Causal and stable first-order chiral hydrodynamics

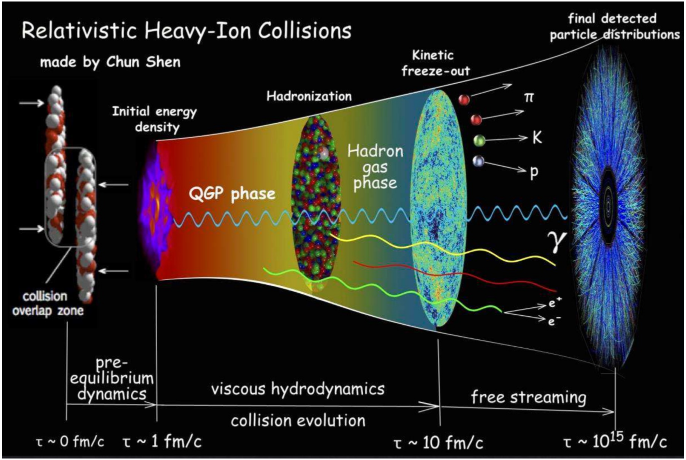
Enrico Speranza

Based on
ES, Bemfica, Disconzi, Noronha, PRD **107**, no.5, 054029 (2023)
Abboud, ES, Noronha, arXiv:2308.02928

QCD Theory Seminar – December 7, 2023

How do quantum effects manifest in relativistic fluids?

Hot and dense QCD matter

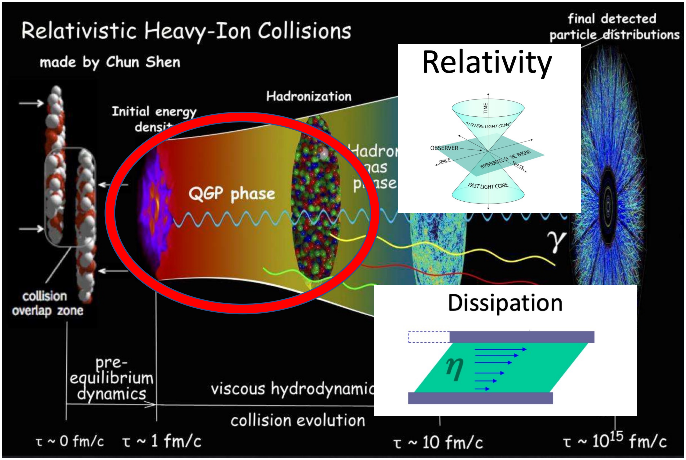


Picture by Chun Shen

- Quark-gluon plasma (QGP) - Universe microseconds after Big Bang
- Relativistic hydrodynamics: Main tool for model-to-data comparison

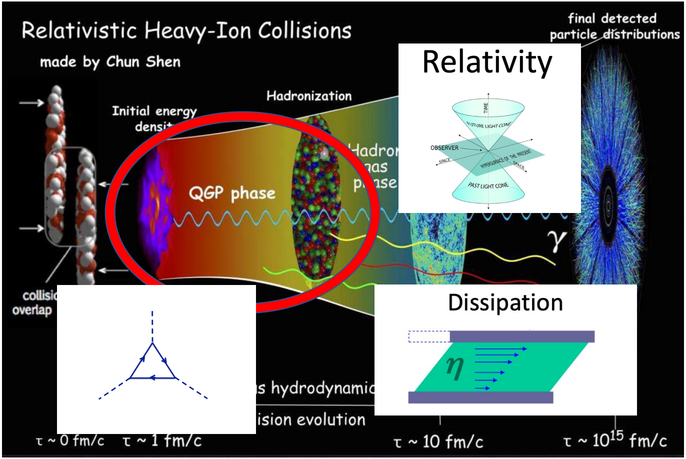
Heinz, Snellings, Ann. Rev. Nucl. Part. Sci. 63, 123 (2013)

How to combine relativity and dissipation?



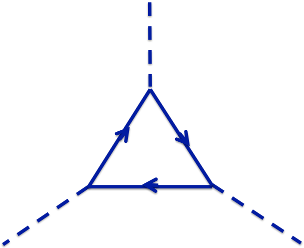
- Heavy-ion collisions define the state fo the art of relativistic dissipative hydrodynamics research

Macroscopic quantum effects in hydrodynamics



- Quantum anomalies can affect behavior of fluids

Chiral (Anomalous) Hydrodynamics



Chiral (Anomalous) Hydrodynamics

Assumptions of hydrodynamics

- **Conservation laws** in long-wavelength, low-energy limit
- **Separation of length scales**: Microscopic λ_{mfp}, Macroscopic L_{hydro}

Knudsen number: $\text{Kn} \equiv \dfrac{\lambda_{\text{mfp}}}{L_{\text{hydro}}} \ll 1$ → **FLUID**

- Expansion in Kn $\Longrightarrow$ Gradient expansion

Nonrelativistic Navier-Stokes equations

- Conservation linear momentum

$$\rho(\partial_t + v_j\partial_j)v_i = \partial_j T_{ji}$$

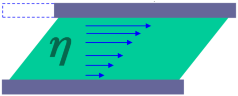
Assumption: Stress tensor is symmetric $T_{ij} = T_{ji}$

