Solitonic Symmetry beyond Homotopy Groups

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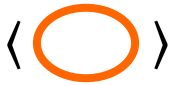
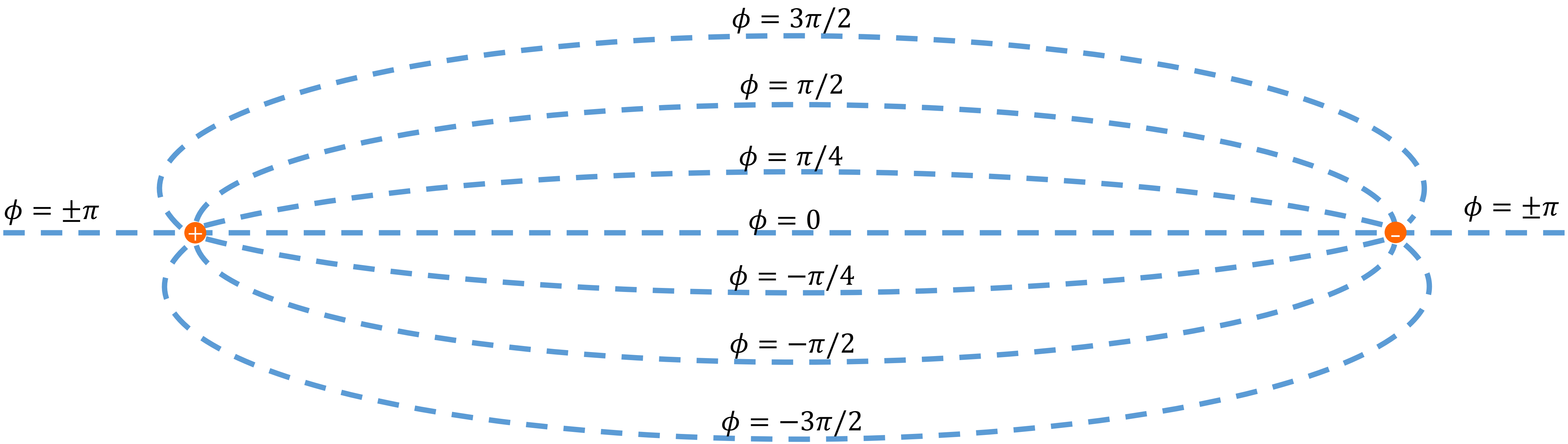
$$S = \frac{1}{2g^2}\int d\phi \wedge \star\, d\phi - \lambda \int (1 - \cos\phi)\,,\ \ \phi \sim \phi + 2\pi$$

$$\pi_1(S^1) = \mathbb{Z}$$

$$U_\alpha(M_1) = \exp\left\{ \mathrm{i}\alpha \int_{M_1} \frac{\mathrm{d}\phi}{2\pi} \right\},\ \ \alpha \sim \alpha + 2\pi$$

- **Example: 2D S^1 boson**  $\sim \exp(-mr)$ ➡ Particle excitation corresponding to solitons. Solitonic **0-form** U(1) is not spontaneously broken

- **Example: 3D S^1 boson**  $\sim \exp(-m\sigma)$ ➡ Stringy excitation corresponding to solitons. Solitonic **1-form** U(1) is not spontaneously broken



$$S = \frac{1}{2g^2}\int da \wedge \star\, da$$

$$\pi_2(BU(1)) = \mathbb{Z}$$

$$U_\alpha(M_2) = \exp\left\{ \mathrm{i}\alpha \int_{M_2} \frac{\mathrm{d}a}{2\pi} \right\}, \quad \alpha \sim \alpha + 2\pi$$

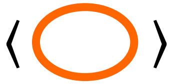
- **Example: 3D U(1) gauge theory**

 ~ Coulomb → Magnetic **0-form** U(1) is spontaneously broken

monopoles

Photons as Goldstone bosons

- **Example: 4D U(1) gauge theory**

 ~ Coulomb → Magnetic **1-form** U(1) is spontaneously broken

't Hooft loops

Photons as Goldstone bosons

Solitonic symmetry is believed to be classified by **Homotopy Group**.

