When is Axial Gauge Applicable?

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Bruno Scheihing-Hitschfeld, XY, 2205.04477

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Introduction

- **Gauge invariance —> redundant gauge degrees of freedom**
- **Eliminate them in the Faddeev-Popov quantization**

$$\int \mathcal{D}\omega\, e^{-\frac{i}{2\xi}\int d^4x\,\omega^a\omega^a} \int \mathcal{D}A\,\det\Big(\frac{\delta G^a(x)}{\delta\theta^b(y)}\Big)\prod_{x,a}\delta\big(G^a(x)-\omega^a(x)\big)e^{iS_{\mathrm{YM}}[A^a]}$$

Axial gauge condition $\quad G^a_A[A] = n^\mu A^a_\mu(x)$

- **Action with the gauge fixing part**

$$\frac{i}{2}\int d^4k\, A^{\mu a}(-k)\Big(-g_{\mu\nu}(k^2+i\varepsilon)+k_\mu k_\nu-\frac{1}{\xi}n_\mu n_\nu\Big)A^{\nu a}(k)$$

Boundary condition

- **Useful in some perturbative calculations, e.g. TMD** Belitsky, Ji, Yuan, hep-ph/0208038

$$[D_T(k)]^{ab}_{\mu\nu} = \frac{i\delta^{ab}}{k^2+i\varepsilon}\left[-g_{\mu\nu}+\frac{n\cdot k\,(k_\mu n_\nu+n_\mu k_\nu)-n^2 k_\mu k_\nu}{(n\cdot k)^2+i\varepsilon}\right]$$

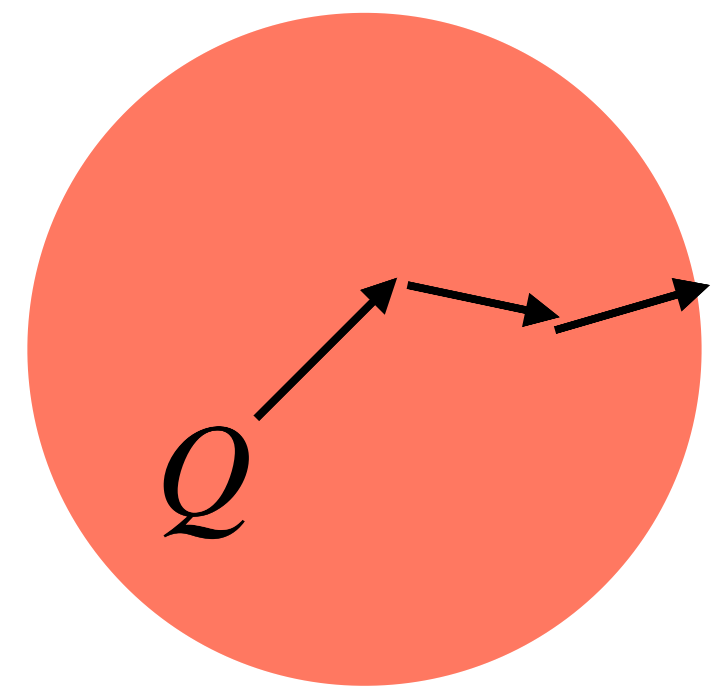
- **But sometimes give inconsistent results**

Example: Heavy Quark and Quarkonium Transport

- **Heavy quark diffusion in quark-gluon plasma**

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\eta_D p_i + \xi_i(t)$$

$$\langle \xi_i(t)\xi_j(t')\rangle = \kappa_{\mathrm{fund}}\delta_{ij}\delta(t-t')$$



- **Heavy quark diffusion coefficient = chromoelectric correlator**

Casalderrey-Solana, Teaney, hep-ph/0605199

$$\kappa_{\mathrm{fund}} = \frac{g^2}{3N_c}\mathrm{Re}\int \mathrm{d}t \langle \mathrm{Tr}_c[U(-\infty,t)E_i(t)U(t,0)E_i(0)U(0,-\infty)]\rangle_{T,Q}$$

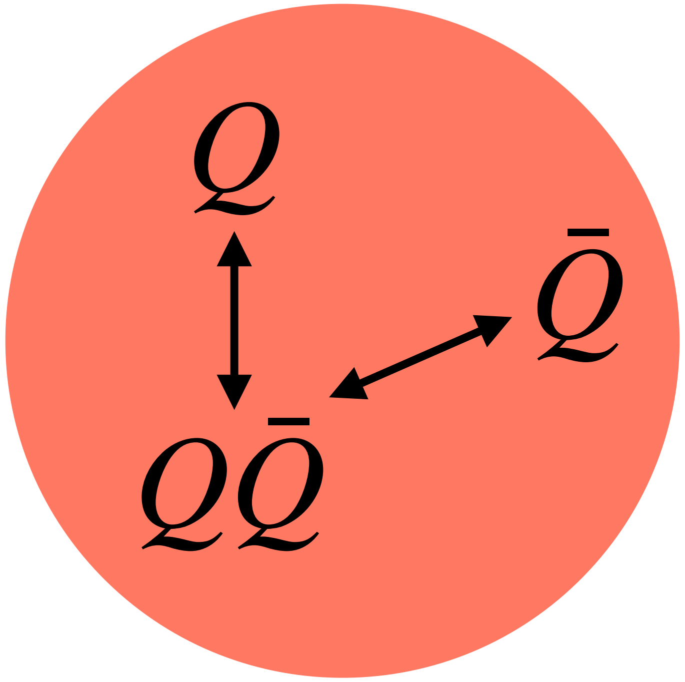
$$\langle O\rangle_{T,Q} \equiv N_c \frac{\mathrm{Tr}[U(-i\beta-\infty,-\infty)Oe^{-\beta H}]}{\mathrm{Tr}[U(-i\beta-\infty,-\infty)e^{-\beta H}]}$$