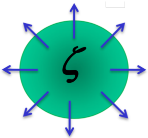
$$T_{ij} = \underbrace{-P\delta_{ij}}_{\text{ideal } \mathcal{O}(\mathrm{Kn}^0)} + \underbrace{\Pi_{ij}}_{\text{dissipation } \mathcal{O}(\mathrm{Kn})+\mathcal{O}(\mathrm{Kn}^2)+\cdots}$$

- Truncation at first order $\mathcal{O}(\mathrm{Kn})$

$$\Pi_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v_k \delta_{ij}\right) + \zeta \partial_k v_k \delta_{ij}$$

η - Shear viscosity , ζ - Bulk viscosity

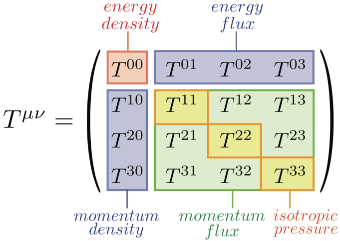




Relativistic hydrodynamics

- Energy-momentum tensor $T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle$, Current $J^{\mu} = \langle \hat{J}^{\mu} \rangle$
- Conservation of energy, momentum and charge

$$\partial_\mu T^{\mu\nu} = 0$$
$$\partial_\mu J^{\mu} = 0$$



picture from Rezzolla, Zanotti, Relativistic Hydrodynamics

$$T^{\mu\nu} = \underbrace{\varepsilon u^\mu u^\nu + P\Delta^{\mu\nu}}_{\text{ideal part}} + \underbrace{\Pi^{\mu\nu}}_{\text{dissipation}}$$

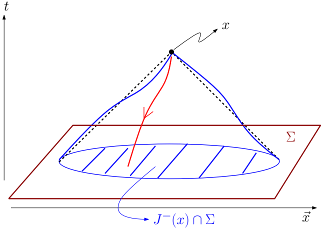
$$J^{\mu} = \underbrace{n u^\mu}_{\text{ideal part}} + \underbrace{\mathcal{J}^{\mu}}_{\text{dissipation}}$$

- Variables ideal fluid: ε, u^μ ($u^\mu u_\mu = -1$), n

Consistency of relativistic hydrodynamics

Any physical theory of hydrodynamics must be:

- **Local well-posed** – Unique solutions for arbitrary initial conditions
- **Causal** – Information cannot propagate at speeds greater than the speed of light
- **Stable** – Systems slightly away from equilibrium will return to it



Bemfica, Disconzi, Noronha, 2018

Landau-Lifhsitz theory is acausal and unstable
$\Longrightarrow$ cannot be used for numerical simulations

How to formulate local well-posed, causal and stable relativistic hydrodynamics?

Initial-value problem

- **Nonlinear** system of second order partial differential equations

$$A^{\mu\nu}_{IJ}(\Psi)\partial_\mu\partial_\nu\Psi_J + B^{\mu\nu}_{IJK}(\Psi)(\partial_\mu\Psi_J)(\partial_\nu\Psi_K) + C^{\mu}_{IJ}(\Psi)\partial_\mu\Psi_J = 0$$

  - $I, J, K = 1, \dots, N$
  - $\Psi_i(x^0, x^i)$ - vector of N unknowns
  - $A^{\mu\nu}_{IJ}(\Psi)$, $B^{\mu\nu}_{IJK}(\Psi)$, $C^{\mu}_{IJ}(\Psi)$ - Coefficients
  - $A^{\mu\nu}_{IJ}(\Psi)$ - Principal part

- Arbitrary initial data on hypersurface $x^0 = 0$

$$\Psi(x^0 = 0, x^i) = f(x^i), \qquad \partial_0\Psi(x^0, x^i)|_{x^0=0} = g(x^i)$$

- Find solution: Need to express $\partial_0\partial_0\Psi$ in terms of initial data $f(x^i)$, $g(x^i)$

$\Longrightarrow A^{00}_{IJ}$ must be invertible

$$\text{Characteristic determinant} = \det A^{00}_{IJ}(\Psi(x^0 = 0, x^i)) \neq 0$$

Local well-posedness

(see e.g., Courant and Hilbert, Choquet-Bruhat, Wald,...)

Initial-value problem is locally well-posed in some function space (e.g., analytic functions) along a hypersurface Σ if

1) **Arbitrary** initial data on $\Sigma \Rightarrow$ Exists unique solution in a neighborhood of Σ
2) "Small changes" of initial data $\Rightarrow$ "Small changes" of solution

- Examples of well-posed theories: Ideal hydrodynamics, BDNK, GR, ...

- Given **arbitrary** initial data, if

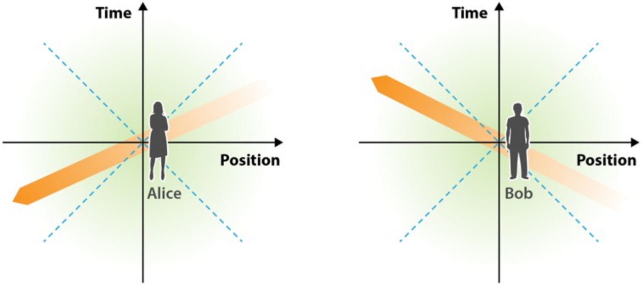
$$\det A^{00}_{IJ}(\Psi) = 0 \quad \Longrightarrow \quad \text{System is ill-posed and acausal}$$

$\Longrightarrow$ No unique solution for **arbitrary** initial data

Causality is necessary for covariant stability

Bemfica, Disconzi, Noronha, PRX 12, 021044 (2022)); Gavassino, PRX 12, 041001 (2022)

$v > c$



When Alice sees a decaying perturbation...