In this talk…

Solitons of different dimensions

⬇

Categorical solitonic symmetry

beyond homotopy groups

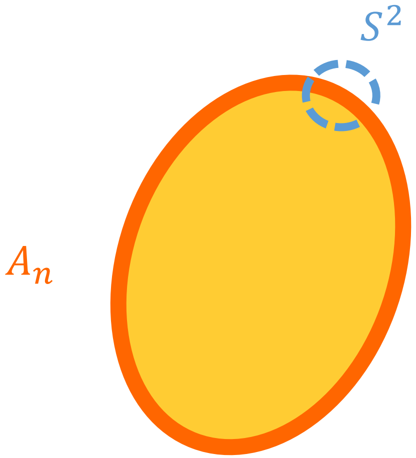
- **4D $\mathbb{C}P^1$ sigma model**

	$n=1$	$n=2$	$n=3$
$\pi_n(\mathbb{C}P^1)$	0	$\mathbb{Z}$	$\mathbb{Z}$

Vortex ---- $\pi_2(\mathbb{C}P^1)$

2D Soliton ---- **stringy excitation**

Operator ---- **line defect** A_n

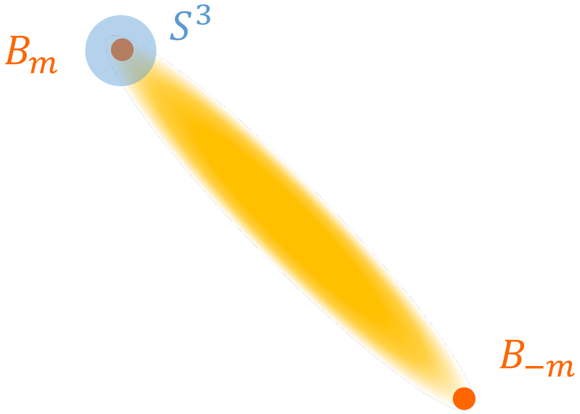


Local conserved current ✓

Hopfion ---- $\pi_3(\mathbb{C}P^1)$

1D Soliton ---- **particle excitation**

Operator ---- **point defect** B_m



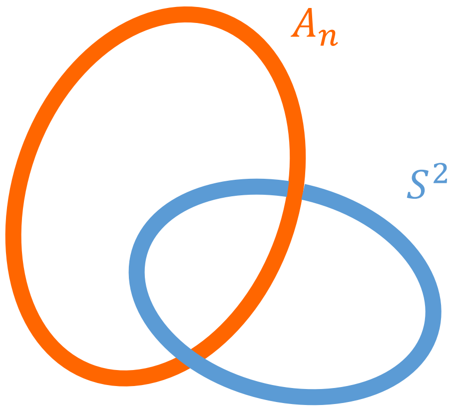
~~Local conserved current~~ ✗

$\mathbb{C}\mathrm{P}^1$ = unit $\mathbb{C}^2$ vector $\vec{z}(x)$ + U(1) gauge redundancy $\vec{z}(x) \sim e^{i\alpha(x)}\vec{z}(x)$

Auxiliary U(1) gauge field ----- $a \equiv i\vec{z}^\dagger \cdot d\vec{z}$ $\quad da \wedge da = 0$

Vortex ---- $\pi_2(\mathbb{C}\mathrm{P}^1)$

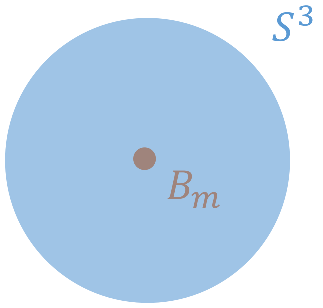
charge: $\displaystyle\int_{S^2} \frac{da}{2\pi} = n$



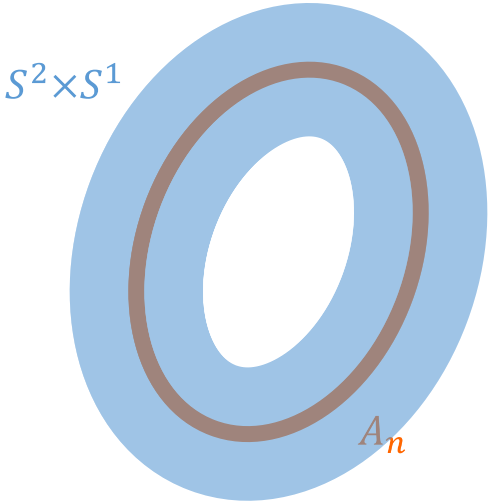
symmetry: $\displaystyle\mathcal{V}_\beta(S^2) = \exp\left\{i\beta \int_{S^2} \frac{da}{2\pi}\right\}, \qquad \beta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$

Hopfion ---- $\pi_3(\mathbb{C}\mathrm{P}^1)$

charge: $\displaystyle\int_{S^3} \frac{ada}{4\pi^2} = m$



symmetry: $\displaystyle\mathcal{H}_\alpha(S^3) = \exp\left\{i\alpha \int_{S^3} \frac{ada}{4\pi^2}\right\}, \qquad \alpha \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}$