Heavy quark effect
on the thermal state

Example: Heavy Quark and Quarkonium Transport

- **Quarkonium dynamics in high T quark-gluon plasma**

$$\frac{\mathrm{d}\rho_S(t)}{\mathrm{d}t} = -i\big[H_S + \gamma_{\mathrm{adj}}\Delta h_S,\ \rho_S(t)\big] + \kappa_{\mathrm{adj}}\Big(L_{\alpha i}\rho_S(t)L^\dagger_{\alpha i} - \frac{1}{2}\{L^\dagger_{\alpha i}L_{\alpha i},\ \rho_S(t)\}\Big)$$



Brambilla, Escobedo, Soto, Vairo, 1612.07248

Dissociation/recombination = quantum decoherence   Akamatsu, Rothkopf, 1110.1203 and more

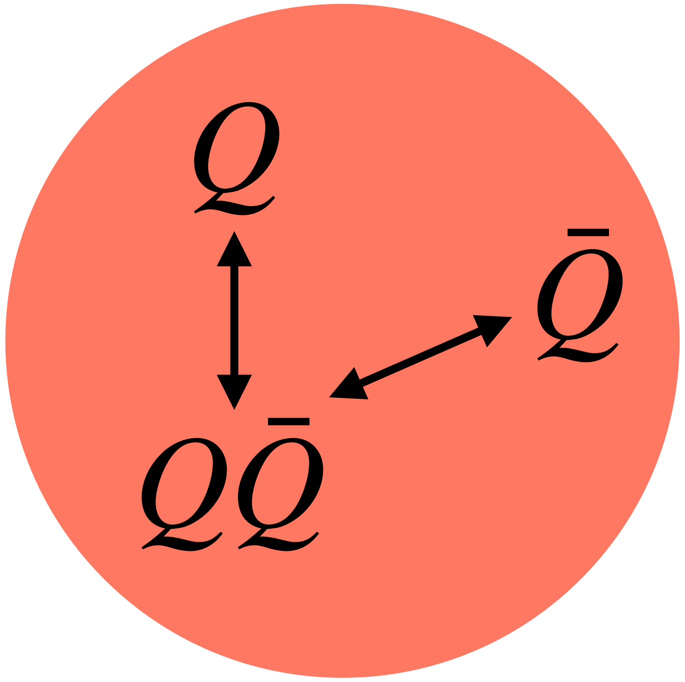
- **Quarkonium transport coefficient = a new chromoelectric correlator**

$$\kappa_{\mathrm{adj}} + i\gamma_{\mathrm{adj}} = \frac{g^2 T_F}{3N_c}\int \mathrm{d}t\,\big\langle \mathcal{T} E^a_i(t) W^{ab}(t,0) E^b_i(0)\big\rangle_T$$

$$\langle O\rangle_T = \mathrm{Tr}(O\rho_T)$$

Example: Heavy Quark and Quarkonium Transport

- **Quarkonium dynamics in low T quark-gluon plasma**



$$\frac{\mathrm{d}n_b(t,\boldsymbol{x})}{\mathrm{d}t} = -\Gamma\, n_b(t,\boldsymbol{x}) + F(t,\boldsymbol{x})$$

$$\Gamma = \int \frac{\mathrm{d}^3 p_{\mathrm{rel}}}{(2\pi)^3} |\langle \psi_b | \boldsymbol{r} | \Psi_{\boldsymbol{p}_{\mathrm{rel}}} \rangle|^2 [g_E^{++}]^{>}(-\Delta E)$$

$$F = \int \frac{\mathrm{d}^3 p_{\mathrm{cm}}}{(2\pi)^3} \frac{\mathrm{d}^3 p_{\mathrm{rel}}}{(2\pi)^3} |\langle \psi_b | \boldsymbol{r} | \Psi_{\boldsymbol{p}_{\mathrm{rel}}} \rangle|^2 [g_E^{--}]^{>}(\Delta E) f_{Q\bar{Q}}(t, \boldsymbol{x}, \boldsymbol{p}_{\mathrm{cm}}, \boldsymbol{x}_{\mathrm{rel}} = 0, \boldsymbol{p}_{\mathrm{rel}})$$