...Bob sees it as an explosion.

acausal $\Longrightarrow$ unstable

Lorentz-covariant linear stability: Stability of ALL homogeneous equilibria

Fourier decomposition: $\delta\varepsilon, \delta n, \delta u^\mu \sim e^{-i(\Omega t - \vec{k}\cdot\vec{x})}$

$$\mathrm{Im}[\Omega(\vec{k})] \leq 0 \quad \forall \vec{k}$$

for all modes

If theory is causal, stability in one Lorentz frame implies stability in any frame!

Hydrodynamic frames

Eckart, Phys. Rev. 58 919 (1940); Landau, Lifshitz, Fluid Mechanics (1987);
Israel, Stewart, Ann. Phys. 118, 341 (1979); Tsumura, Kunihiro, Ohnishi, Phys. Lett. B656, 274 (2007)

- Most general decomposition

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu$$
$$J^\mu = (n + \mathcal{N})u^\mu + \mathcal{J}^\mu$$

- Hydrodynamic variables out of equilibrium can be defined in many ways
- Hydrodynamic frames are specific choices of hydrodynamic variables
- Examples:
  - **Landau** frame: $T^{\mu\nu}u_\nu = -\varepsilon u^\mu$
  - **Eckart** frame: $J^\mu = n u^\mu$
- Causality must hold regardless how you define out of equilibrium variables

How does one formulate causal and stable relativistic hydrodynamics?

Israel-Stewart theory

Israel, Stewart, Ann. Phys. 118 341?372 (1979)

- Energy-momentum tensor out of equilibrium

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$

- Π, $\pi^{\mu\nu}$ are promoted to be dynamical variables
  $\Longrightarrow$ Relaxation-type equations

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta \partial_\mu u^\mu + \dots$$
$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} + \dots$$

- Causal and stable
  Hiscock, Lindblom, Annals of Physics (1983); Bemfica, Disconzi, Hoang, Noronha, Radosz, PRL (2021)

- Different formulations: BRSSS, DNMR, ...
  Baier, Romatschke, Son, Starinets, Stephanov JHEP (2008); Denicol, Niemi, Molnar, Rischke, PRD (2012)

In heavy-ion simulations one solves Israel-Stewart hydrodynamics

State of the art of conventional hydrodynamics

- Most general decomposition out of equilibrium.

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu$$

- **Generalized Israel-Stewart theory**: $\mathcal{A}$, Π, $\mathcal{Q}^\mu$, $\pi^{\mu\nu}$ are promoted to be dynamical variables, e.g.,
  Noronha, Spalinski, ES, PRL **128**, 252302 (2022)

$$\tau_{\mathcal{A}} u_\alpha \partial^\alpha \mathcal{A} + \mathcal{A} = \varphi T u_\alpha \partial^\alpha \left(\frac{1}{T}\right)$$

- **First-order (BDNK) theory**: write $\mathcal{A}$, Π, $\mathcal{Q}^\mu$, $\pi^{\mu\nu}$ with the most general expression using first-order derivatives of u^μ and T, e.g.
  Bemfica, Disconzi, Noronha, PRD **98**, 104064 (2018); PRD **100**, 104020 (2019); PRX **12**, 021044 (2022); Kovtun JHEP **10**, 034 (2019); Hoult, Kovtun, JHEP **06**, 067 (2020)

$$\mathcal{A} = a_1 u_\alpha \partial^\alpha \left(\frac{1}{T}\right) + a_2 \partial_\alpha u^\alpha$$

Navier-Stokes – Relativistic diffusion equation

- Diffusion of charge density n on stationary background velocity u^μ

$$J^\mu = nu^\mu \underbrace{-d\Delta^{\mu\alpha}\partial_\alpha n}_{\mathcal{O}(\mathsf{Kn})}$$

$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, $d =$ diffusion coefficient

- Conservation law

$$\partial_\mu J^\mu = u_\alpha \partial^\alpha n + n\partial_\alpha u^\alpha - d\Delta^{\mu\alpha}\partial_\mu\partial_\alpha n = 0$$

- **ACAUSAL!**
- Stability in the "rest" frame: $\delta n \sim e^{-i(\Omega t - \vec{k}\cdot\vec{x})}$

$$\Omega = -id\vec{k}^2$$

Looks stable... **HOWEVER** for a general u^μ it is **UNSTABLE!**

NOT suitable for numerical simulations!

