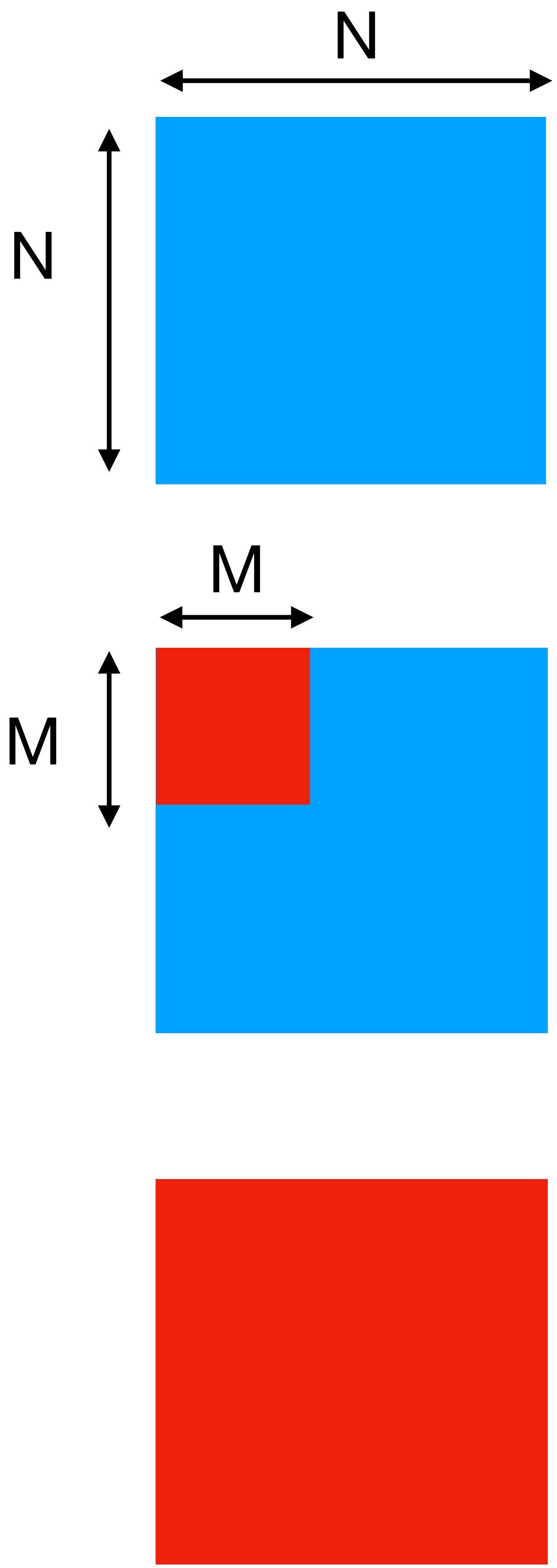
QCD phase transition



Light quark mass



Heavy quark mass



Completely Confined

Partially Confined
(= Partially Deconfined)

MH-Maltz, 2016
Berenstein, 2018
MH-Ishiki-Watanabe, 2018
MH-Jevicki-Peng-Wintergerst, 2019
MH-Shimada-Wintergerst, 2020

Completely Deconfined

lower
energy

higher
energy

From Matrix Model
to
Quantum Field Theory

(MH, Shimada, Wintergerst, 2020; MH, Watanabe, 2023)

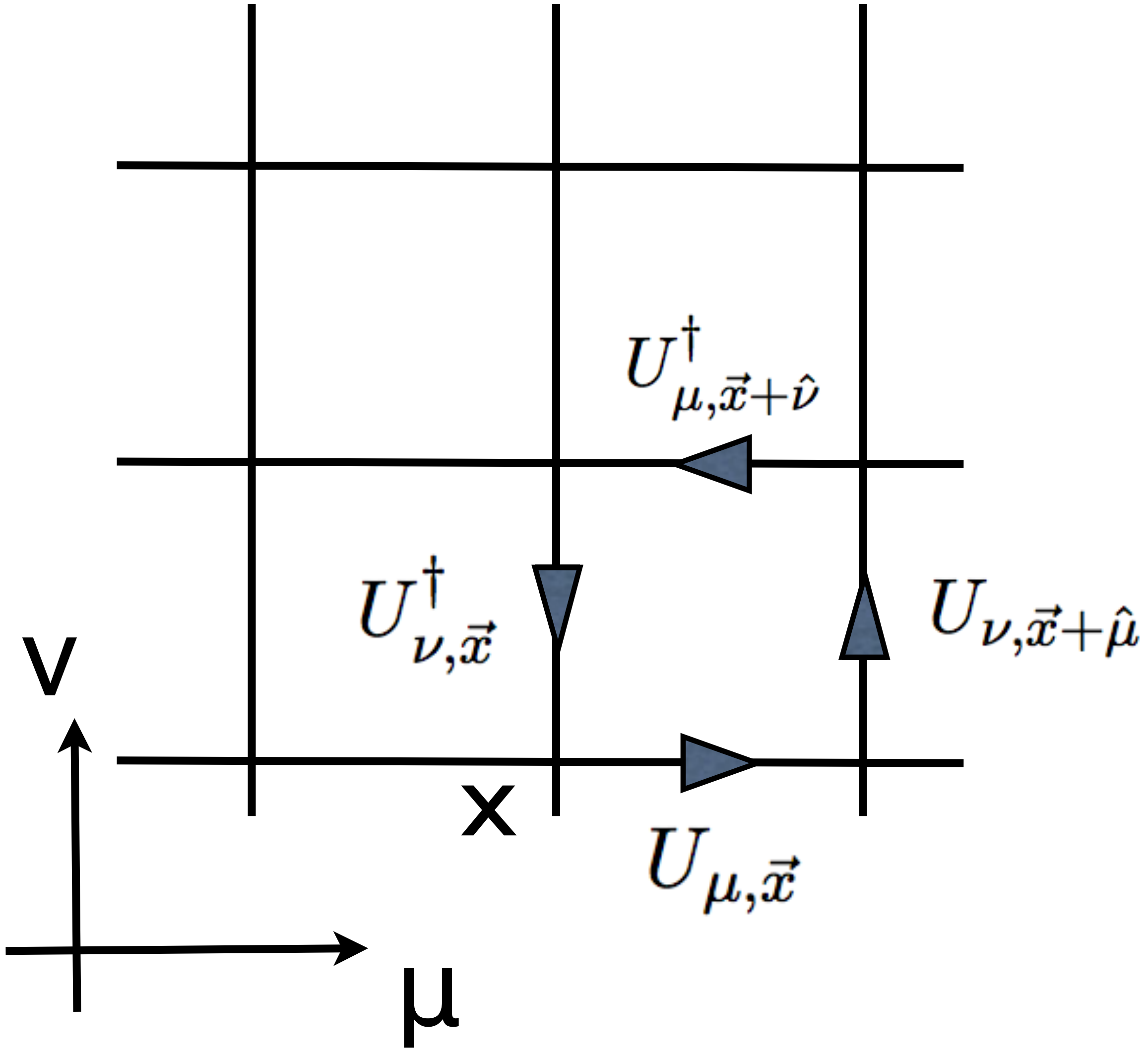
(Spatial) Lattice Regularization

Unitary link variable

$$U_{\mu,\vec{x}} = e^{iaA_\mu(x)}$$

a : lattice spacing

$\beta = 1/(g_{YM}^2(a) \cdot N)$



Ground state is a wave packet around $U_{\vec{n},\mu} = \mathbf{1}$

up to gauge transformation

$$U_{\vec{n},\mu} = \mathbf{1} \longrightarrow U_{\vec{n},\mu} = \Omega_{\vec{n}}^{-1}\Omega_{\vec{n}+\mu}$$

Ground state of cannot be local SU(N) invariant!!

<u>Global</u> SU(N) invariance:

$$P_{\vec{n}} \equiv \Omega_{\vec{n}}^{-1} V \Omega_{\vec{n}}$$

$$P_{\vec{n}}^{-1} U_{\vec{n},\mu} P_{\vec{n}+\mu} = \Omega_{\vec{n}}^{-1} V^{-1} \Omega_{\vec{n}} (\Omega_{\vec{n}}^{-1}\Omega_{\vec{n}+\mu}) \Omega_{\vec{n}+\mu}^{-1} V \Omega_{\vec{n}+\mu} = \Omega_{\vec{n}}^{-1}\Omega_{\vec{n}+\mu}$$

More generally, any *slowly varying* SU(N) transformation leads to enhancement.

$$Z(T) = \frac{1}{\mathrm{vol}G}\int_G dg\,\mathrm{Tr}_{\mathcal{H}_{\mathrm{ext}}}\left(\hat{g}e^{-\hat{H}/T}\right)$$

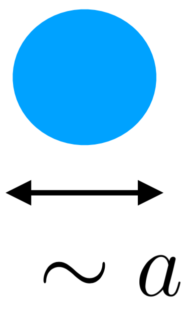
$$\sim \frac{1}{\mathrm{vol}G}e^{-E_{\mathrm{typical}}/T}\int_G dg\,\underline{\langle\mathrm{typical}|}\hat{g}\underline{|\mathrm{typical}\rangle}$$

Sufficiently large
overlap is needed

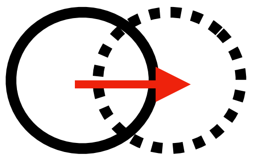
$$P_{\vec{n}} \equiv \Omega_{\vec{n}}^{-1}V_{\vec{n}}\Omega_{\vec{n}}$$

$$P_{\vec{n}}^{-1}U_{\vec{n},\mu}P_{\vec{n}+\mu} = \Omega_{\vec{n}}^{-1}V_{\vec{n}}^{-1}\Omega_{\vec{n}}(\Omega_{\vec{n}}^{-1}\Omega_{\vec{n}+\mu})\Omega_{\vec{n}+\mu}^{-1}V_{\vec{n}+\mu}\Omega_{\vec{n}+\mu} = \Omega_{\vec{n}}^{-1}\underline{V_{\vec{n}}^{-1}V_{\vec{n}+\mu}}\Omega_{\vec{n}+\mu}$$

should be close to 1



$\sim a$



$$V_{\vec{n}}^{-1}V_{\vec{n}+\hat{\mu}} \sim e^{iaX}$$

Polyakov line is slowly-varying Haar random

Renormalization

- Polyakov loop/Wilson loop receives renormalization
- <Pol> = 0 in the continuum limit

We cannot distinguish excited states and ground state if we zoom in to very short distance

- work at fixed lattice spacing

We do this for QCD thermodynamics.