Mehen, XY, 2009.02408

- **Generalized gluon distribution for small-size quarkonium dynamics**

$$[g_{\mathrm{adj}}^{++}]^{>}(\omega) \equiv \frac{g^2 T_F}{3N_c} \int \frac{\mathrm{d}\omega}{2\pi} e^{i\omega t} \langle E_i^a(t) W^{ab}(t,0) E_i^b(0) \rangle_T$$

- **General KMS relation** $[g_{\mathrm{adj}}^{++}]^{>}(\omega) = e^{\omega/T} [g_{\mathrm{adj}}^{--}]^{>}(-\omega)$

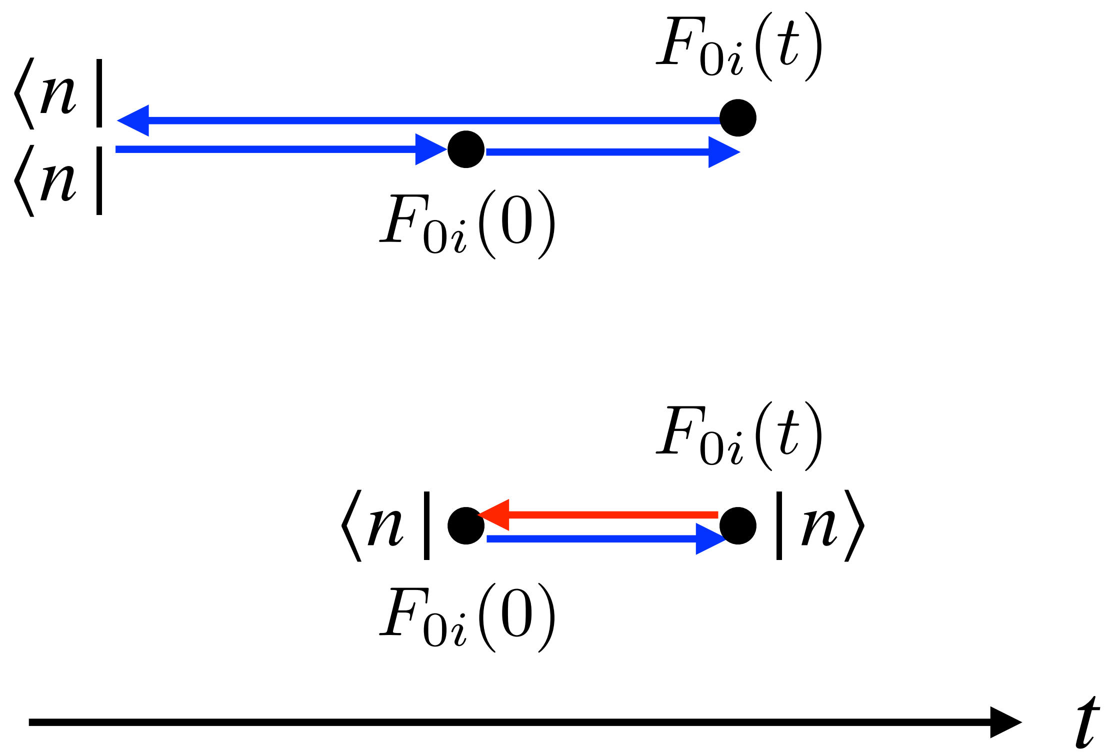
Binder, Mukaida, Scheihing-Hitschfeld, XY, 2107.03945

Two Chromoelectric Field Correlators

- **Different operator orderings**

$g_E^{\mathrm{Q}}(t)$:

$g_E^{\mathrm{Q\bar{Q}}}(t)$:



Color interactions in **both** initial **and** final states since HQ carries color

Color interactions in **either** initial **or** final state since quarkonium colorless

Mehen, XY, 2009.02408

- **NLO perturbative results in Feynman gauge**

$$\left(g_E^{\mathrm{Q\bar{Q}}}(p_0) - g_E^{\mathrm{Q}}(p_0)\right)_{\mathrm{vac}} = \frac{g^4 N_c(N_c^2-1)T_F p_0^3}{(2\pi)^3}\pi^2$$

- **But in temporal axial gauge, they would look identical**

$g^2\langle 0|g^2\mathcal{T}(E_i^a(t)E_i^a(0))|0\rangle$ agrees in axial gauge w/ $g_E^{Q\bar{Q}}$

Origin of Axial Gauge Puzzle

- **Study a more general gauge choice**

$$G^a_M[A] = \frac{1}{\lambda} n^\mu A^a_\mu(x) + \partial^\mu A^a_\mu(x)$$

Feynman gauge: $\lambda \to \infty,\ \xi = 1$ Axial gauge: $\lambda \to 0,$ any ξ