BDNK approach for relativistic diffusion

Bemfica, Disconzi, Noronha, PRD **98**, 104064 (2018); PRD **100**, 104020 (2019); PRX **12**, 021044 (2022); Kovtun JHEP **10**, 034 (2019); Hoult, Kovtun, JHEP **06**, 067 (2020)

Effective field theory approach: Include all possible terms $\mathcal{O}(\mathrm{Kn})$

$$J^\mu = (n + \underbrace{\tau u_\alpha \partial^\alpha n) u^\mu - d\Delta^{\mu\nu}\partial_\nu n}_{\mathcal{O}(\mathrm{Kn})}$$

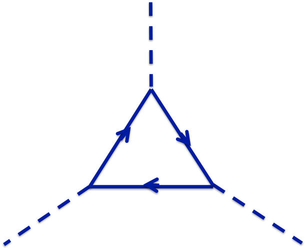
Additional term $\tau u_\alpha \partial^\alpha n \sim \mathcal{O}(\mathrm{Kn}) \implies$ Wave-like equation

$\partial_\mu J^\mu = 0$ causal and stable if, and only if:

$$\tau > 0 \quad \text{and} \quad \underbrace{\sqrt{\frac{d}{\tau}} \leq 1}_{\text{wave propagation speed}}$$

Suitable for numerical simulations!

Chiral (Anomalous) Hydrodynamics



Quantum Anomaly

- Symmetry of Lagrangian $\Longrightarrow$ Noether's theorem $\Longrightarrow$ Conservation law

$$\partial_\mu J^\mu = 0$$

Associated charge Q constant in time

- Is this ensured when we consider quantum corrections?
Not necessarily!

$$\partial_\mu J^\mu \neq 0$$

> **Quantum anomaly**: Classical symmetry destroyed by quantum corrections

Chiral anomaly

- **Massless** QED/QCD Lagrangian is invariant under axial $U_A(1)$

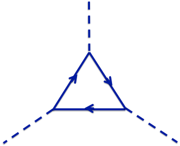
$$\psi = \begin{pmatrix} \text{up} \\ \text{down} \\ \text{strange} \end{pmatrix} \quad \rightarrow \psi' = e^{-i\theta\gamma_5}\psi$$

- Noether's theorem $\Longrightarrow$ Conservation axial vector $J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$

$$\partial_\mu J_A^\mu = 0$$

- Quantum corrections

$$\partial_\mu J_A^\mu = C E_\mu B^\mu$$

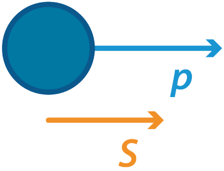


E^μ - electric field, B^μ - magnetic field, C - Anomaly coefficient

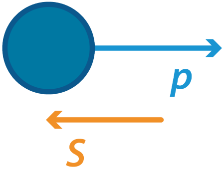
$$\frac{d(\text{Right-handed - Left-handed})}{dt} \neq 0$$

Chirality

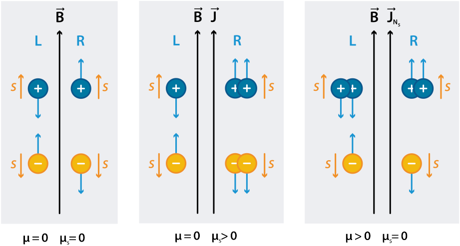
Right-handed:



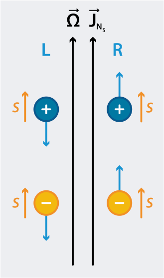
Left-handed:



- Chiral magnetic effect: $\vec{J}_V = \frac{1}{2\pi^2}\mu_A \vec{B}$ (μ_A - Axial chemical potential)



- Chiral vortical effect



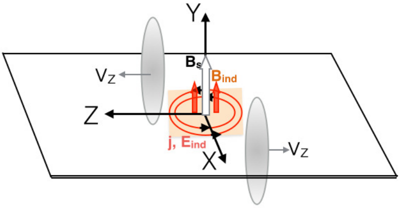
$$\vec{J}_V = \frac{1}{\pi^2}\mu_5 \mu \vec{\omega}$$

$$\vec{J}_A = \left(\frac{\mu_V^2 + \mu_A^2}{2\pi^2} + \frac{T^2}{6}\right)\vec{\omega}$$

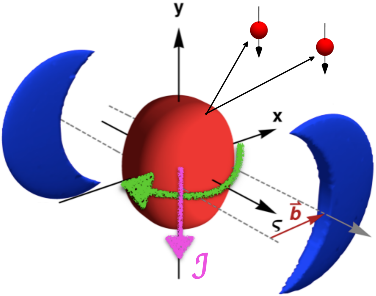
$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{v}$$

Kharzeev, Liao, Voloshin, Wang, Prog. Part. Nucl. Phys. 88, 1-28 (2016)

Polarization in magnetic and vortical field



Roy et al PRC 96 (2017) 054909



Florkowski et al Prog. Part. Nucl. Phys. 108, 103709 (2019)

Noncentral nuclear collisions $\Rightarrow$ Large vorticity and magnetic field
$\Rightarrow$ Particle polarization

Basic polarization mechanisms

Electromagnetic interaction

$$\sim -q\vec{S}\cdot\vec{B}$$

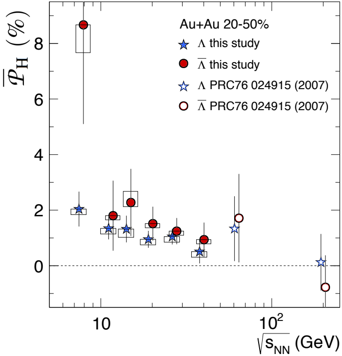
Spin-vorticity coupling

$$\sim -\vec{S}\cdot\vec{\omega}$$

$$\sim \text{Quantum}\cdot\text{Classical}$$

$\vec{S}$ - Particle spin, $\vec{B}$ - Magnetic field, $\vec{\omega}$ - Medium rotation, q - Charge