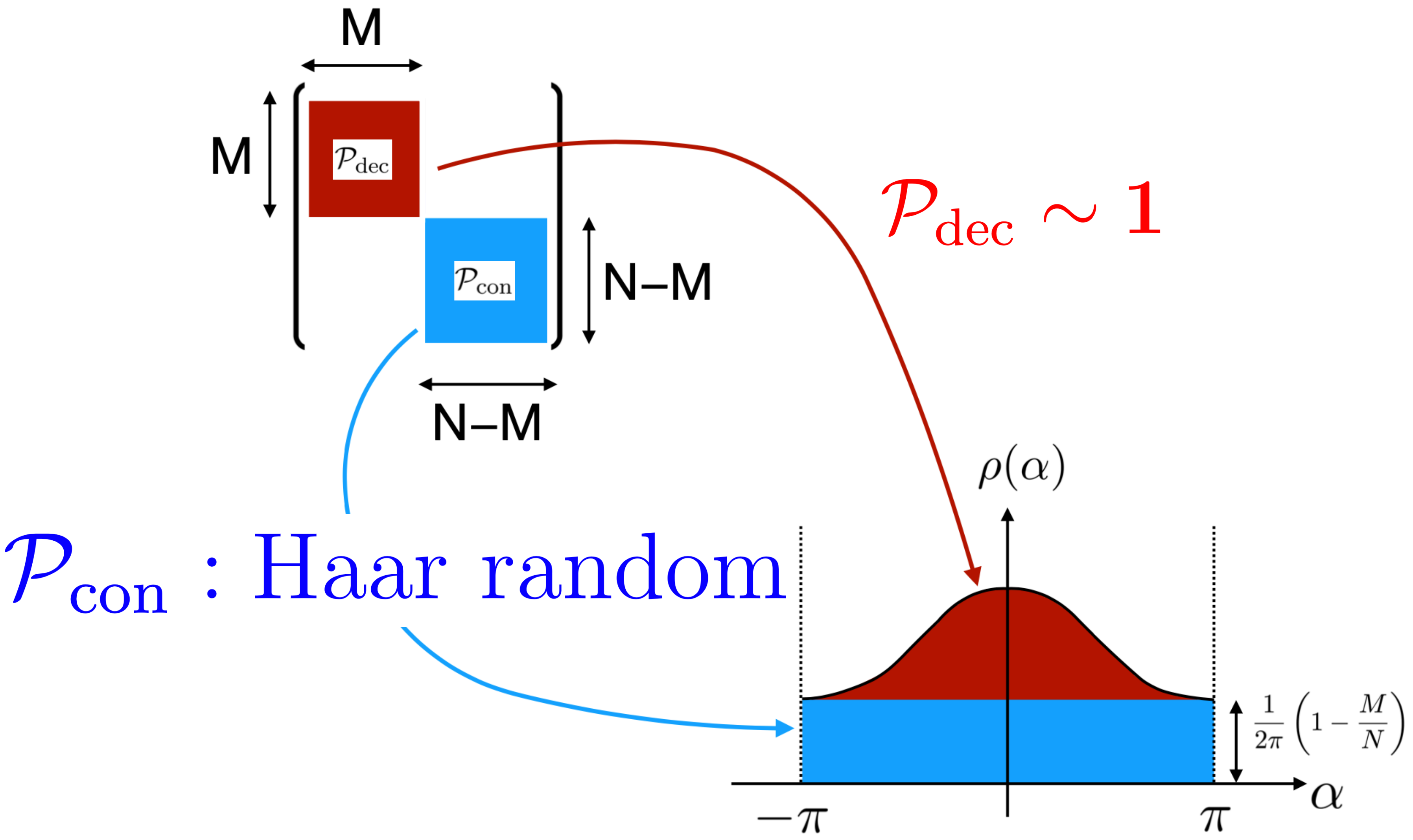
or...

- use renormalized or smeared loop

We do this for Casimir scaling.

Finite-N theories

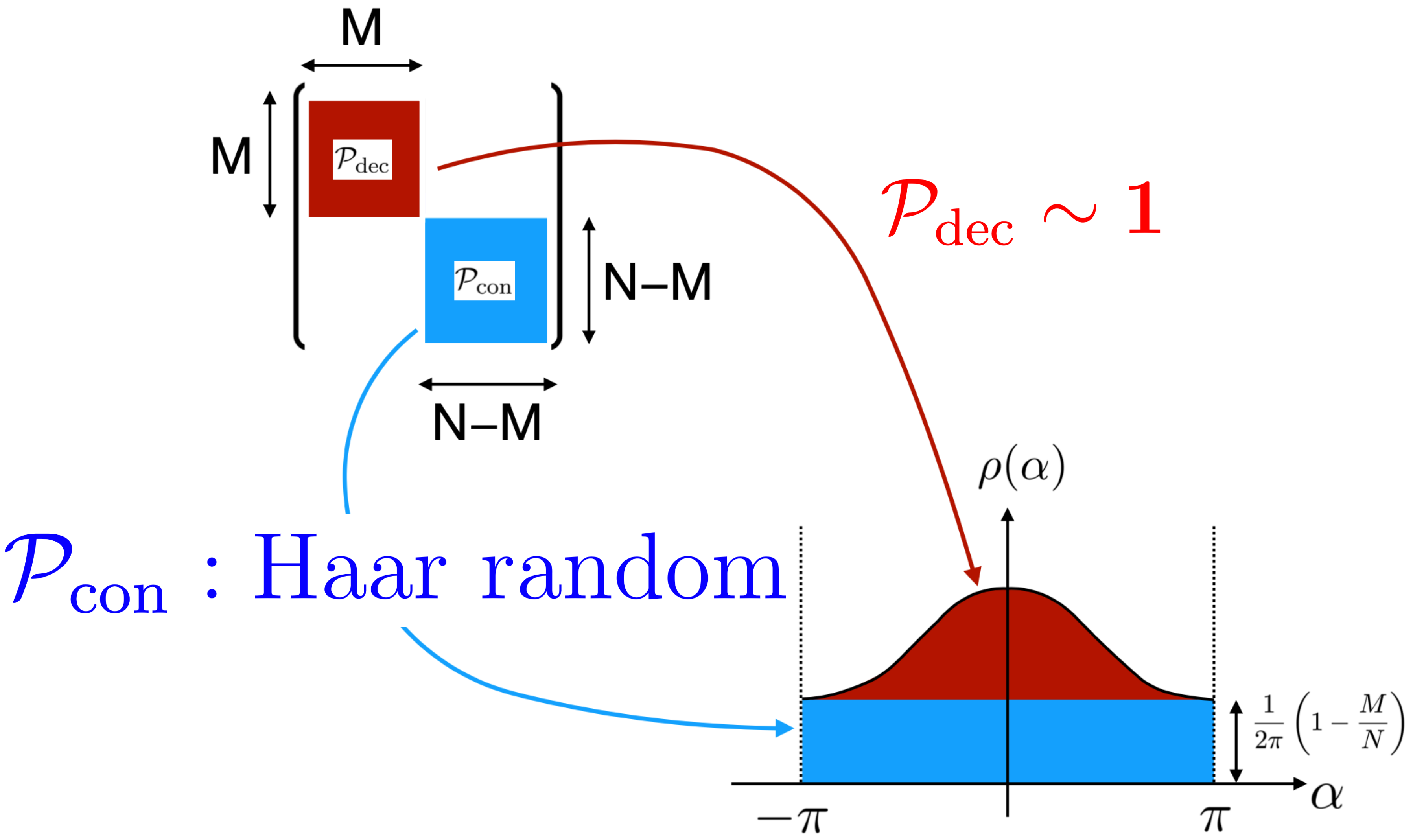
MH, Ohata, Shimada, Watanabe, 2023 (hep-th)
MH, Watanabe, 2023 (hep-th)



$\mathcal{P}_{\rm dec} \sim 1$

$\mathcal{P}_{\rm con}$: Haar random

1/N correction makes "M" ambiguous...



$\mathcal{P}_{\rm dec} \sim 1$

$\mathcal{P}_{\rm con}$: Haar random

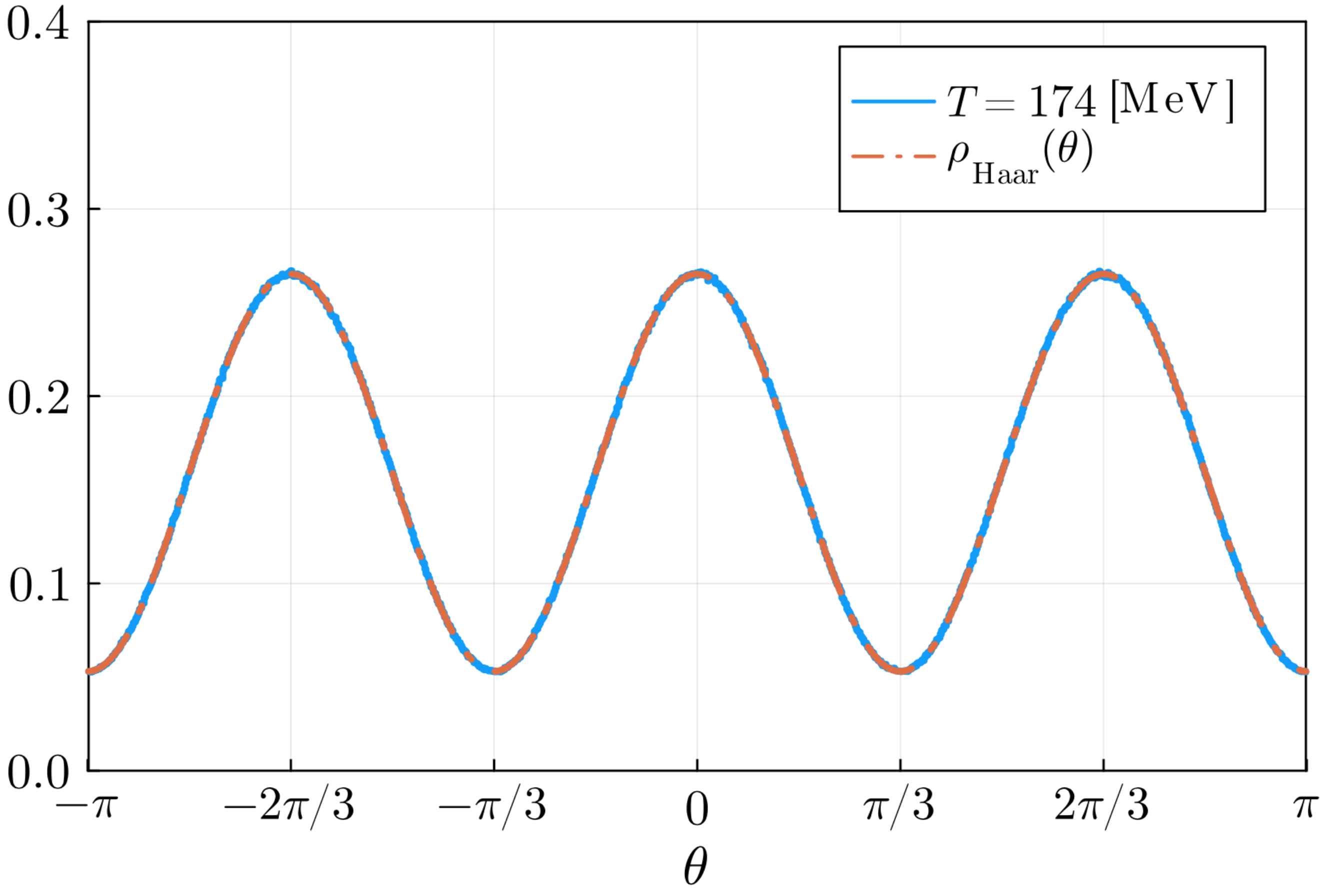
1/N correction makes "M" ambiguous...