For $\xi = 1$

$$[D_T(k)]^{ab}_{\mu\nu} = \frac{i\delta^{ab}}{k^2 + i\varepsilon}\left[-g_{\mu\nu} + \frac{k_\mu n_\nu\left(n\cdot k - i\lambda k^2\right) + n_\mu k_\nu\left(n\cdot k + i\lambda k^2\right) - n^2 k_\mu k_\nu}{(n\cdot k)^2 + \lambda^2 (k^2)^2 + (1 + 2\lambda^2 k^2) i\varepsilon}\right]$$

- **Issue arises in the order of taking limits**

$$\int \frac{d^4k}{(2\pi)^4} \frac{\eta}{(n\cdot k)^2 + \eta^2} \left[D_T(k)\right]_{\nu\mu} n^\mu N(p,k)$$

$$U_{[(+\infty)n^\mu, 0]} = \mathrm{P}\exp\left(ig\int_0^{+\infty} ds\, e^{-\eta s} n^\mu A_\mu(sn^\mu)\right)$$

If $\lambda \to 0$ is taken first, vanishing result

If $\eta \to 0$ is taken first, non-vanishing result

Axial gauge puzzle associated w/ Wilson lines of infinite extent

Nonperturbative Perspective: Abelian

- **Consider a gauge transformation from Feynman to axial in Abelian case:**

$$G_F(x) = \partial_\mu A^\mu(x) \to \partial_\mu A^\mu(x) - \partial^2\theta(x) = n_\mu A^\mu(x) = G_A(x)$$

In momentum space $$\theta(k) = \frac{1}{k^2}\left(n_\mu A^\mu(k) + ik_\mu A^\mu(k)\right)$$

- **Gauge field transforms as**

$$A^\mu(k) \to M^\mu{}_\nu A^\nu(k)$$

$$M^\mu{}_\nu = g^\mu{}_\nu + \frac{ik^\mu}{k^2}\left(n_\nu + ik_\nu\right)$$

- **Transformation matrix has zero eigenvalue**

$$M^\mu{}_\nu k^\nu = i\frac{n\cdot k}{k^2}k^\mu$$

Zero eigenvalue for $n\cdot k = 0$, Jacobian = 0

Obstruction at infinite "time" $$A(\bar{n}\cdot x) = \int d(n\cdot k)\, e^{i(\bar{n}\cdot x)(n\cdot k)} A(n\cdot k)$$

Nonperturbative Perspective: Non-Abelian

- **In non-Abelian case:**

$$A'_\mu(x) = V(x)A_\mu(x)V^{-1}(x) - \frac{i}{g}\big(\partial_\mu V(x)\big)V^{-1}(x) \qquad V(x) = e^{i\theta^a(x)T_F^a}$$

For $V(x)$ properly defined at $\bar{n}\cdot x \to \infty$ $\qquad \lim_{\bar{n}\cdot x\to\infty} n^\mu \partial_\mu \theta^a(x) = 0$

At $\bar{n}\cdot x \to \infty$ $\qquad n\cdot A'(x) = V(x) n\cdot A(x) V^{-1}(x)$

Thus $\mathrm{Tr}\big[\big(n^\mu A_\mu(\bar{n}\cdot x = \infty)\big)^2\big]$ cannot be changed by gauge transformation

Cannot smoothly go from a gauge with $n\cdot A(\bar{n}\cdot x \to \infty) \neq 0$ to axial gauge $n\cdot A = 0$

Applicability Condition

- Gauge transformation towards axial gauge breaks down at $\bar{n} \cdot x \to \infty$

- Not always a problem in calculations

$$\frac{\int \mathcal{D}A\, e^{iS[A]}\, O[A]}{\int \mathcal{D}A\, e^{iS[A]}}$$

  - If operator $O[A]$ contains no fields at $\bar{n} \cdot x \to \infty$ —> **canceled** ✅

  - If operator $O[A]$ contains fields at $\bar{n} \cdot x \to \infty$ —> **problem!** ❌

- Axial gauge works fine quarkonium chromoelectric correlator because only finite-extended Wilson lines involved

- Axial gauge breaks down for heavy quark chromoelectric correlator because gauge field at infinite time involved