$\tau \in \frac{\mathbb{R}}{2\pi}$

Consider a gauge transformation: $\vec{z} \to \vec{z}' e^{-ik\tau} \implies a \to a' + k\mathrm{d}\tau$

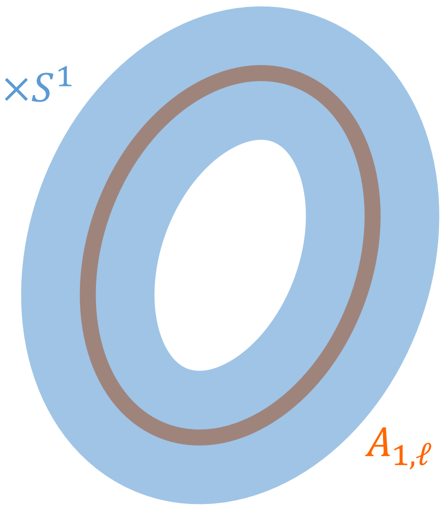
$$\int_{S^2\times S^1} \frac{a'\mathrm{d}a'}{4\pi^2} - \int_{S^2\times S^1} \frac{a\mathrm{d}a}{4\pi^2} = 2kn$$

charge: $\int_{S^2\times S^1} \frac{a\mathrm{d}a}{4\pi^2} \in \mathbb{Z}_{2|n|}$

symmetry: $\mathcal{H}_{\frac{q}{n}\pi}(S^2\times S^1) = \exp\left\{ i\frac{q}{n}\int_{S^2\times S^1} \frac{a\mathrm{d}a}{4\pi} \right\}$, $q \in \mathbb{Z}_{2|n|}$

- $A_{n\neq 0}$ has $2|n|$ deformation classes, classified by the $\mathbb{Z}_{2|n|}$ hopfion charge, denoted by $A_{n,\ell}$ with $\ell \sim \ell + 2|n|$.
- The existence of these deformation classes can also be studied via algebraic topology. e.g. [Pontryagin, 1941]

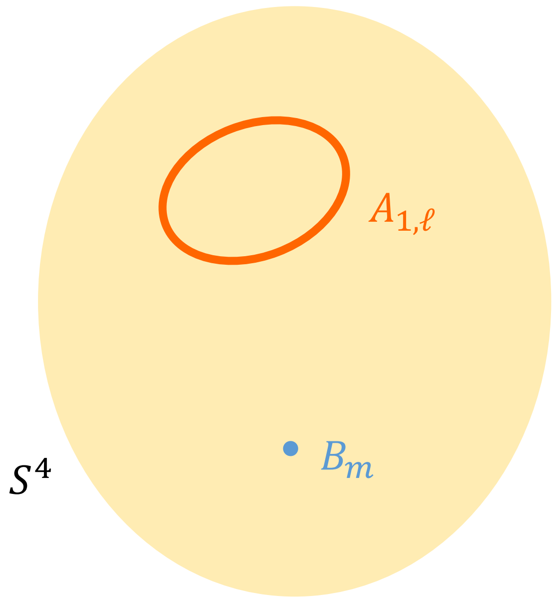
An explicit description of the 2 deformation classes of $A_{1,\ell}$



- At each τ, we have a map ϕ_τ: $S^2 \mapsto \mathrm{CP}^1$.
- $\tau \mapsto \phi_\tau$ describes a rotation process of S^2, which is reduced to $\tau \mapsto \mathrm{SO}(3)$.
- Due to $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$, we have two deformation classes.

$$\begin{cases} A_{1,0} \equiv \text{the untwisted class} \\ A_{1,1} \equiv \text{the twisted class} \end{cases}$$

Symmetry generator always well-defined: $\mathcal{H}_\pi(M^3) = \exp\left\{ \mathrm{i} \int_{M^3} \frac{ada}{4\pi} \right\} \to \pm 1 \quad \Longrightarrow \quad \mathbb{Z}_2$ symmetry



[Chen, Tanizaki, 2022]

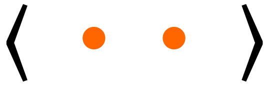
$$\begin{cases} \text{even } m: & \langle A_{1,0}\, B_m \rangle \neq 0 & \langle A_{1,1}\, B_m \rangle = 0 \\ \text{odd } m: & \langle A_{1,0}\, B_m \rangle = 0 & \langle A_{1,1}\, B_m \rangle \neq 0 \end{cases}$$

- $A_{1,0}$ absorbs/emits any **even** number of hopfions.
- $A_{1,1}$ absorbs/emits any **odd** number of hopfions.
- B_m and B_{m+2} must share the **same** hopfion charge, provided **invertibility**.

