1-form symmetry versus large N QCD

Aleksey Cherman
University of Minnesota

Collaborators:

Theo Jacobson

Maria Neuzil

Confinement

This talk is about color confinement. The basic question is: what's the best way to define it?

The first way we learn about in classes is that confinement means that:

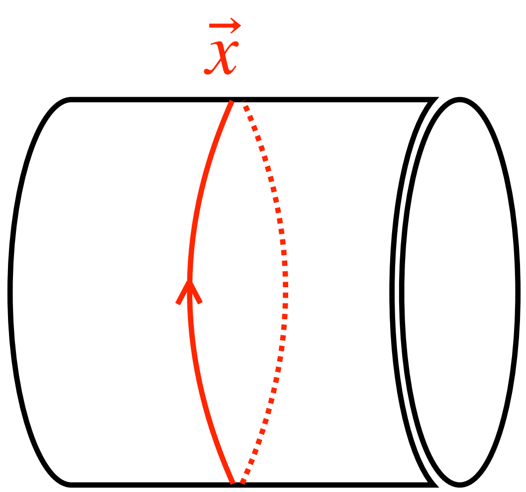
1. Isolated fundamental test quarks have an **infinite free energy**.
2. Far-apart test quark-anti-quark pairs feel an **attractive linear potential**.

By these criteria, pure SU(N) YM theory on $\mathbb{R}^4$ confines color.

But there is also a more powerful definition, which requires a little more abstraction.

SU(N) pure Yang-Mills

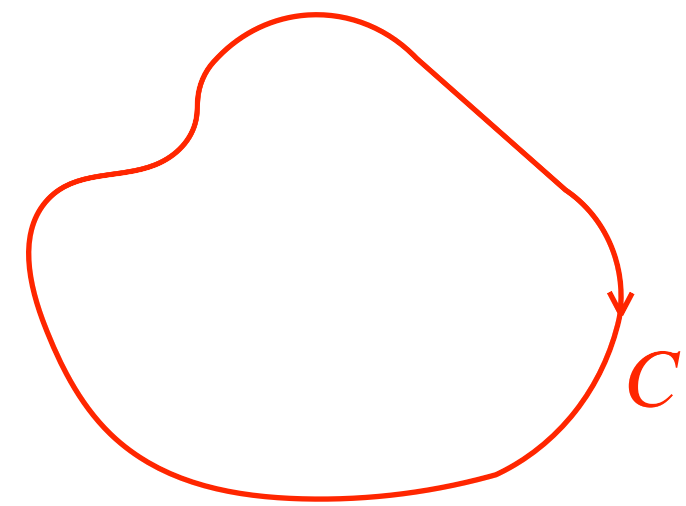
- Confinement means that Wilson **line operators** obey some **selection rules**:



$$\left\langle \frac{1}{N} \mathrm{tr}_F \, P(\vec{x}) \right\rangle = 0$$

in any finite volume

Quark free energy diverges



$$\lim_{A(C)\to\infty} \left\langle \frac{1}{N} \mathrm{tr}_F \, W(C) \right\rangle = 0$$

in any scheme

Linear quark-quark potential

$\mathbb{Z}_N$ 1-form symmetry in YM theory

- We expect that QFT selection rules should be explained by global symmetries.
- Confinement involves selection rules for line operators, rather than the local operators.
- The necessary symmetry is called a ``$\mathbb{Z}_N$ 1-form symmetry”.
  - To understand 1-form symmetries, it’s important to define symmetries via the existence of appropriate symmetry generators, which are topological operators.
  - $\mathbb{Z}_N$ structure arises since collections of N quarks are not confined - they bind into a color-singlet baryon.