Example: Two Gluon TMDs

- **Weizsacker-Williams TMD (in standard TMD factorization)**

TMD handbook, 2304.03302

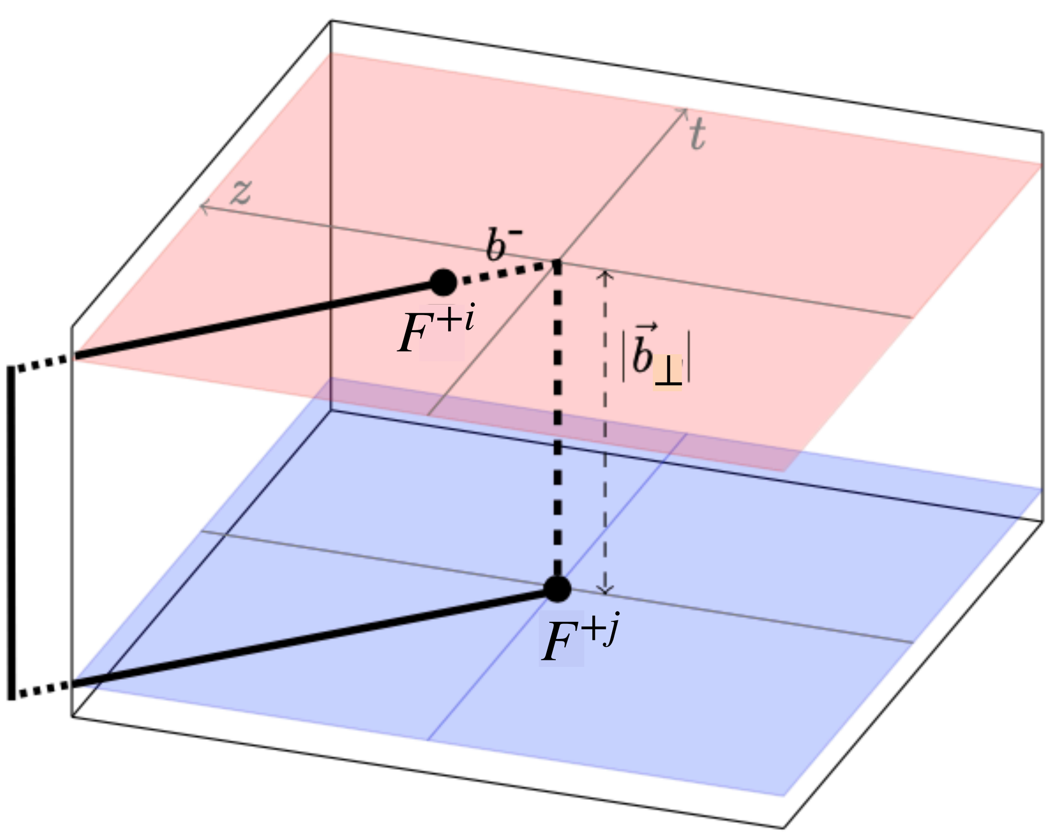
$$T_F \langle p(P,S) \big| F^{a+i}(b^-, b_\perp) \mathcal{W}^{ab}_{[(b^-,b_\perp),(-\infty^-,b_\perp)]} \mathcal{W}^{bc}_{[(-\infty^-,b_\perp),(-\infty^-,0_\perp)]} \mathcal{W}^{cd}_{[(-\infty^-,0_\perp),(0^-,0_\perp)]} F^{d+j}(0^-, 0_\perp) \big| p(P,S) \rangle$$