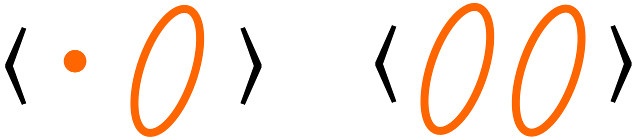
The $\mathbb{Z}_2$ charge is classified by **reduced spin bordism group**.

$$\widetilde{\Omega}_3^{Spin}(\mathbb{C}\mathrm{P}^1) = \mathbb{Z}_2$$

We have shown…



Selection rule → U(1)



Selection rule → $\mathbb{Z}_2$

non-group-like selection rule

To encode all above…

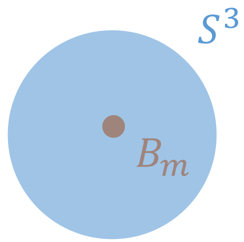
We need **non-invertible charge**.

We need **bordism covariant** (i.e. TQFT) instead of **bordism invariant** to construct $\mathcal{H}_\alpha(M^3)$.

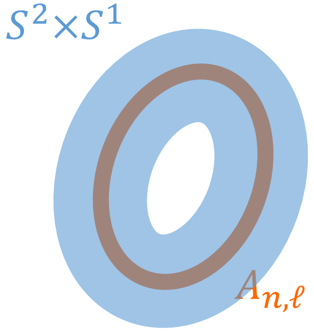
This is possible for **rational** coefficients $\alpha \in 2\pi \frac{\mathbb{Q}}{\mathbb{Z}}$.

For $\alpha = \frac{\pi}{N}$

$$\mathcal{H}_{\frac{\pi}{N}}(M^3) = \int \mathfrak{D}b \, \exp\left\{-\mathrm{i}\int_{M^3}\left(\frac{N}{4\pi} b\mathrm{d}b + \frac{1}{2\pi} b\mathrm{d}a\right)\right\}$$



$$\mathcal{H}_{\frac{\pi}{N}}(S^3) = \exp\left\{\frac{\mathrm{i}}{N}\int_{S^3}\frac{a\mathrm{d}a}{4\pi}\right\} = \mathrm{e}^{\mathrm{i}\frac{\pi}{N}m}$$



$$\mathcal{H}_{\frac{\pi}{N}}(S^2 \times S^1) = \begin{cases} \exp\left\{\frac{\mathrm{i}}{N}\int_{S^2\times S^1}\frac{a\mathrm{d}a}{4\pi}\right\} = \mathrm{e}^{\mathrm{i}\frac{\pi}{N}\ell}, & \text{if } n = 0 \mod N \\ 0, & \text{if } n \neq 0 \mod N \end{cases} \quad \ell \sim \ell + 2|n|$$

For $\alpha = \frac{p}{N}\pi$ $$\mathcal{H}_{\frac{p}{N}\pi}(M^3) = \mathcal{A}^{N,p}(M^3, \mathbb{C}\mathrm{P}^1)$$

$\mathcal{A}^{N,p}$ denotes the **minimal** spin TQFT$_3$ with $\mathbb{Z}_N$ 1-form symmetry whose ‘t Hooft anomaly is labeled by p.

e.g. $$\mathcal{A}^{N,1} \simeq U(1)_N$$

Symmetry indeed becomes **non-invertible**

$$\mathcal{H}_\alpha \times \mathcal{H}_\alpha^\dagger \neq 1 \qquad \mathcal{H}_\alpha \times \mathcal{H}_{-\alpha} \neq 1 \qquad \mathcal{H}_\alpha \times \mathcal{H}_\beta \neq \mathcal{H}_{\alpha+\beta}$$

4D $\mathbb{C}\mathrm{P}^1$ sigma model

- $\mathrm{Hom}\left(\widetilde{\Omega}_3^{Spin}(\mathbb{C}\mathrm{P}^1), U(1)\right)$ gives **invertible** 0-form solitonic symmetry.
- Minimal spin $\mathrm{TQFT}_3(\mathbb{C}\mathrm{P}^1)$ gives **non-invertible** 0-form solitonic symmetry.

3D $\mathbb{C}\mathrm{P}^1$ sigma model

- $\mathrm{Hom}\left(\widetilde{\Omega}_3^{Spin}(\mathbb{C}\mathrm{P}^1), U(1)\right)$ classifies couplings to **invertible** topological phase ($\mathbb{Z}_2$ θ-angle).
- Minimal spin $\mathrm{TQFT}_3(\mathbb{C}\mathrm{P}^1)$ classifies couplings to **non-invertibl**e topological phase (topological order).

$\Longrightarrow$ "(-1)-form solitonic symmetry"

Thank you for listening!