- Polarization along global angular momentum

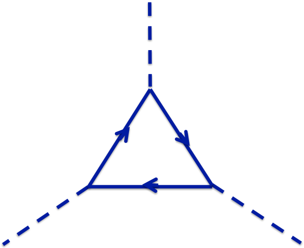


L. Adamczyk et al. (STAR), Nature 548 62-65 (2017)

- Quark-gluon plasma is the "most vortical fluid ever observed"

$$\omega = (P_\Lambda + P_{\bar{\Lambda}})k_B T/\hbar \approx 10^{21}\mathrm{s}^{-1}$$

Chiral (Anomalous) Hydrodynamics



Chiral viscous hydrodynamics from entropy

Son, Surowka, PRL **103**, 191601 (2009); Erdmenger, Haack, Kaminski, Yarom, JHEP **01**, 055 (2009); Banerjee, Bhattacharya, Bhattacharyya, Dutta, Loganayagam, Surowka, JHEP **01**, 094 (2011)

- Constitutive relations (Landau frame)

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (P - \zeta \partial_\lambda u^\lambda)\Delta^{\mu\nu} - 2\eta\sigma^{\mu\nu}$$
$$J^\mu = n u^\mu - \sigma T \Delta^{\mu\nu}\partial_\nu(\mu/T) + \sigma E^\mu + \xi\omega^\mu + \xi_B B^\mu$$

$\omega^\mu = (1/2)\epsilon^{\mu\nu\alpha\beta}u_\nu(\partial_\alpha u_\beta)$ - vorticity
$F^{\nu\lambda}$ - Maxwell tensor, $E^\mu = F^{\mu\nu}u_\nu$, $B^\mu = (1/2)\epsilon^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}$
ξ, ξ_B - **Ideal** chiral coefficients, C - anomaly coefficient

- Equations of motion

$$\partial_\mu T^{\mu\nu} = F^{\nu\lambda}J_\lambda \qquad \partial_\mu J^\mu = C E_\mu B^\mu$$

with constraint $u_\alpha u^\alpha = -1$

Chiral hydrodynamics from kinetic theory

e.g., Chen, Son, Stephanov, PRL **115**, no.2, 021601 (2015); Yang, PRD **98**, no.7, 076019 (2018)

- Consider ensemble of particles (and antiparticles) with chirality $\pm$
- Distribution function

$$f_{\text{eq},\pm}(x,p) = [\exp(g_\pm) + 1]^{-1}$$

$$g_\pm(x,p) = -\beta\cdot p - \frac{\mu_\pm}{T} - \underbrace{\frac{1}{2}S^{\mu\nu}\varpi_{\mu\nu}}_{\text{Spin-vorticity coupling}}$$

  $S^{\mu\nu}$ - Rank-2 spin tensor,
  Thermal vorticity - $\varpi^{\mu\nu} = -\frac{1}{2}(\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)$
- Hydrodynamic densities from distribution function: energy-momentum tensor $T^{\mu\nu}$, vector and axial vector currents J^μ_V, J^μ_A

$$\partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu J^\mu_V = 0 \qquad \partial_\mu J^\mu_A = 0$$

This is great. But can one solve it?

Is chiral hydro a local well-posed, causal and stable theory?

Ideal chiral hydrodynamics from kinetic theory

ES, Bemfica, Disconzi, Noronha, PRD **107**, no.5, 054029 (2023)

Consider the full nonlinear system: $\partial_\mu T^{\mu\nu} = 0,\ \partial_\mu J^\mu_V = 0,\ \partial_\mu J^\mu_A = 0$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + P\Delta^{\mu\nu} + \xi_T(\omega^\mu u^\nu + \omega^\nu u^\mu)$$
$$J^\mu_V = n_V u^\mu + \xi_V\,\omega^\mu$$
$$J^\mu_A = n_A u^\mu + \xi_A\,\omega^\mu$$

e.g., Chen, Son, Stephanov, PRL **115**, no.2, 021601 (2015); Yang, PRD **98**, no.7, 076019 (2018)

Vorticity - $\omega^\mu = (1/2)\epsilon^{\mu\nu\alpha\beta}u_\nu\partial_\alpha u_\beta$

- One can prove:

System is ill posed:
No unique solution exists for **arbitrary** initial data!

- **Conventional ideal case**: $\xi_T, \xi_V, \xi_A = 0 \Longrightarrow$ Well-posed, causal and stable

Proof of ill-posedness

ES, Bemfica, Disconzi, Noronha, PRD **107**, no.5, 054029 (2023)