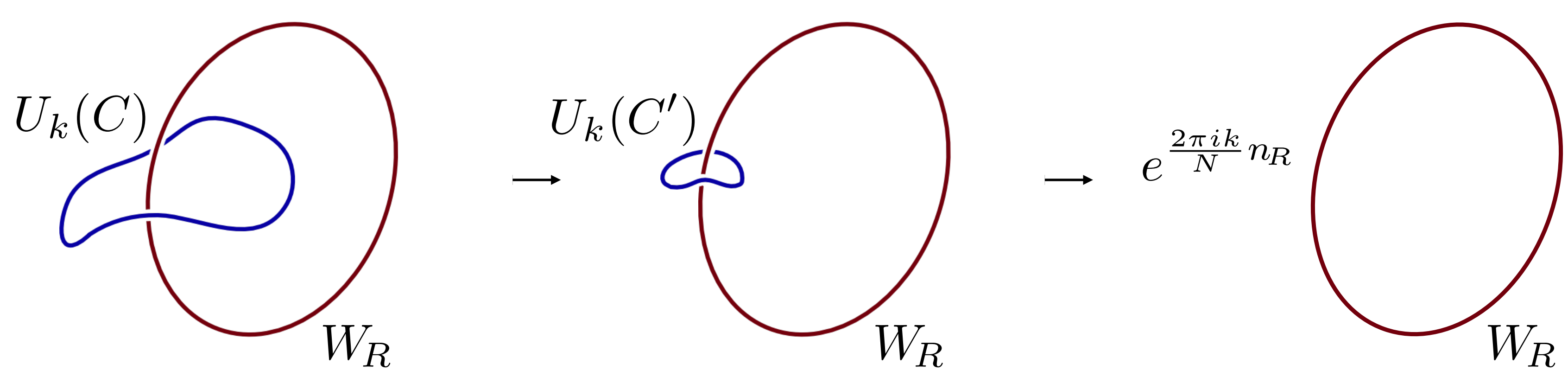
$\mathbb{Z}_N$ 1-form symmetry in YM theory

Gaiotto, Kapustin, Seiberg, Willett 2014

The symmetry is generated by N co-dimension-2 topological symmetry generators U_k:

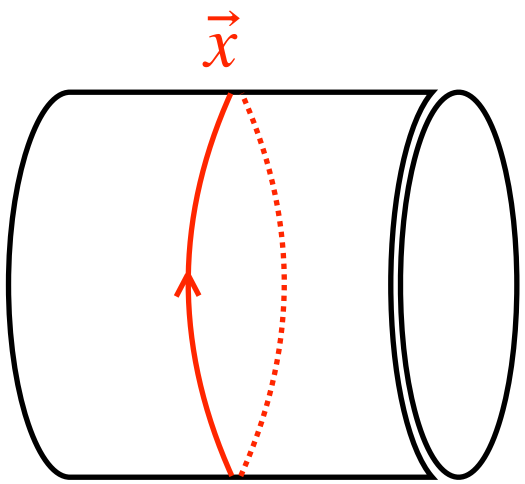
$$U_k(M_{d-2})U_n(M_{d-2}) = U_{k+n \bmod N}(M_{d-2})$$

$$\langle U_k(M_{d-2})W_F(C)\rangle = e^{2\pi i \frac{k}{N}\mathrm{Link}(C,M_{d-2})}\langle W_F(C)\rangle$$



- Symmetry geometry multiply Wilson loops by $\mathbb{Z}_N$ phases…

Confinement selection rules from symmetry



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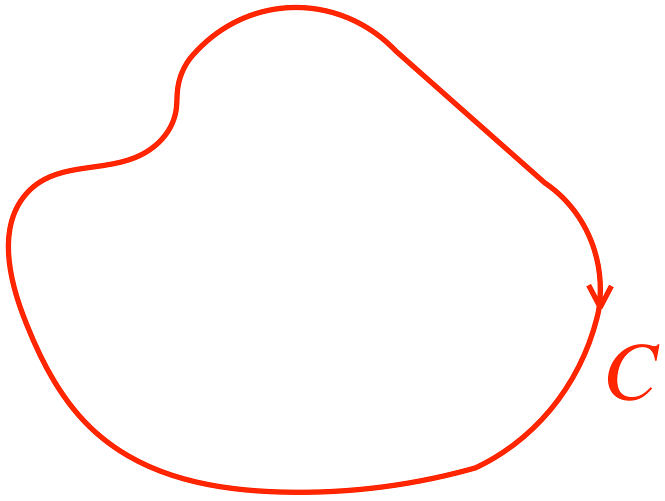


$$\lim_{A(C)\to\infty}\left\langle \frac{1}{N}\mathrm{tr}_F\, W(C)\right\rangle = 0$$

in any scheme

These selection rules are consequences of the $\mathbb{Z}_N$ 1-form symmetry.