But no ambiguity for M=0

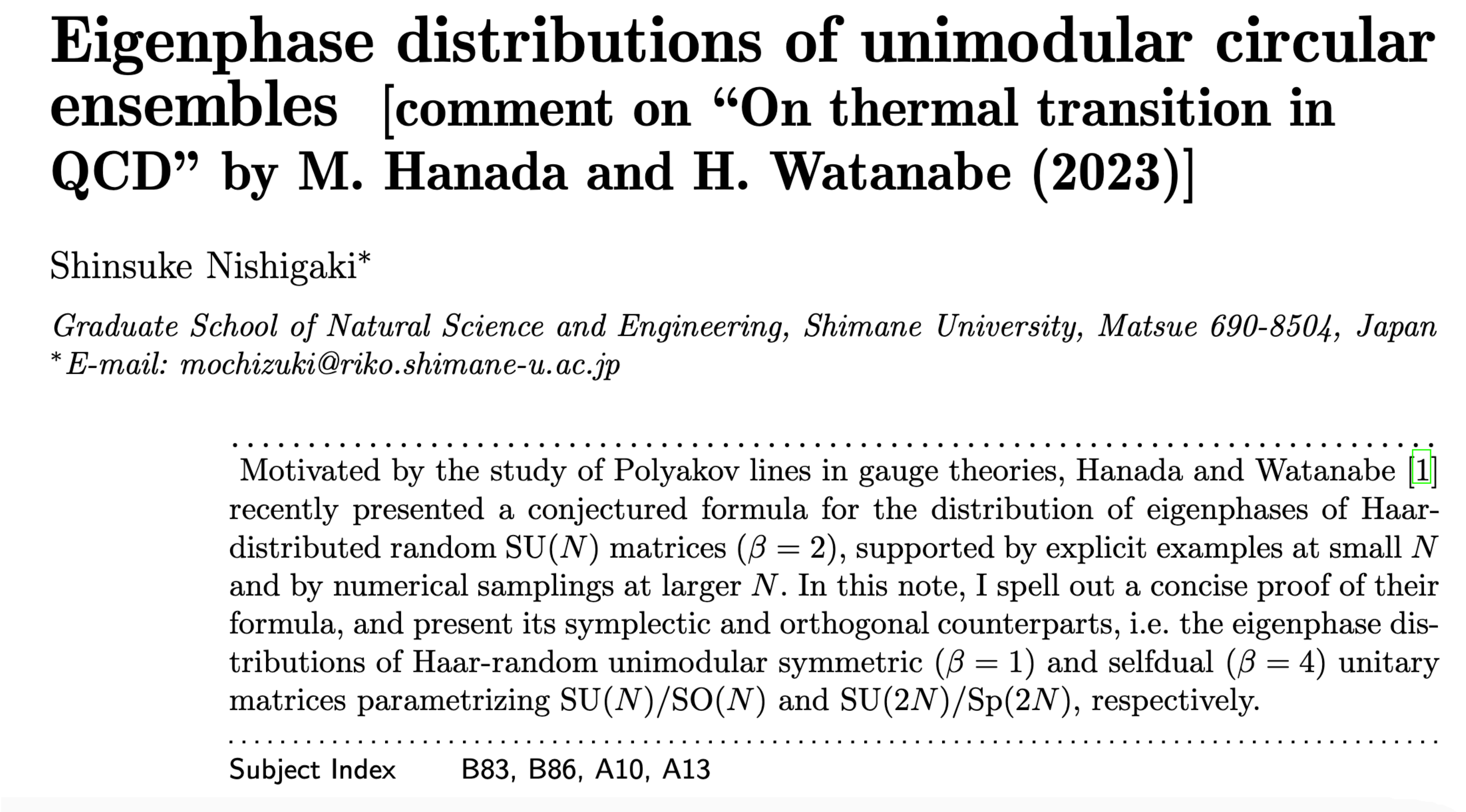
Completely-confined → SU(N) Haar random

(At sufficiently strong coupling)

N=3
WHOT-QCD
configugration



$$\rho_{\mathrm{Haar}}(\theta) = \frac{1}{2\pi}\left(1 - (-1)^N \cdot \frac{2}{N}\cos(N\theta)\right)$$



Eigenphase distributions of unimodular circular ensembles [comment on “On thermal transition in QCD” by M. Hanada and H. Watanabe (2023)]

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Motivated by the study of Polyakov lines in gauge theories, Hanada and Watanabe [1] recently presented a conjectured formula for the distribution of eigenphases of Haar-distributed random SU(N) matrices ($\beta = 2$), supported by explicit examples at small N and by numerical samplings at larger N. In this note, I spell out a concise proof of their formula, and present its symplectic and orthogonal counterparts, i.e. the eigenphase distributions of Haar-random unimodular symmetric ($\beta = 1$) and selfdual ($\beta = 4$) unitary matrices parametrizing SU(N)/SO(N) and SU($2N$)/Sp($2N$), respectively.

Subject Index B83, B86, A10, A13

$$\rho_{\beta,N}(\theta) = \frac{N}{2\pi} \times \left\{ \begin{array}{ll} 1-(-1)^N \dfrac{2}{N} \cos N\theta & (\beta=2) \\ 1-(-1)^N \dfrac{\sqrt{\pi}(N-1)!}{2^{N-1}\Gamma(N/2+3/2)\Gamma(N/2+1)} \cos N\theta & (\beta=1) \\ 1-(-1)^N \dfrac{(2N)!!}{(2N-1)!!N} \cos N\theta + \dfrac{2}{(2N-1)N} \cos 2N\theta & (\beta=4) \end{array} \right\}.$$

$$\begin{aligned}\rho_{\text{Polyakov}}(\theta) &= \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n>0}\left(\tilde{\rho}_n e^{-in\theta} + \tilde{\rho}_{-n} e^{in\theta}\right) \\ &= \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{n>0} 2\tilde{\rho}_n \cos(n\theta)\,.\end{aligned}$$

$$\tilde{\rho}_n = \begin{cases} \frac{(-1)^N}{N} & (n = \pm N) \\ 0 & (n \neq \pm N) \end{cases}$$

$$\tilde{\rho}_n = \frac{1}{N}\left\langle \mathrm{Tr}(\mathcal{P}^n) \right\rangle$$

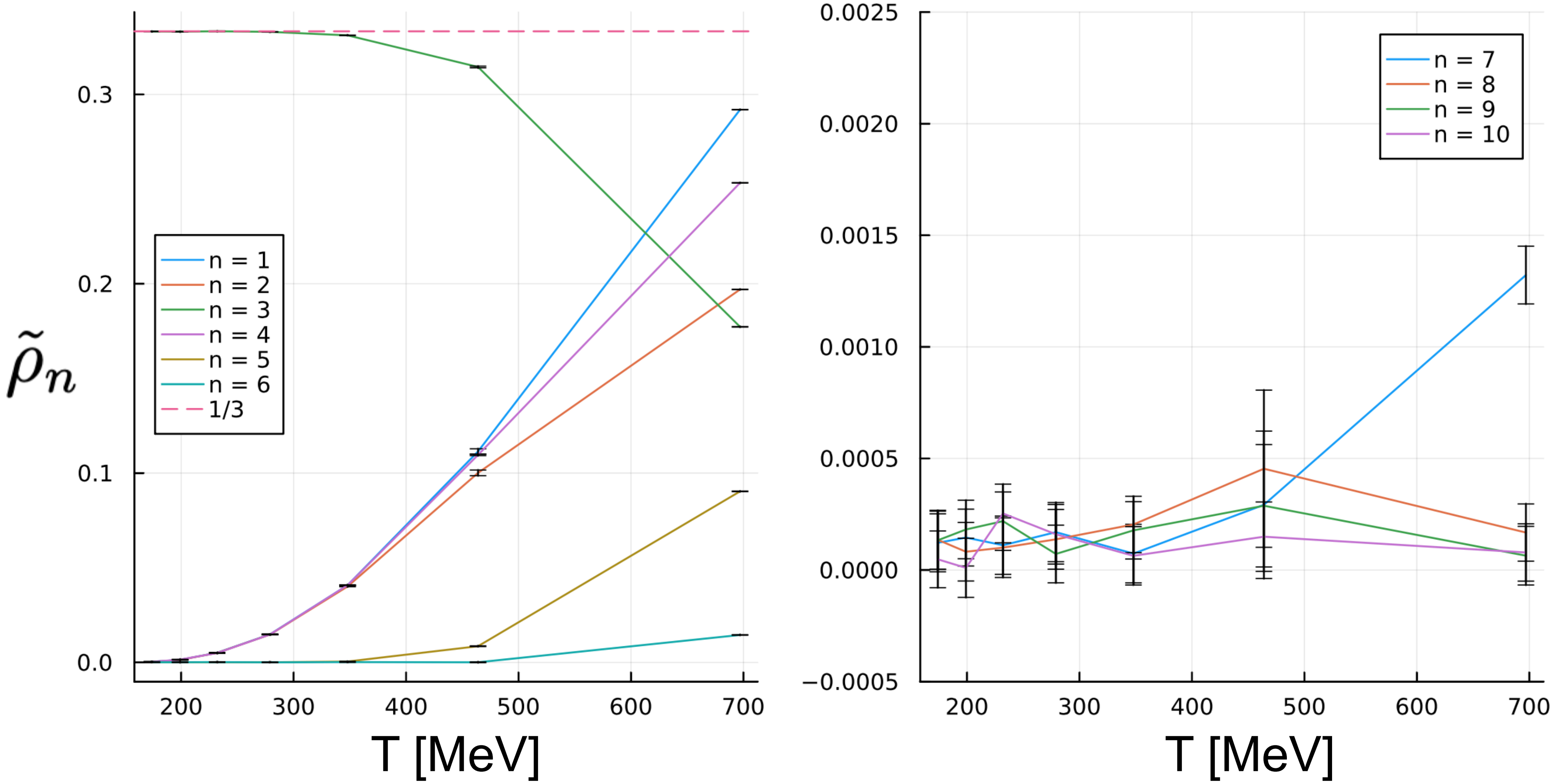
related to baryon

Corrections to Haar-random distribution will be discussed later.