Calculate characteristic determinant

$$
\det[\mathcal{A}(\Psi,\varphi)] = \det\begin{bmatrix} b+\xi_{T,\varepsilon}c & \xi_{T,n_V}c & \xi_{T,n_A}c & 0_{1\times 4} \\ \frac{1}{3}v^{\mu}+\xi_{T,\varepsilon}b\,\omega^{\mu} & (\xi_T)'_{n_V}b\,\omega^{\mu} & \xi_{T,n_A}b\,\omega^{\mu} & \frac{1}{2}\xi_T b\,u_{\lambda}v_{\alpha}\epsilon^{\lambda\alpha\mu}{}_{\nu} \\ \xi_{V,\varepsilon}c & b+\xi_{V,n_V}c & \xi_{V,n_A}c & 0_{1\times 4} \\ \xi_{A,\varepsilon}c & \xi_{A,n_V}c & b+\xi_{A,n_A}c & 0_{1\times 4} \end{bmatrix}
$$

$$
= \left(\frac{\xi_T b}{2}\right)^4 \det\begin{bmatrix} b+\xi_{T,\varepsilon}c & \xi_{T,n_V}c & \xi_{T,n_A}c \\ \xi_{V,\varepsilon}c & b+\xi_{V,n_V}c & \xi_{V,n_A}c \\ \xi_{A,\varepsilon}c & \xi_{A,n_V}c & b+\xi_{A,n_A}c \end{bmatrix} \det\left[u_{\lambda}v_{\alpha}\epsilon^{\lambda\alpha\mu}{}_{\nu}\right] = 0,
$$

$= 0$

- Valid both at linear and nonlinear regime
- Ill-posedness because of vorticity

Why is the bad term bad

Gavassino, Abboud, ES, Noronha, to appear

Consider only heat diffusion term

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu$$

Perturbation around equilibrium

$$u^\mu = (1, \vec{\delta u}) \quad \Longrightarrow \quad \delta T^{0j} = (\varepsilon + P)\delta u^j + \delta \mathcal{Q}^j \not\propto \delta u^j$$

In ideal chiral hydro

$$\delta \mathcal{Q}^j \sim \frac{1}{2}\xi_T(\vec{\nabla} \times \vec{\delta u})^j \quad \Longrightarrow \quad \vec{\delta u} = f(t)(\sin(kz), \cos(kz), 0)$$

with $k = 2(\varepsilon + P)/\xi_T$

$f(t)$ can be any function of time $\Longrightarrow$ System is ill-posed!

Local well-posedness demands heat diffusion along vorticity to vanish

$$\xi_T \omega^\mu$$

How do we fix it?

We change hydrodynamic frame

Ideal chiral hydrodynamics in Landau frame

ES, Bemfica, Disconzi, Noronha, PRD **107**, no.5, 054029 (2023)

- Shift of velocity

$$u^\mu = u_L^\mu - \frac{\xi_T \omega_L^\mu}{\varepsilon + P}$$

- Landau frame (consider one current for simplicity)

$$T^{\mu\nu} = \varepsilon u_L^\mu u_L^\nu + P\Delta^{\mu\nu}$$
$$J_A^\mu = n_A u_L^\mu + \xi_{AL}\omega_L^\mu$$

The theory is well-posed and causal if

$$\frac{\partial \xi_{AL}}{\partial n_A}|\omega_L| \leq 1$$

- Definition of hydrodynamic variables (hydrodynamic frames) matter even in the ideal case

BDNK First-order viscous chiral hydrodynamics

Abboud, ES, Noronha, arXiv:2308.02928

For simplicity, we switch off electromagnetic fields

$$\text{d.o.f.} = \varepsilon, u^\mu, n \qquad \partial_\mu T^{\mu\nu} = 0 \qquad \partial_\mu J^\mu = 0$$

- Constitutive relations in a general hydrodynamic frame

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})u^\mu u^\nu + (P + \Pi)u^\mu u^\nu + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \pi^{\mu\nu}$$
$$J^\mu = (n + \mathcal{N})u^\mu + \mathcal{J}^\mu$$

- Consider the theory

$$\mathcal{A} = \varepsilon_1 D\varepsilon + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 Dn, \quad \pi^{\mu\nu} = -2\eta\sigma^{\mu\nu}$$
$$\Pi = \pi_1 D\varepsilon + \pi_2 \partial_\lambda u^\lambda + \pi_3 Dn$$
$$\mathcal{Q}^\mu = \theta_1 \Delta^{\mu\lambda}\partial_\lambda \varepsilon + \theta_2 Du^\mu + \theta_3 \Delta^{\mu\lambda}\partial_\lambda n + \xi_T \omega^\mu$$
$$\mathcal{N} = \nu_1 D\varepsilon + \nu_2 \partial_\lambda u^\lambda + \nu_3 Dn$$
$$\mathcal{J}^\mu = \gamma_1 \Delta^{\mu\lambda}\partial_\lambda \varepsilon + \gamma_2 Du^\mu + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda n + \xi_J \omega^\mu$$

When is the theory causal and stable?

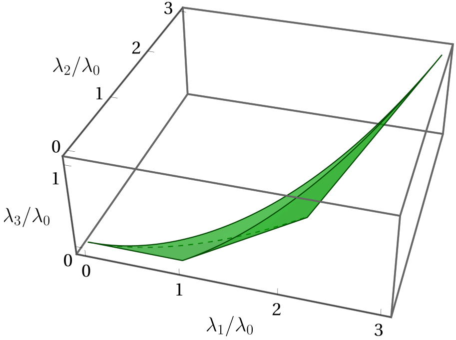
Causality demands heat diffusion along vorticity to vanish

$$\xi_T \omega^\mu$$

Causality conditions

Abboud, ES, Noronha, arXiv:2308.02928

Causal if, and only if, $0 \leq \eta/\theta_2 \leq 1$ and



- λ_i's are combinations of transport parameters
- Causality only depends on three combinations
- These conditions imply local well-posedness too