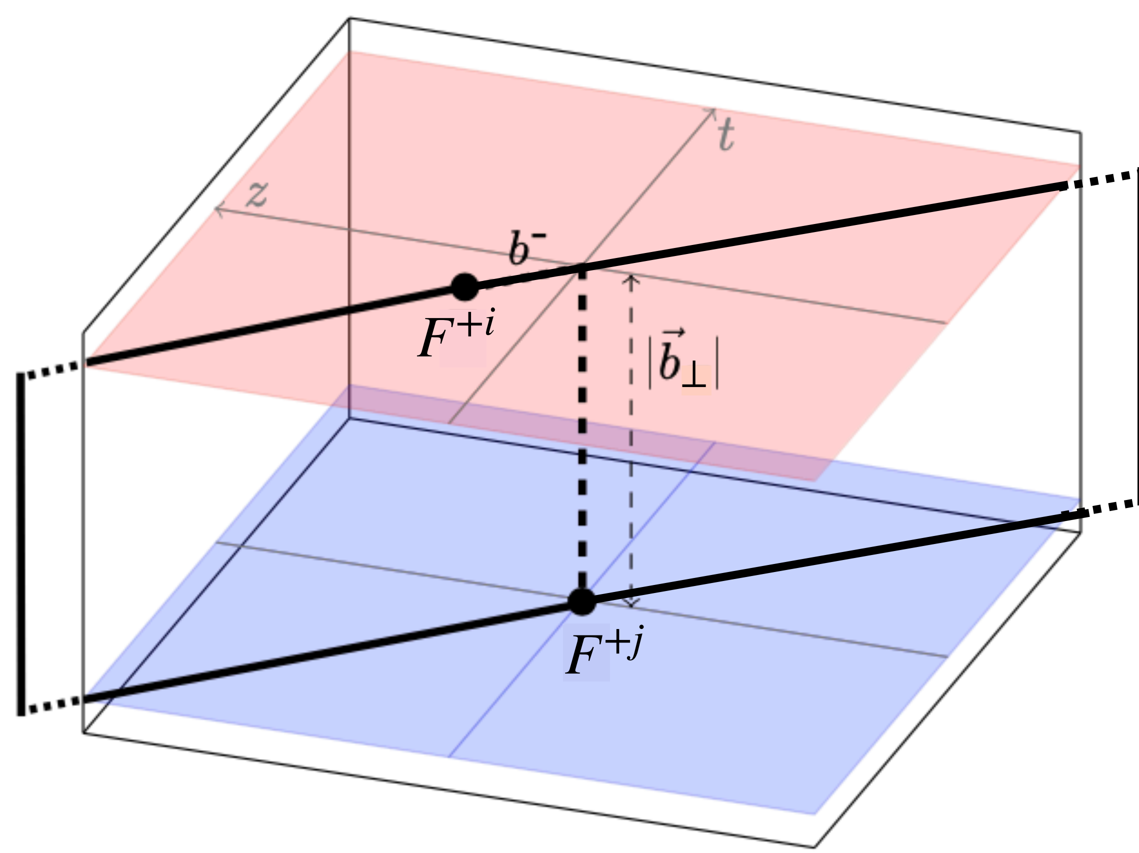
- **Dipole TMD (in small-x physics)**

D. Kharzeev, Y.V. Kovchegov, K. Tuchin, hep-ph/0307037

F. Dominguez, B.-W. Xiao, F. Yuan, 1009.2141

$$\langle p(P,S) \big| \mathrm{Tr}_c \big[U_{[(-\infty^-,0_\perp),(-\infty^-,b_\perp)]} U_{[(-\infty^-,b_\perp),(b^-,b_\perp)]} F^{+i}(b^-, b_\perp) U_{[(b^-,b_\perp),(+\infty^-,b_\perp)]} U_{[(+\infty^-,b_\perp),(+\infty^-,0_\perp)]} U_{[(+\infty^-,0_\perp),(0^-,0_\perp)]} F^{+j}(0^-, 0_\perp) U_{[(0^-,0_\perp),(-\infty^-,0_\perp)]} \big] \big| p(P,S) \rangle$$





Example: Two Gluon TMDs

- In practical calculations, to properly write down Feynman rules for transverse/spatial Wilson lines need a time cutoff, which serves a regulator for the Wilson lines in the time direction, as in the case of Belitsky, Ji, Yuan, hep-ph/0208038 ✅

- If we integrate over transverse positions
  - Standard gluon PDF has finite Wilson line ✅
  - The integrated dipole PDF has infinite Wilson lines ❌

Example: Jet Quenching Parameter

- **Jet quenching parameter**

$$\hat{q} = \lim_{L^- \to +\infty^-} \frac{\sqrt{2}}{L^-} \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} k_\perp^2 \int \mathrm{d}^2 x_\perp e^{-i x_\perp \cdot k_\perp} \frac{\langle \mathrm{Tr}_c W_\square^{\mathcal{R}} \rangle}{d_{\mathcal{R}}}$$

$$W_\square^{\mathcal{R}} = \mathrm{Tr}(W^{\mathcal{R}}(0^-, x_\perp) W^{\mathcal{R}}(L^-, x_\perp) W^{\mathcal{R}}(L^-, 0_\perp) W^{\mathcal{R}}(0^-, 0_\perp))$$

Kovner, Wiedemann, hep-ph/0304151

Liu, Rajagopal, Wiedemann, hep-ph/0605178

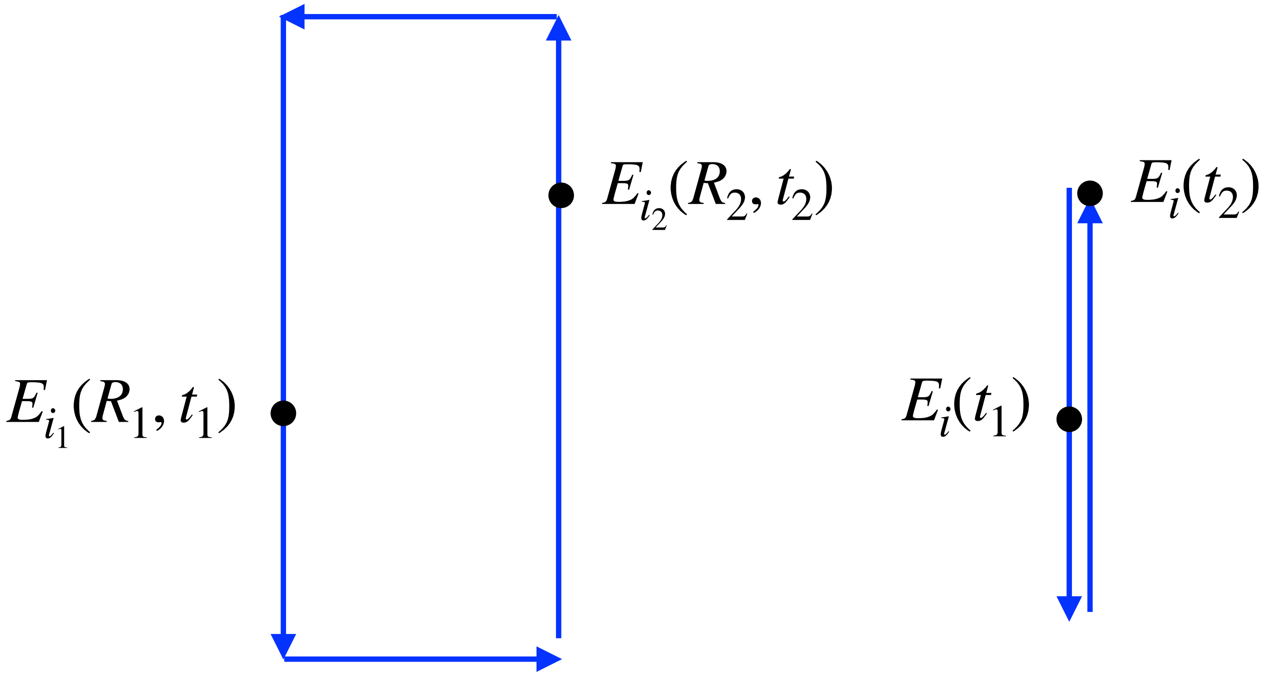
- Calculation of the Wilson loop of finite extent, then take limit ✅