QCD
Thermodynamics

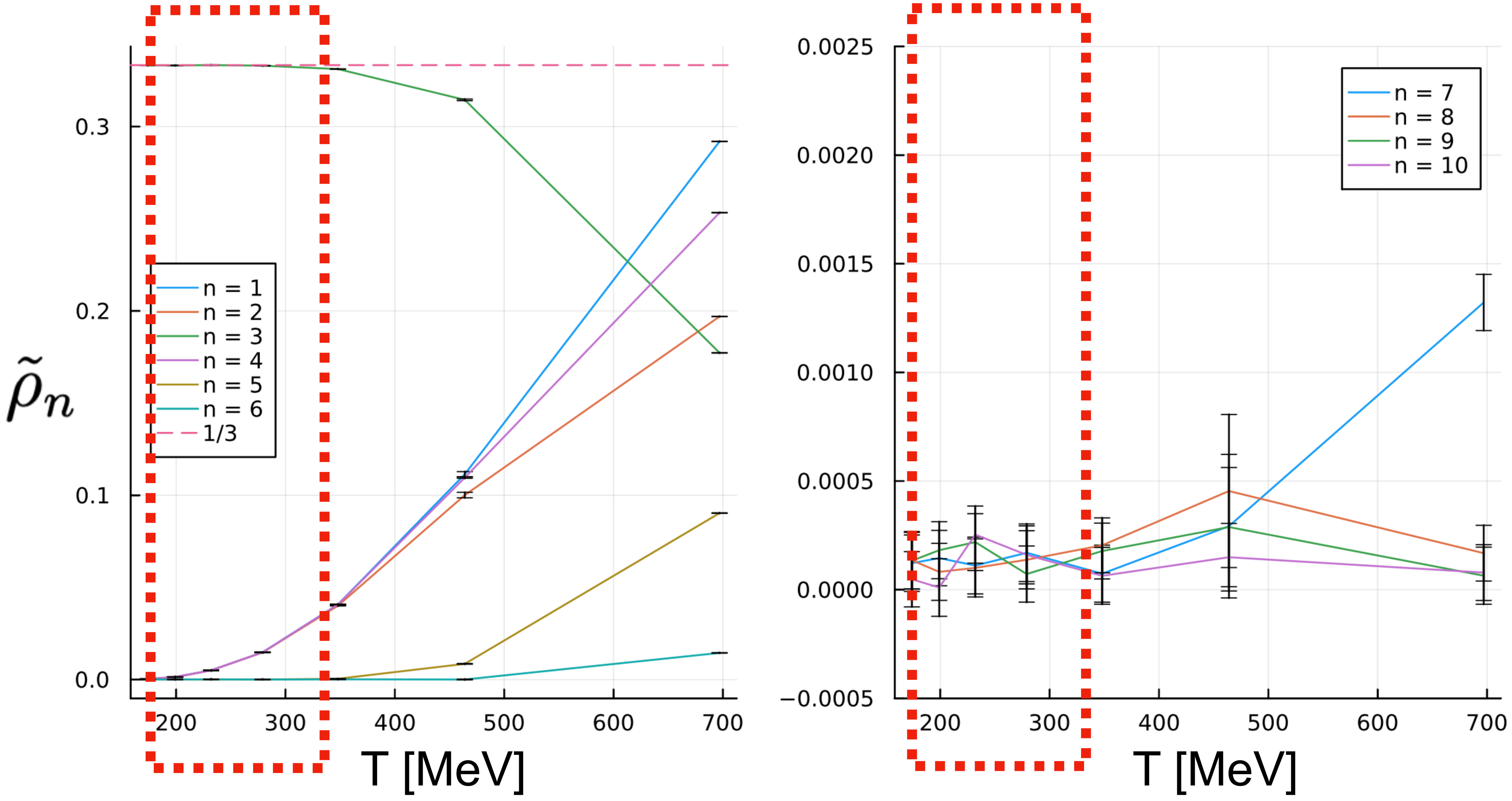
Finite-N counterpart of GWW

Formation of gap → condensation of higher-order coefficients



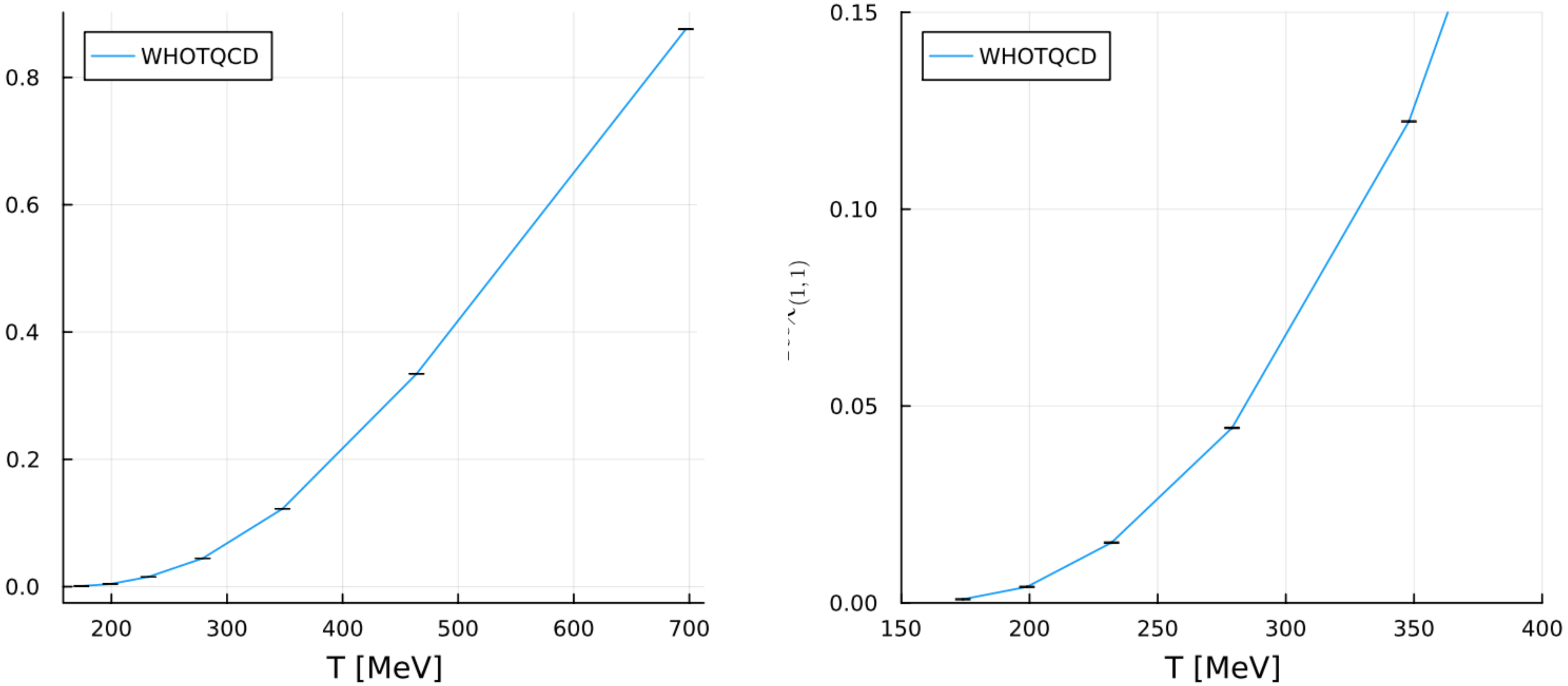
Finite-N counterpart of GWW

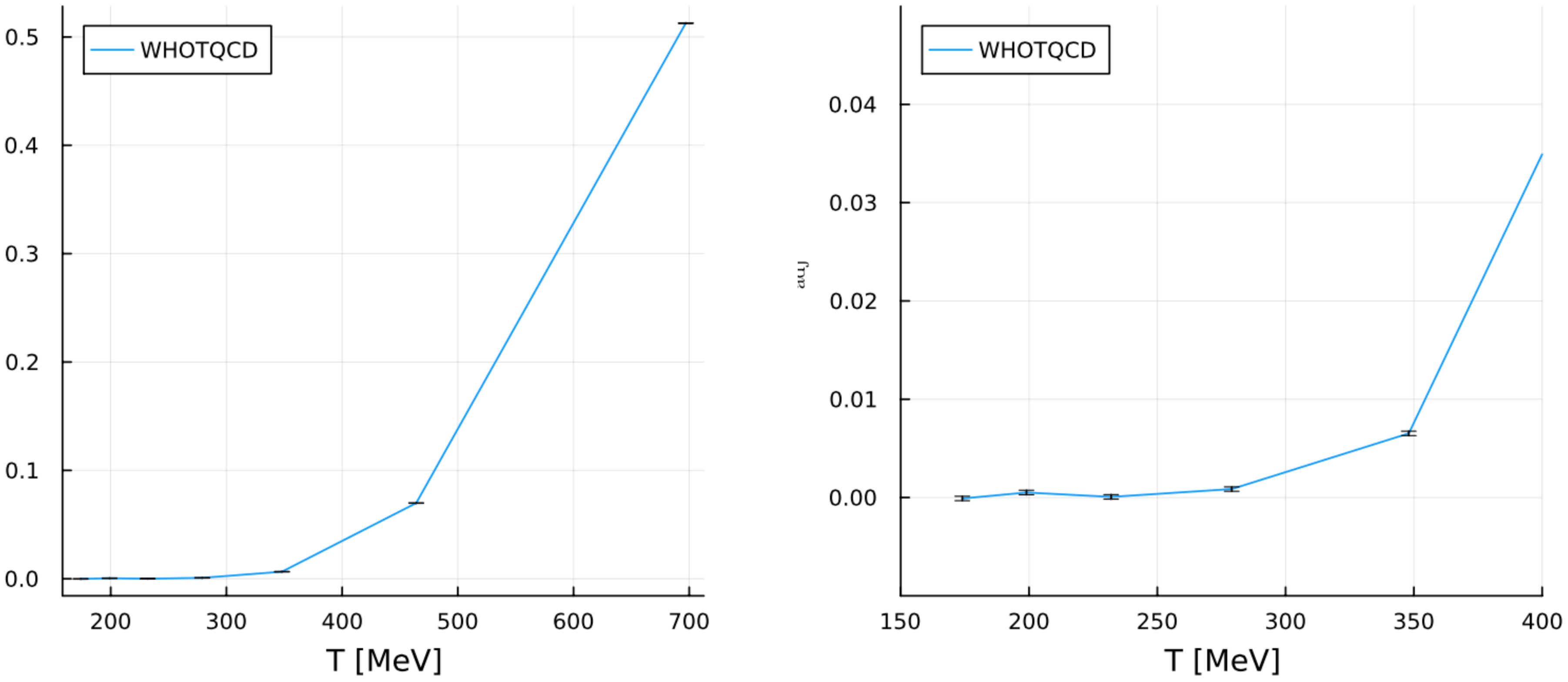
Formation of gap → condensation of higher-order coefficients



Partially deconfined?