Necessary and sufficient conditions for stability

Abboud, ES, Noronha, arXiv:2308.02928

$$\begin{aligned}
&\theta_2 > 0 \quad \text{and} \quad \eta \geq 0,\\
&\overline{\lambda}_0 > 0 \quad \text{and} \quad \overline{A} \geq 0 \quad \text{and} \quad \overline{F} \geq 0 \quad \text{and} \quad \overline{\lambda}_3 \geq 0,\\
&\Delta_{(1,0)} \geq 0 \quad \text{and} \quad \Delta_{(1,2)} \geq 0,\\
&(\Delta_{(2,4)}, \Delta_{(2,2)}, \Delta_{(2,0)}) \in S_2,\\
&(\Delta_{(3,8)}, \Delta_{(3,6)}, \Delta_{(3,4)}, \Delta_{(3,2)}) \in S_3,\\
&(\Delta_{(4,12)}, \Delta_{(4,10)}, \Delta_{(4,8)}, \Delta_{(4,6)}, \Delta_{(4,4)}) \in S_4.
\end{aligned}$$

$$\begin{aligned}
\Delta_{(1,0)} &= \overline{A}\,\overline{B} - \overline{\lambda}_0,\\
\Delta_{(1,2)} &= \overline{\lambda}_1\overline{A} - \overline{\lambda}_0\overline{C},\\
\Delta_{(2,0)} &= \Delta_{(1,0)},\\
\Delta_{(2,2)} &= \overline{C}\Delta_{(1,0)} + \Delta_{(1,2)} - \overline{A}(\overline{A}\,\overline{D} - \overline{\lambda}_0 c_s^2),\\
\Delta_{(2,4)} &= \overline{C}\Delta_{(1,2)} - \overline{A}(\overline{\lambda}_2\overline{A} - \overline{\lambda}_0\overline{E}),\\
\Delta_{(3,2)} &= (\overline{D} - \overline{B}c_s^2)\Delta_{(1,0)},\\
\Delta_{(3,4)} &= (\overline{A}\,\overline{F} - \overline{B}\,\overline{E} - c_s^2\overline{\lambda}_1 + \overline{\lambda}_2)\Delta_{(1,0)} - \overline{B}c_s^2\Delta_{(1,2)} + \overline{D}\Delta_{(2,2)} + c_s^2\overline{\lambda}_0(\overline{A}\,\overline{D} - c_s^2\overline{\lambda}_0),\\
\Delta_{(3,6)} &= (\overline{\lambda}_3\overline{A} - \overline{\lambda}_1\overline{E})\Delta_{(1,0)} + (\overline{A}\,\overline{F} - \overline{B}\,\overline{E} - c_s^2\overline{\lambda}_1)\Delta_{(1,2)} + \overline{\lambda}_2\Delta_{(2,2)} + \overline{D}\Delta_{(2,4)}\\
&\quad + \overline{\lambda}_0[\overline{E}(\overline{A}\,\overline{D} - \overline{\lambda}_0 c_s^2) + c_s^2(\overline{\lambda}_2\overline{A} - \overline{\lambda}_0\overline{E})],\\
\Delta_{(3,8)} &= (\overline{\lambda}_3\overline{A} - \overline{\lambda}_1\overline{E})\Delta_{(1,2)} + \overline{\lambda}_2\Delta_{(2,4)} + \overline{\lambda}_0\overline{E}(\overline{\lambda}_2\overline{A} - \overline{\lambda}_0\overline{E}),\\
\Delta_{(4,4)} &= -\overline{F}\Delta_{(1,0)} + c_s^2\Delta_{(3,2)},\\
\Delta_{(4,6)} &= [\overline{F}(\overline{A}c_s^2 - \overline{C}) - \overline{\lambda}_3]\Delta_{(1,0)} - \overline{F}\Delta_{(2,2)} + \overline{E}\Delta_{(3,2)} + c_s^2\Delta_{(3,4)},\\
\Delta_{(4,8)} &= [\overline{\lambda}_3(\overline{A}c_s^2 - \overline{C}) + \overline{A}\,\overline{E}\,\overline{F}]\Delta_{(1,0)} + \overline{A}\,\overline{F}c_s^2\Delta_{(1,2)} - (\overline{\lambda}_3 + \overline{C}\,\overline{F})\Delta_{(2,2)} - \overline{F}\Delta_{(2,4)},\\
&\quad + \overline{E}\Delta_{(3,4)} + c_s^2\Delta_{(3,6)} - \overline{A}^3\overline{F}^2,\\
\Delta_{(4,10)} &= \overline{\lambda}_3\overline{A}\,\overline{E}\Delta_{(1,0)} + \overline{A}(\overline{E}\,\overline{F} + \overline{\lambda}_3 c_s^2)\Delta_{(1,2)} - \overline{\lambda}_3\overline{C}\Delta_{(2,2)} - (\overline{\lambda}_3 + \overline{C}\,\overline{F})\Delta_{(2,4)}\\
&\quad + \overline{E}\Delta_{(3,6)} + c_s^2\Delta_{(3,8)} - 2\overline{\lambda}_3\overline{A}^3\overline{F},\\
\Delta_{(4,12)} &= \overline{\lambda}_3\overline{A}\,\overline{E}\Delta_{(1,2)} - \overline{\lambda}_3\overline{C}\Delta_{(2,4)} + \overline{E}\Delta_{(3,8)} - \overline{\lambda}_3^2\overline{A}^3.
\end{aligned}$$

$$S_2 = \{(a,b,c) \in \mathbb{R}^3 | a \geq 0, c \geq 0, b \geq -2\sqrt{ac}\}.$$

S_3 :
- $a \geq 0$ and $b \geq 0$ and $c \geq 0$ and $d \geq 0$,
- $a > 0$ and $d > 0$ and $\mathrm{Disc}_3(a,b,c,d) \leq 0$,
- $a = 0$ and $(b,c,d) \in S_2$,
- $d = 0$ and $(a,b,c) \in S_2$,