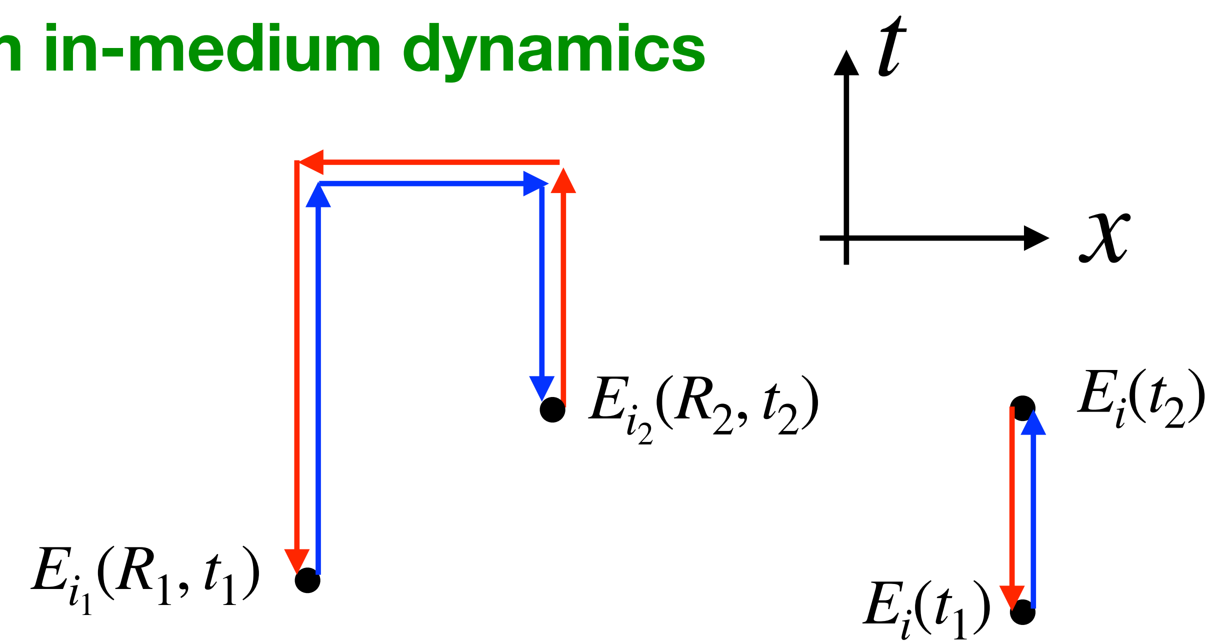
Conclusions

- Axial gauge puzzle:
  - Two chromoelectric field correlators for heavy quark and quarkonium transport look identical in axial gauge, but perturbative calculations in Feynman gauge show they are different
  - Origin of the issue: break down of gauge transformation towards axial gauge at $\bar{n} \cdot x \to \infty$
  - Not always give incorrect results: can be safely applied when gauge fields at infinite "time" are not involved in operators of interest

Backup: Two Chromoelectric Field Correlators

- **Correlators for heavy quark and quarkonium in-medium dynamics**





$$\langle O \rangle_T = \mathrm{Tr}(O\rho_T)$$

Single heavy quark

$$g_E^{\mathrm{Q}}(t) = g^2 \left\langle \mathrm{Tr}_c \left(U_{[-\infty,t]} E_i(t) U_{[t,0]} E_i(0) U_{[0,-\infty]} \right) \right\rangle_T$$

J.Casalderrey-Solana, D.Teaney, hep-ph/0605199

Heavy quark antiquark pair

$$g_E^{\mathrm{Q\bar{Q}}}(t) = g^2 T_F \left\langle \left(E_i^a(t) \mathcal{W}_{[t,0]}^{ab} E_i^b(0) \right) \right\rangle_T$$