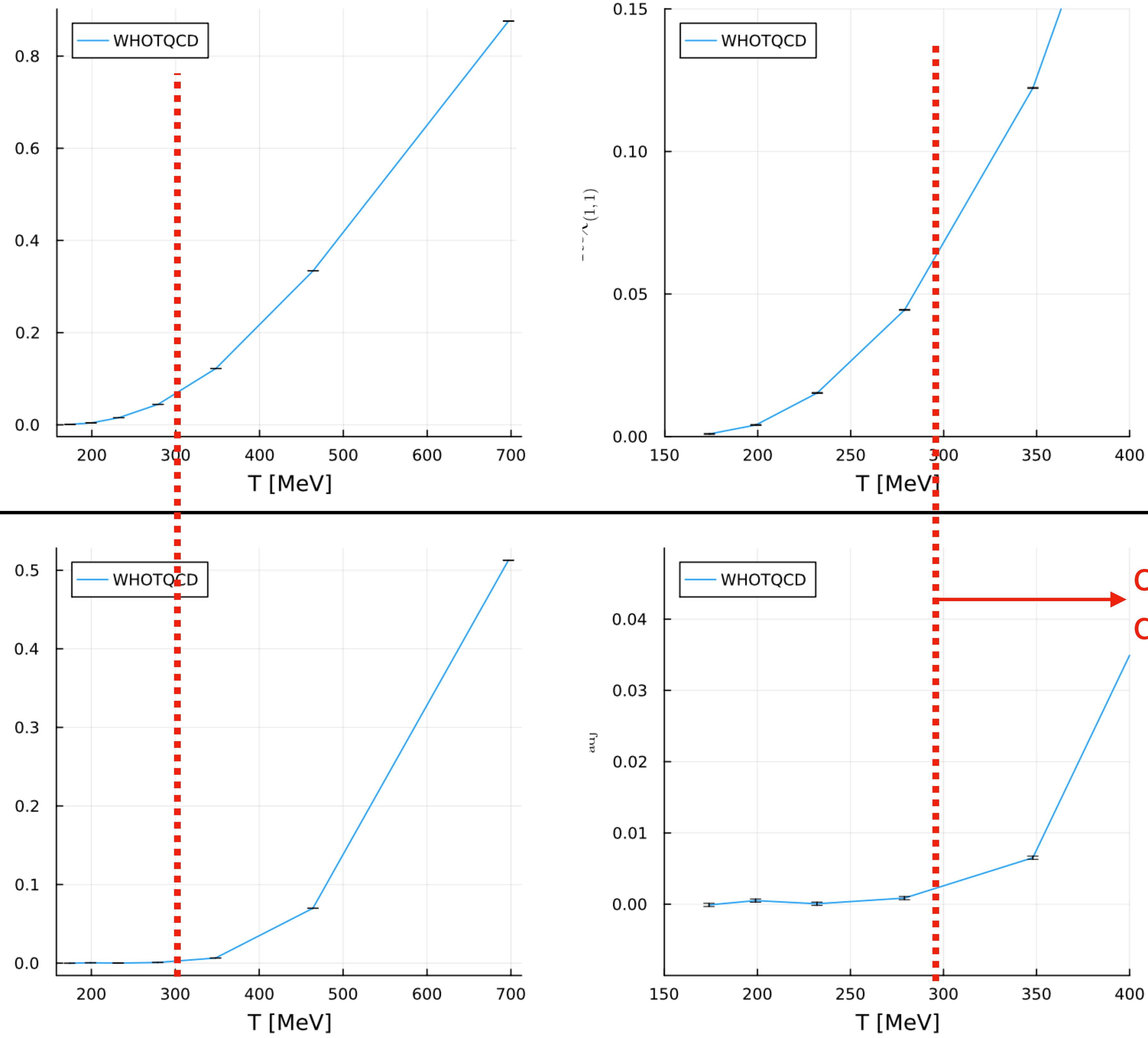
Fundamental representation





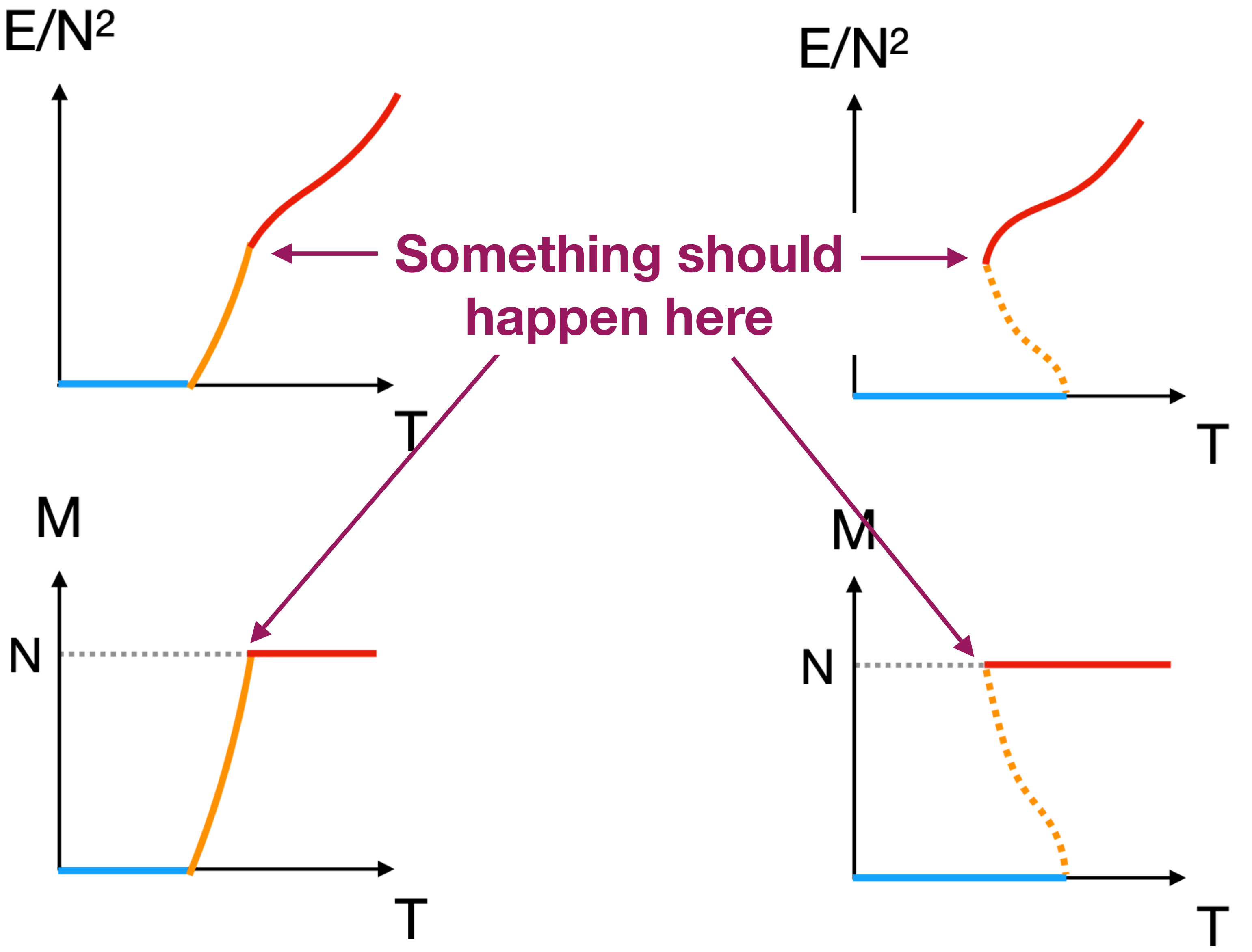
Adjoint representation

Fundamental representation



complete deconfinement?

Adjoint representation



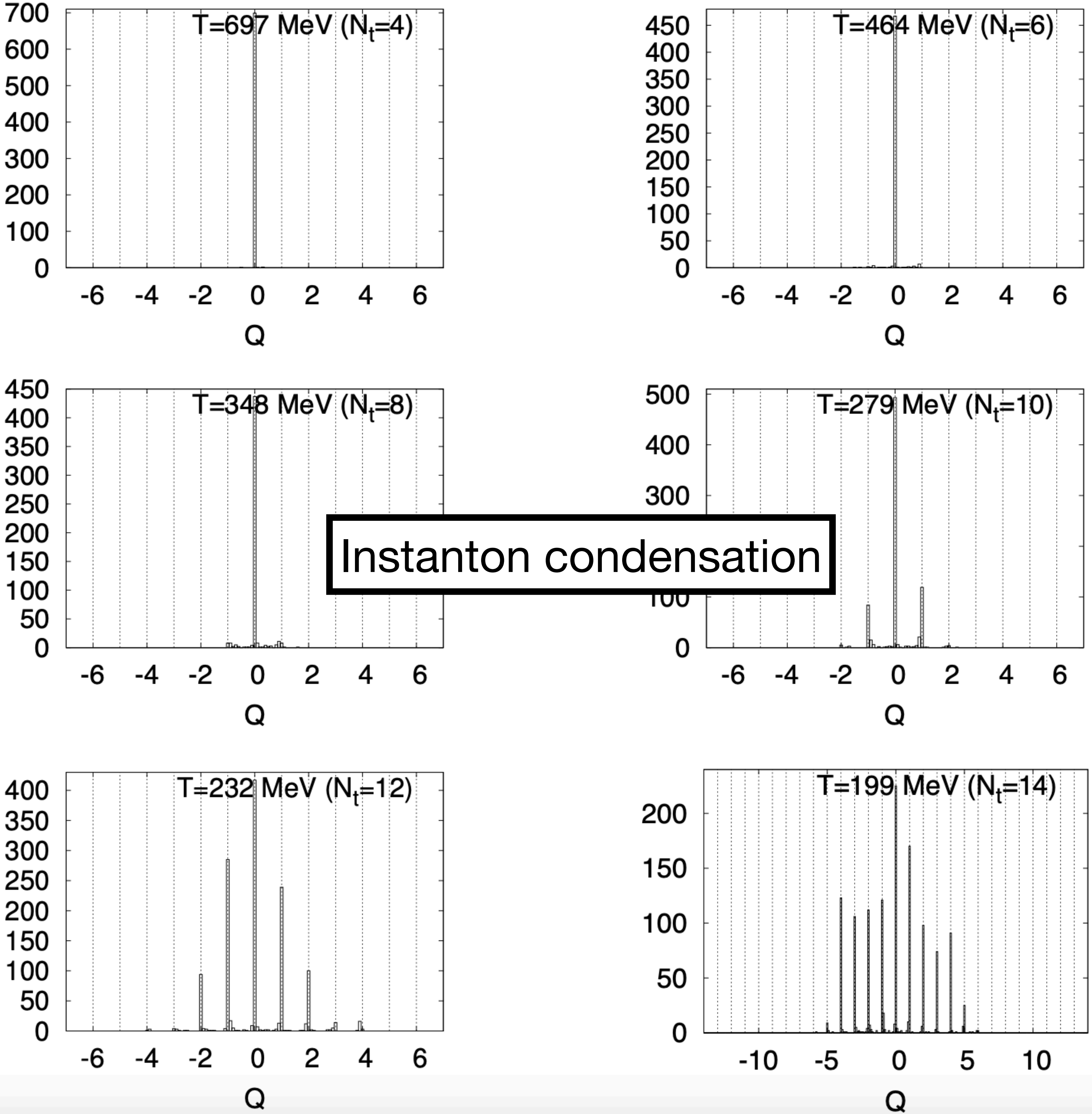
Something should happen here

Light quark mass **Heavy quark mass**

- Chiral symmetry breaking (MH-Robinson, 2019; MH-Knaggs-Holden-O'Bannon, 2021)
- Instanton condensation (MH-Ohata-Shimada-Watanabe, 2023)

Q = topological charge

(WHOT-QCD collaboration, 2016)