S_4 :
- $a > 0$ and $e > 0$ and $\chi_2 < -2$ and $L \leq 0$ and $\chi_1 + \chi_3 > 0$,
- $a > 0$ and $e > 0$ and $-2 \leq \chi_2 \leq 6$ and $\left\{\begin{matrix} L \leq 0 \text{ and } \chi_1 + \chi_3 > 0 \\ \text{or} \\ L \geq 0 \text{ and } K_1 \leq 0 \end{matrix}\right\}$,
- $a > 0$ and $e > 0$ and $\chi_2 > 6$ and $\left\{\begin{matrix} L \leq 0 \text{ and } \chi_1 + \chi_3 > 0 \\ \text{or} \\ \chi_1 > 0 \text{ and } \chi_3 > 0 \\ \text{or} \\ L \geq 0 \text{ and } K_2 \leq 0 \end{matrix}\right\}$,
- $a = 0$ and $(b,c,d,e) \in S_3$,
- $e = 0$ and $(a,b,c,d) \in S_3$,

$$\chi_1 = ba^{-3/4}e^{-1/4}, \quad \chi_2 = ca^{-1/2}e^{-1/2}, \quad \chi_3 = da^{-1/4}e^{-3/4},$$
$$L = \left(\chi_2^2 - 3\chi_1\chi_3 + 12\right)^3 - \left(72\chi_2 + 9\chi_1\chi_2\chi_3 - 2\chi_2^3 - 27\chi_1^2 - 27\chi_3^2\right)^2,$$
$$K_1 = (\chi_1 - \chi_3)^2 - 16(\chi_1 + \chi_2 + \chi_3 + 2), \quad K_2 = (\chi_1 - \chi_3)^2 - \frac{4(\chi_2 + 2)}{\sqrt{\chi_2 - 2}}(\chi_1 + \chi_3 + 4\sqrt{\chi_2 - 2}).$$

$$\begin{aligned}
A &= w\{\tilde{\varepsilon}, \tilde{\nu}\} + \theta_2(\tilde{\varepsilon}_1 + \tilde{\nu}_3),\\
B &= \theta_2 + w(\tilde{\varepsilon}_1 + \tilde{\nu}_3),\\
C &= \left(\frac{4}{3}\eta - \pi_2\right)(\tilde{\varepsilon}_1 + \tilde{\nu}_3) + (\varepsilon_2 + \theta_2)(\tilde{\pi}_1 + \tilde{\theta}_1 - \langle\tilde{\nu}\rangle) + (\nu_2 + \gamma_2)(\tilde{\pi}_3 + \tilde{\theta}_3 + \langle\tilde{\varepsilon}\rangle)\\
&\quad + w(\{\tilde{\pi}, \tilde{\nu}\} + \{\tilde{\gamma}, \tilde{\varepsilon}\}) + n\{\tilde{\varepsilon}, \tilde{\pi} + \tilde{\theta}\} - \theta_2(\tilde{\theta}_1 + \tilde{\gamma}_3),\\
D &= \frac{4}{3}\eta - \pi_2 + w(\tilde{\pi}_1 - \tilde{\gamma}_3 - \langle\tilde{\nu}\rangle) + n(\tilde{\pi}_3 + \tilde{\theta}_3 + \langle\tilde{\varepsilon}\rangle) + \left(\frac{\partial P}{\partial \varepsilon}\right)_n(\varepsilon_2 + \theta_2) + \left(\frac{\partial P}{\partial n}\right)_\varepsilon(\nu_2 + \gamma_2),\\
E &= -\left(\frac{4}{3}\eta - \pi_2\right)(\tilde{\theta}_1 + \tilde{\gamma}_3) + w\{\tilde{\gamma}, \tilde{\pi}\} + n\{\tilde{\pi}, \tilde{\theta}\} + (\varepsilon_2 + \theta_2)\langle\tilde{\gamma}\rangle - (\nu_2 + \gamma_2)\langle\tilde{\theta}\rangle,\\
F &= w\langle\tilde{\gamma}\rangle - n\langle\tilde{\theta}\rangle.
\end{aligned}$$

- Conditions look complicated but can be easily checked numerically!
- Completed causality and stability analysis of conventional BDNK too

Conclusions

- Complete formulation of local well-posed, causal and stable first-order chiral hydrodynamics
- Effective field theory approach
- Consistent first-order theory $\Longrightarrow$ Heat diffusion along vorticity must be absent both in ideal and first-order theories
- **Future:** dynamical magnetic fields; applications in QGP physics, astrophysics, condensed matter, ...