Thomas Mehen, XY, 2009.02408

Color interactions in
both initial **and** final states
since HQ carries color

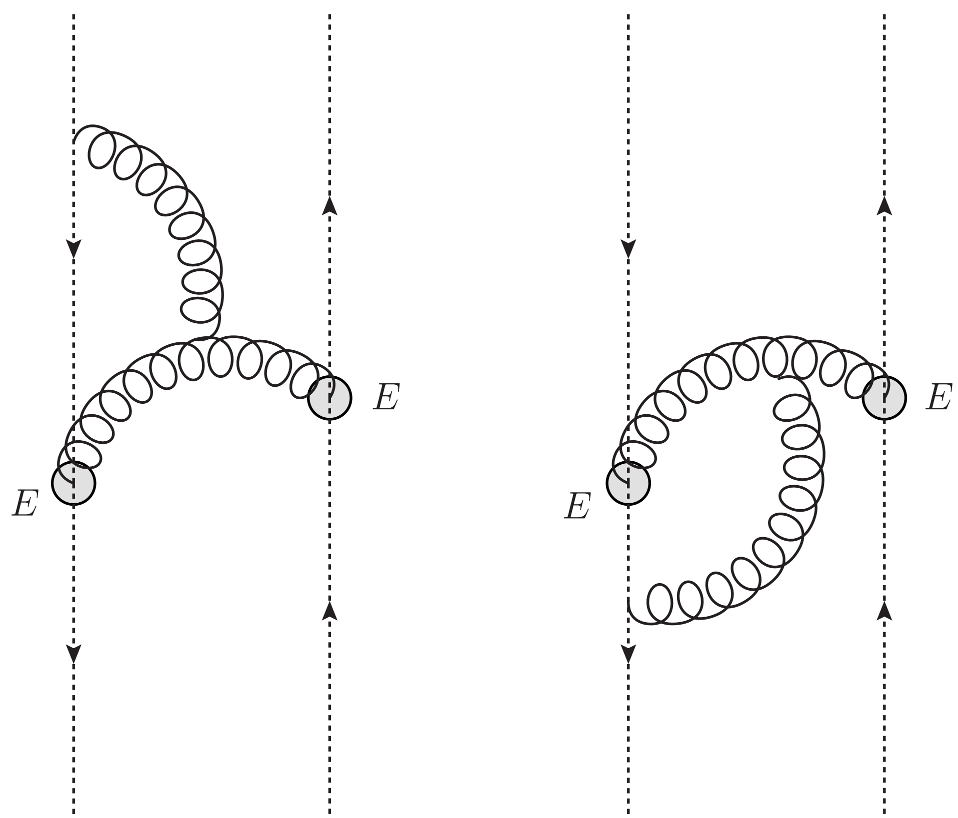
Color interactions in
either initial **or** final state
since quarkonium colorless

Backup: Two Chromoelectric Field Correlators

- **At NLO they have different values**

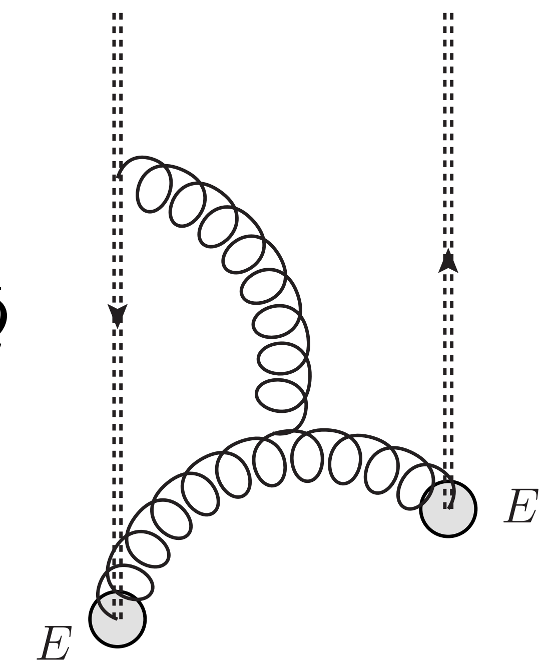
$$\int_{-\infty}^{+\infty} dt\, e^{ip_0 t}\left(g_E^{\mathrm{Q}\bar{\mathrm{Q}}}(t) - g_E^{\mathrm{Q}}(t)\right) = \frac{g^4 N_c (N_c^2 - 1) T_F p_0^3}{(2\pi)^3}\pi^2$$

g_E^Q



Y.Burnier, M.Laine, J.Langelage, L.Mether, 1006.0867

$g_E^{Q\bar{Q}}$



T.Binder, K.Mukaida, B.Scheihing-Hitschfeld, XY, 2107.03945

- **However, they look identical in temporal axial gauge** $A_0 = 0$

$$g^2 T_F \left\langle \mathrm{Tr}_c\left[E_i(t)E_i(0)\right]\right\rangle_T$$

What's wrong?