Instanton condensation

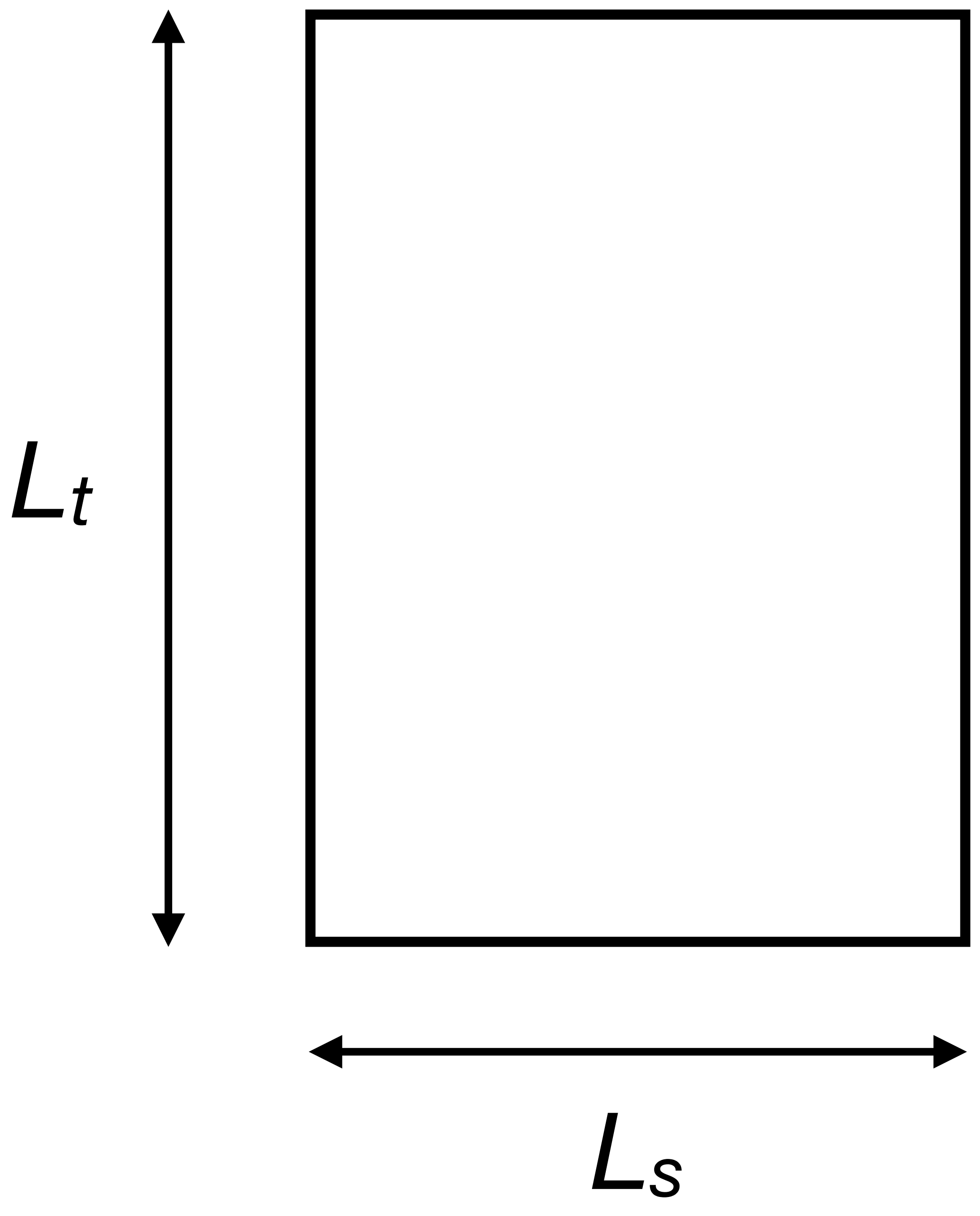
Casimir scaling from random walk

(Bergner, Gautam, MH, 2023)

Previously, 'phenomenological' approach by Brzoska et al 2004,
Arcioni et al 2005, Buividovich et al 2006,2007

Casimir scaling

(Ambjorn-Olsen-Petersen 1984; Del Debbio-Faber-Greensite-Olejnik1995; Bali 2000,...)



$$W \sim \exp(-\sigma_r^2 L_t L_s)$$

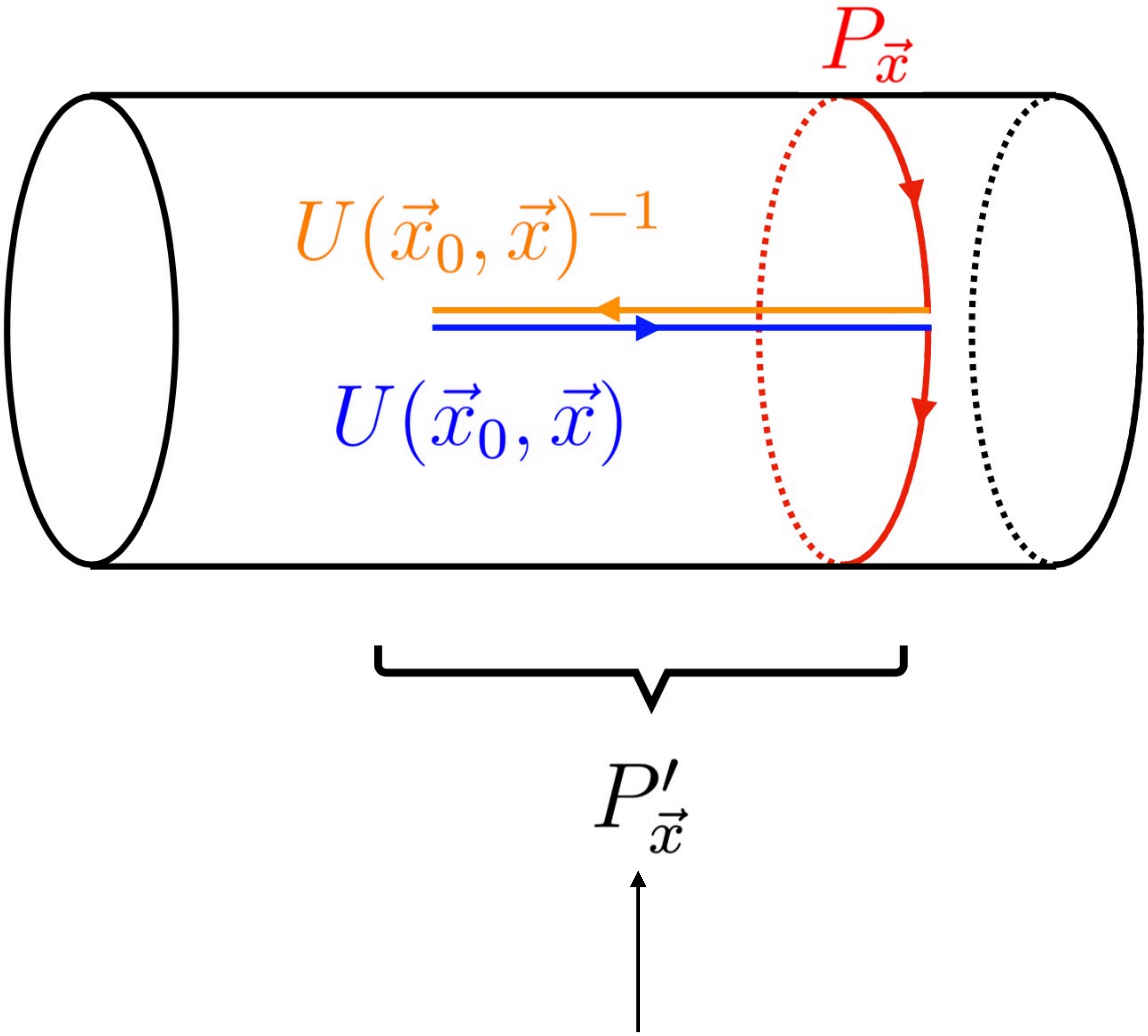
$$\sigma_r^2 = \sigma_0^2 \times C_r$$

C_r : quadratic Casimir

$$\mathrm{Tr}(T_\alpha T_\beta) = 2\delta_{\alpha\beta}$$

$$\sum_\alpha \left(T_\alpha^{(\mathrm{r})}\right)^2 = 4C_\mathrm{r}\mathbf{1}$$

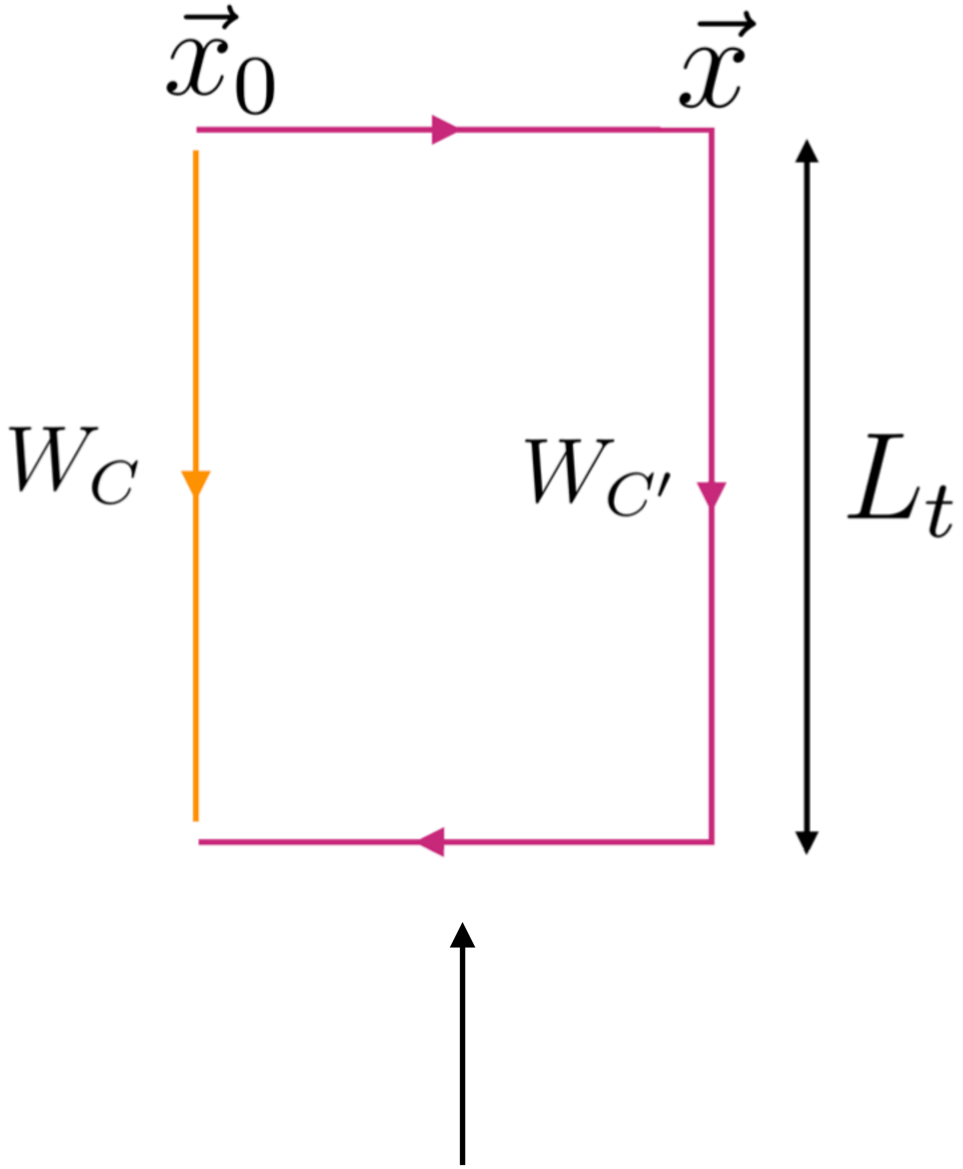
Holds approximately at intermediate distance



This gentleman random walks slowly.

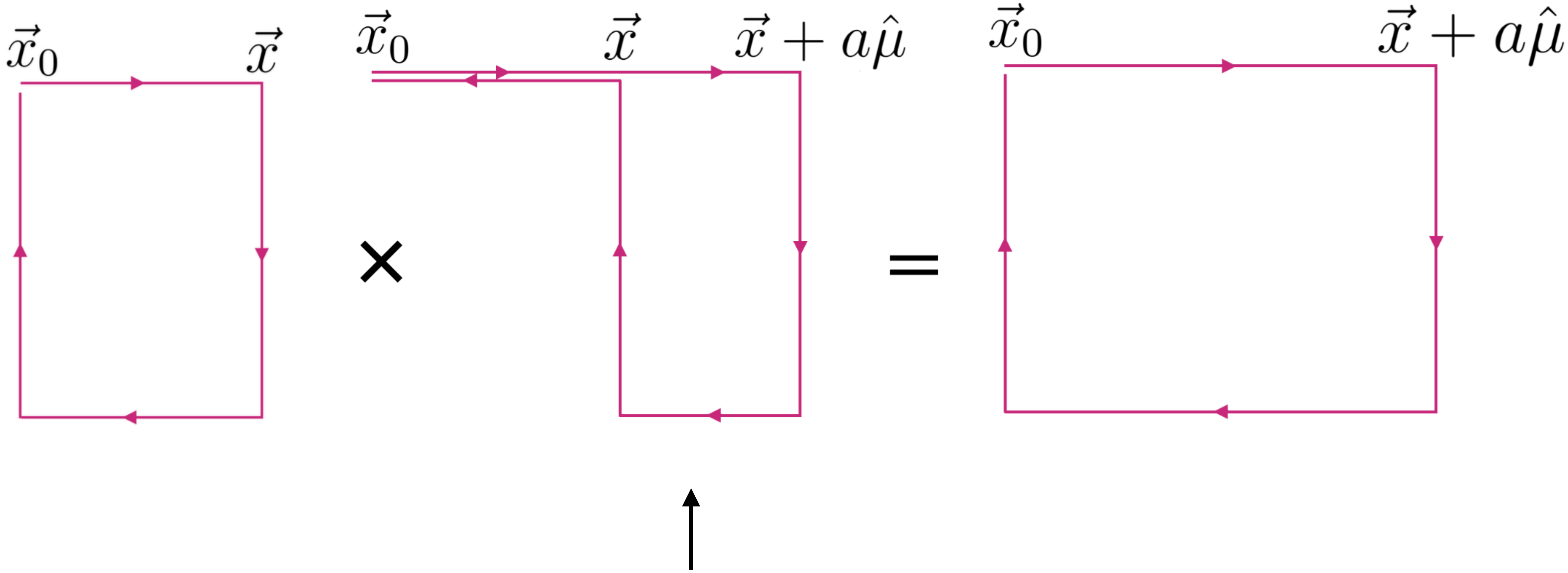
We can check the same random walk for Wilson loop as well.

(Bergner, Gautam, MH, Holden, in progress)



This gentleman random walks slowly.

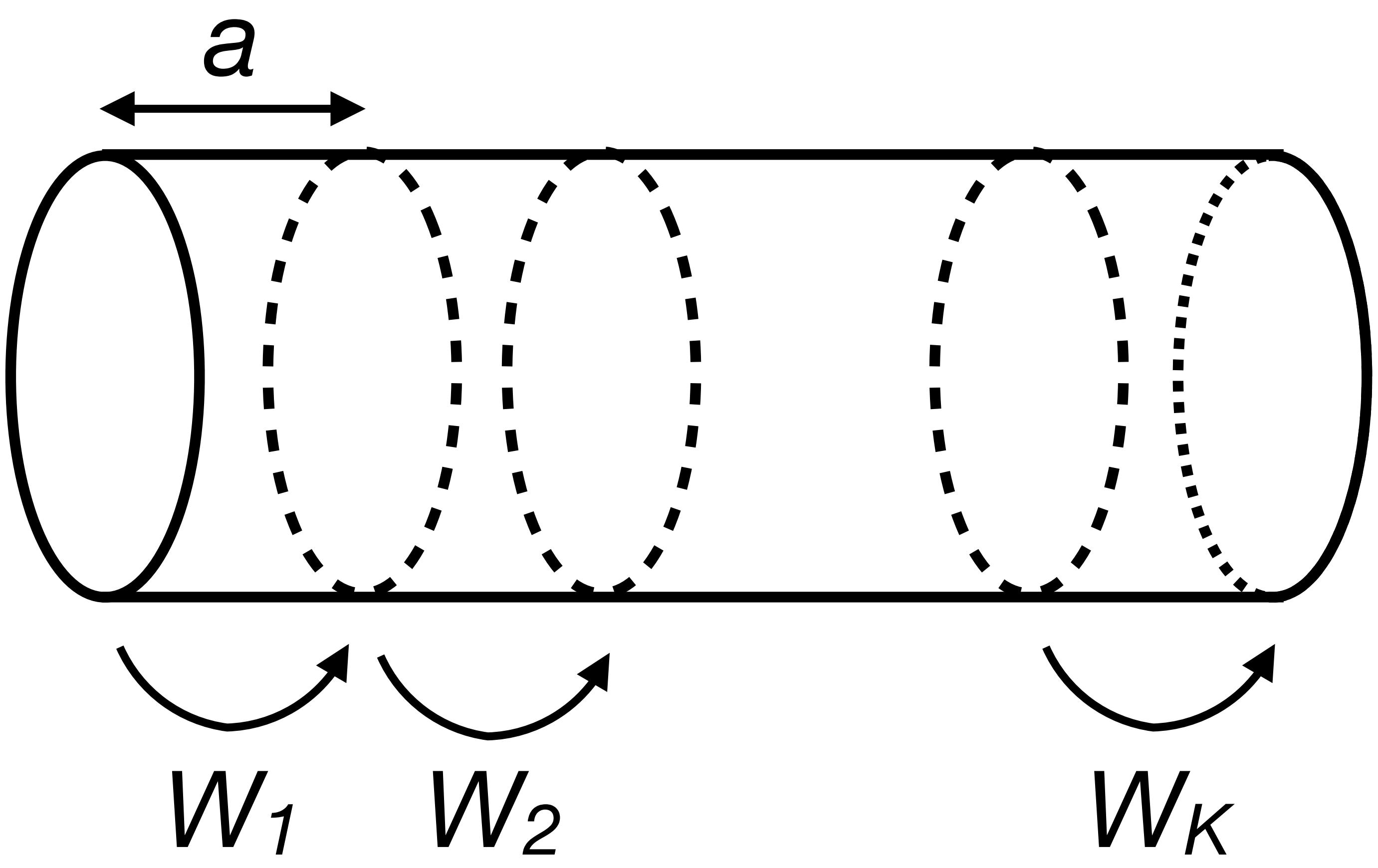
Large Wilson loop = product of many 'thin' Wilson loops



'thin' Wilson loop ~ e^{iX}

X : random matrix

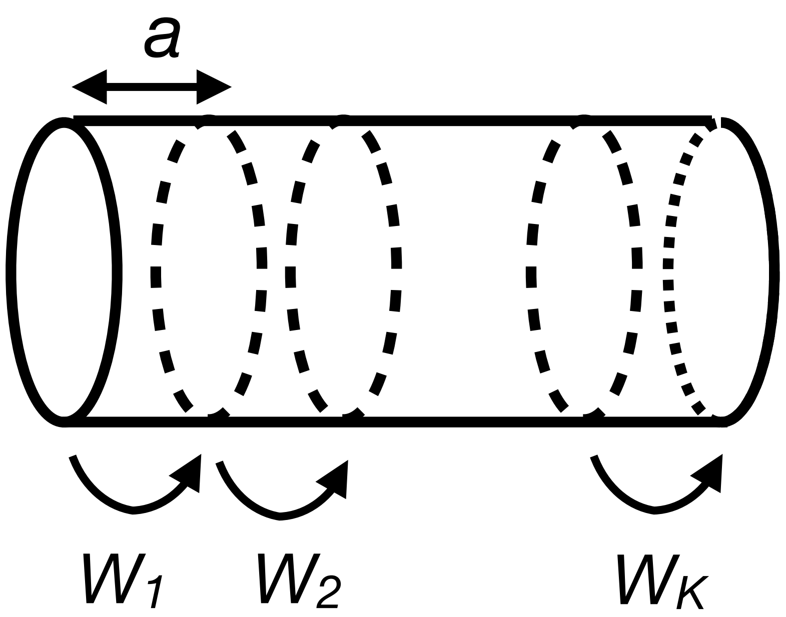
$$\boxed{\left\langle (\chi_{\mathrm{r}}(P_{\vec{x}}))^* \chi_{\mathrm{r}'}(P_{\vec{x}+L\hat{u}}) \right\rangle = d_{\mathrm{r}}^{-1} \delta_{rr'} \left\langle \chi_{\mathrm{r}}(P'^{-1}_{\vec{x}} P'_{\vec{x}+L\hat{u}}) \right\rangle}$$



$$W_j \equiv (P'_{\vec{x}+(j-1)a\hat{u}})^{-1} P'_{\vec{x}+ja\hat{u}}$$

$$\boxed{P'^{-1}_{\vec{x}} P'_{\vec{x}+L\vec{u}} = W_1 W_2 \cdots W_K}$$

Vanilla Random Walk



Suppose all W's are independent
(very crude approximation)

$$W = e^{i\Delta X} \longleftarrow$$ Gaussian random

$$\left\langle W_1^{(\mathrm{r})} W_2^{(\mathrm{r})} \cdots W_k^{(\mathrm{r})} \right\rangle = \left\langle W_1^{(\mathrm{r})} \right\rangle \left\langle W_2^{(\mathrm{r})} \right\rangle \cdots \left\langle W_k^{(\mathrm{r})} \right\rangle = \left(\langle W^{(\mathrm{r})} \rangle \right)^k$$

$$\begin{aligned} \langle W^{(\mathrm{r})} \rangle &= \left\langle \mathbf{1} + i\Delta x^\alpha T_\alpha^{(\mathrm{r})} - \frac{\Delta^2 x^\alpha x^\beta}{2} T_\alpha^{(\mathrm{r})} T_\beta^{(\mathrm{r})} + \cdots \right\rangle \\ &= \mathbf{1} - \frac{\Delta^2}{2} (T_\alpha^{(\mathrm{r})})^2 + \cdots \\ &= \mathbf{1} - 2\Delta^2 C_\mathrm{r} \mathbf{1} + \cdots . \end{aligned}$$

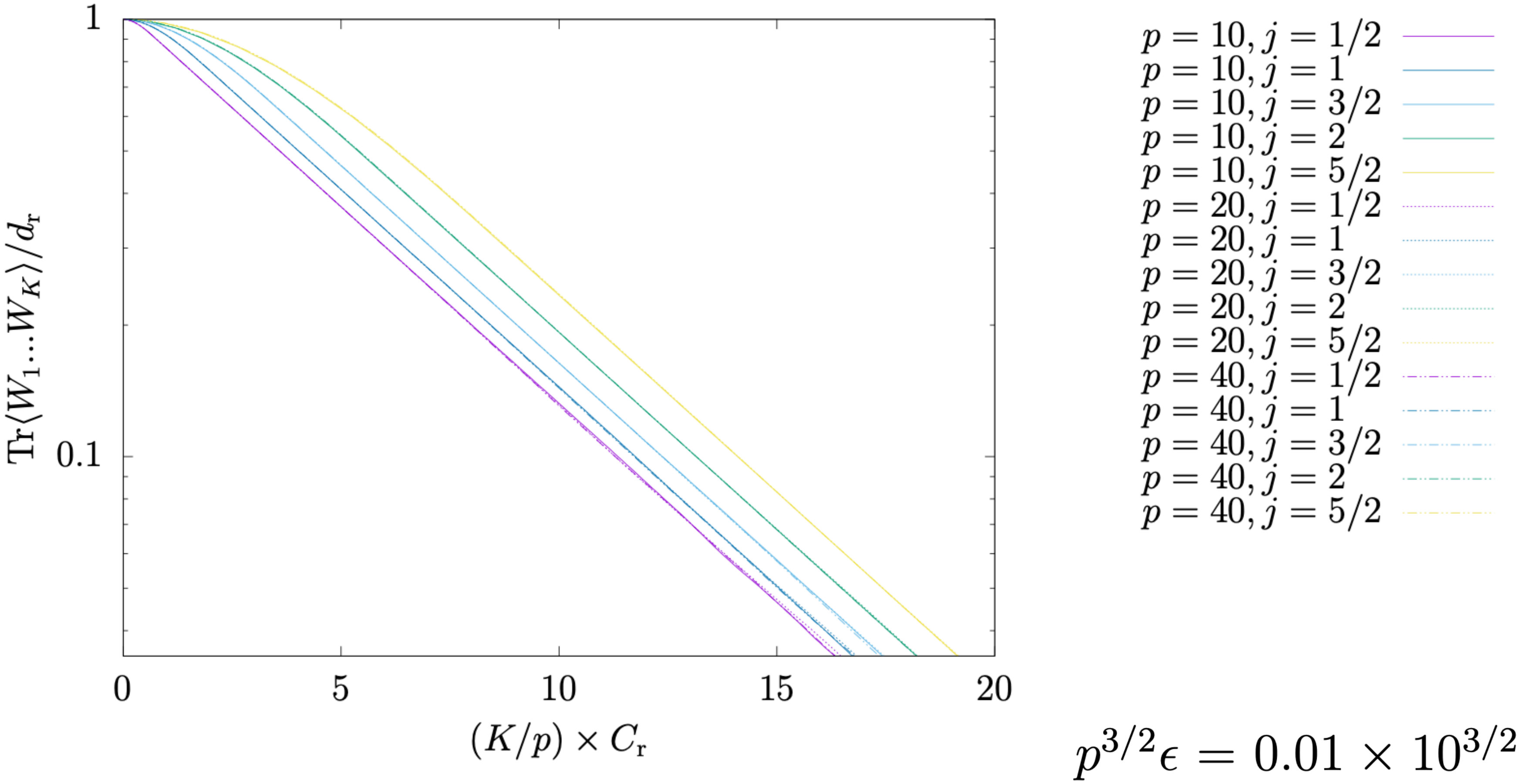
$$\boxed{\left\langle W_1^{(\mathrm{r})} \right\rangle \left\langle W_2^{(\mathrm{r})} \right\rangle \cdots \left\langle W_K^{(\mathrm{r})} \right\rangle \simeq e^{-2\Delta^2 C_\mathrm{r} K} \mathbf{1}}$$

Casimir scaling
up to O(Δ^4)

Random Walk with gradually changing velocity

$$W_i = Z_i \cdots Z_{i+p-1} \qquad\qquad Z_j = e^{i\epsilon X_j}$$

SU(2)

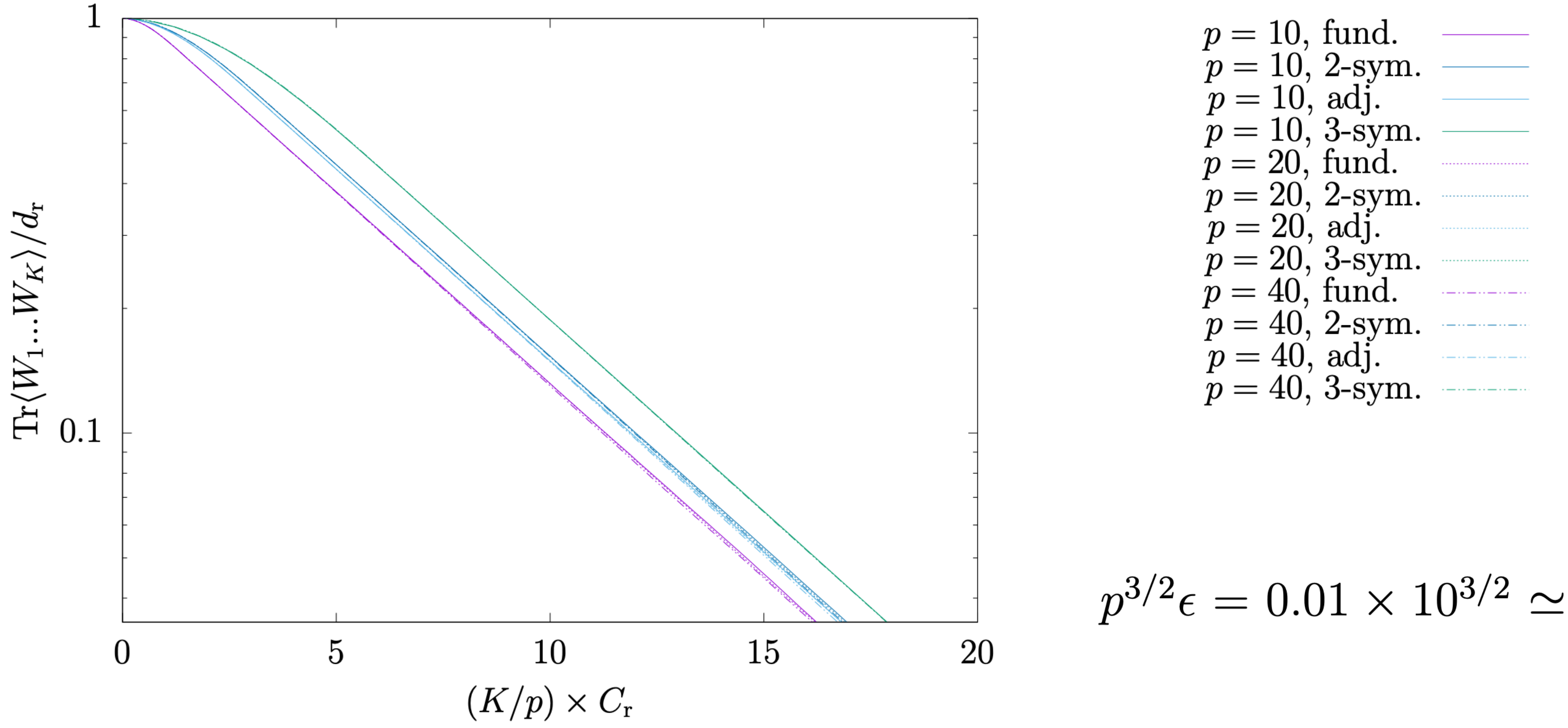


$$p^{3/2}\epsilon = 0.01 \times 10^{3/2} \simeq 0.31622$$

- Well-defined continuum limit with fixed $p^{3/2}\epsilon$
- Casimir scaling if $p^{3/2}\epsilon$ is not too large

$$W_i = Z_i \cdots Z_{i+p-1} \qquad\qquad Z_j = e^{i\epsilon X_j}$$

SU(3)



$$p^{3/2}\epsilon = 0.01 \times 10^{3/2} \simeq 0.31622$$

Correction to Haar randomness →string breaking

$$\langle \chi_{\mathrm{r}}(P_{\vec{x}}) \rangle = \langle \chi_{\mathrm{r}}(P_{\vec{x}+L\hat{u}}) \rangle \sim e^{-\beta m_{\mathrm{r}}}$$

if we assume mass gap

QCD : nonzero m_r for any representation

pure YM : nonzero m_r in center-neutral sector

$$\langle (\chi_{\mathrm{r}}(P_{\vec{x}}))^* \chi_{\mathrm{r}}(P_{\vec{x}+L\hat{u}}) \rangle \sim \underset{\substack{\uparrow \\ \text{connected}}}{e^{-\beta C_{\mathrm{r}} L \sigma_0^2}} + \underset{\substack{\uparrow \\ \text{disconnected}}}{e^{-2\beta m_{\mathrm{r}}}}$$

dominant at long distance

Summary

- Einstein studied large-N limit of non-Abelian gauge theory.
- Color confinement @ large N = Bose-Einstein condensation.
- Polyakov line is 'order parameter' associated with 'gauge symmetry' (or gauge redundancy).
- Gross-Witten-Wadia (GWW) transition = onset of confinement
- GWW = chiral transition, instanton condensation...?
- Polyakov lines random walks slowly.
- Casimir scaling follows from random walk.

Work in progress

(Bergner, MH; Bergner, Gautam, MH, Holden)

- Thermal transitions in 3d SU(2), 3d SU(3), and 4d SU(2) pure Yang-Mills are not first order.
- Partially-deconfined state should be stable, like in QCD.
- Polyakov loops in different representations can deconfine at different temperature.
- Numerical simulations are straightforward.

back up

$$P'^{-1}_{\vec{x}} P'_{\vec{x}+L\hat{u}} = \text{Path ordering}\left[e^{i\sum_\alpha \int_0^L dL' v_\alpha(L') T_\alpha}\right]$$

$$\boxed{R^{(\mathrm{r})}\left(P'^{-1}_{\vec{x}} P'_{\vec{x}+L\hat{u}}\right) = \text{Path ordering}\left[e^{i\sum_\alpha \int_0^L dL' v_\alpha(L') T^{(\mathrm{r})}_\alpha}\right]}$$

$$\chi_{\mathrm{r}}(P_{\vec{x}}) = \mathrm{Tr}_{\mathrm{r}}\left(R^{(\mathrm{r})}(P_{\vec{x}})\right)$$

$$\chi_{\mathrm{r}'}(P_{\vec{x}+L\hat{u}}) = \chi_{\mathrm{r}'}(P'_{\vec{x}+L\hat{u}}) = \chi_{\mathrm{r}'}\left(P'_{\vec{x}} \cdot P'^{-1}_{\vec{x}} P'_{\vec{x}+L\hat{u}}\right)$$

$$\langle(\chi_{\mathrm{r}}(P_{\vec{x}}))^* \cdot \chi_{\mathrm{r}'}(P_{\vec{x}+L\hat{u}})\rangle = \left\langle(\chi_{\mathrm{r}}(P'_{\vec{x}}))^* \cdot \chi_{\mathrm{r}'}\left(P'_{\vec{x}} \cdot P'^{-1}_{\vec{x}} P'_{\vec{x}+L\hat{u}}\right)\right\rangle$$

$$\frac{1}{\mathrm{Vol}(\mathrm{SU}(N))}\int dP (R^{(\mathrm{r})}_{ij}(P))^* R^{(\mathrm{r}')}_{kl}(P) = d_{\mathrm{r}}^{-1}\delta_{\mathrm{rr}'}\delta_{il}\delta_{kj}$$

$$\boxed{\langle(\chi_{\mathrm{r}}(P_{\vec{x}}))^* \chi_{\mathrm{r}'}(P_{\vec{x}+L\hat{u}})\rangle = d_{\mathrm{r}}^{-1}\delta_{rr'}\left\langle\chi_{\mathrm{r}}(P'^{-1}_{\vec{x}} P'_{\vec{x}+L\hat{u}})\right\rangle}$$