- The confining phase is thus identified as a phase where the $\mathbb{Z}_N$ 1-form symmetry is not spontaneously broken.

Why care about $\mathbb{Z}_N$ 1-form symmetry?

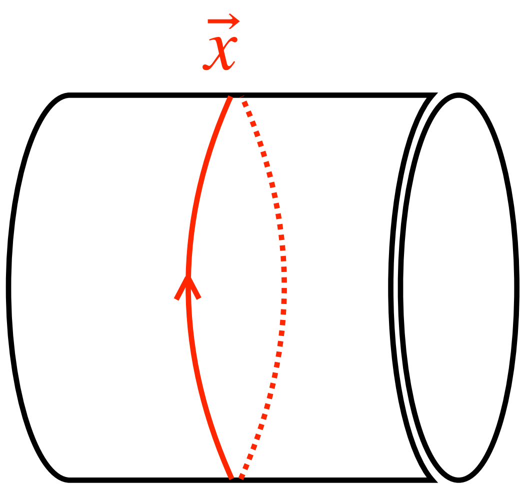
- Q: What does one gain by rephrasing color confinement in this formal way? A: A lot of nice things!
  - It's generally useful to have a symmetry picture of any phenomenon.
  - It's been very fruitful since 2014: allows predictions about phase boundaries in parameter space, anomaly-matching, etc.
  - New results, new insights into old results.
    - Example: SU(N) QCD with 1 quark flavor has a point in parameter space where gap vanishes (uses results on YM)

Gaiotto, Komargodski, Seiberg, 2017

SU(N) QCD at large N

With $N_F \sim \mathcal{O}(1)$ fundamental-rep quark fields, quark loops are suppressed at large N.

- Large N QCD obeys the same selection rules as pure YM:



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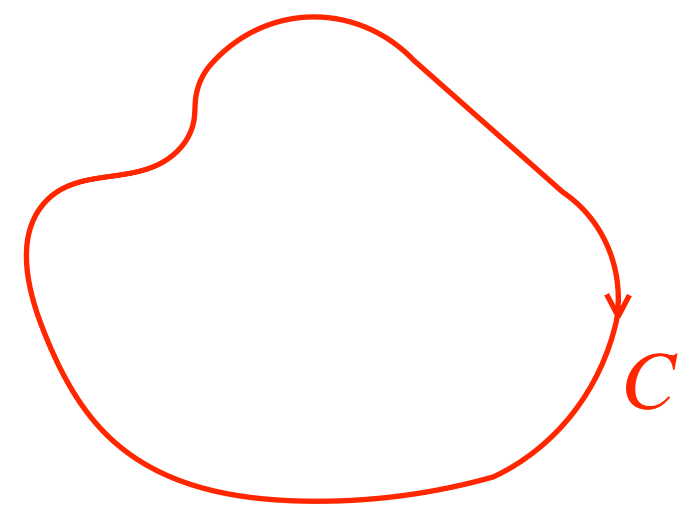


$$\lim_{A(C)\to\infty} \left\langle \frac{1}{N} \mathrm{tr}_F\, W(C) \right\rangle = 0$$

in any scheme

- Confinement is well-defined in large N QCD!

Confinement in large N QCD

We usually expect that selection rules are consequences of symmetries.

So are the Wilson loop selection rules consequences of a symmetry in large N QCD?

- Natural guess: at large N, there's a $\mathbb{Z}_N$ 1-form symmetry which explains the selection rules, just as in YM theory.
- Curiously, this guess is not right.
    - Rest of the talk will explain why.

Obstructions to 1-form symmetry

Two basic issues:

- Existence of open Wilson lines in large N QCD.
- Large N quark loop suppression isn't quite universal.

I'll explain these issues, then show how things work explicitly in a calculable example, 2d scalar QCD on the lattice using the hopping expansion.

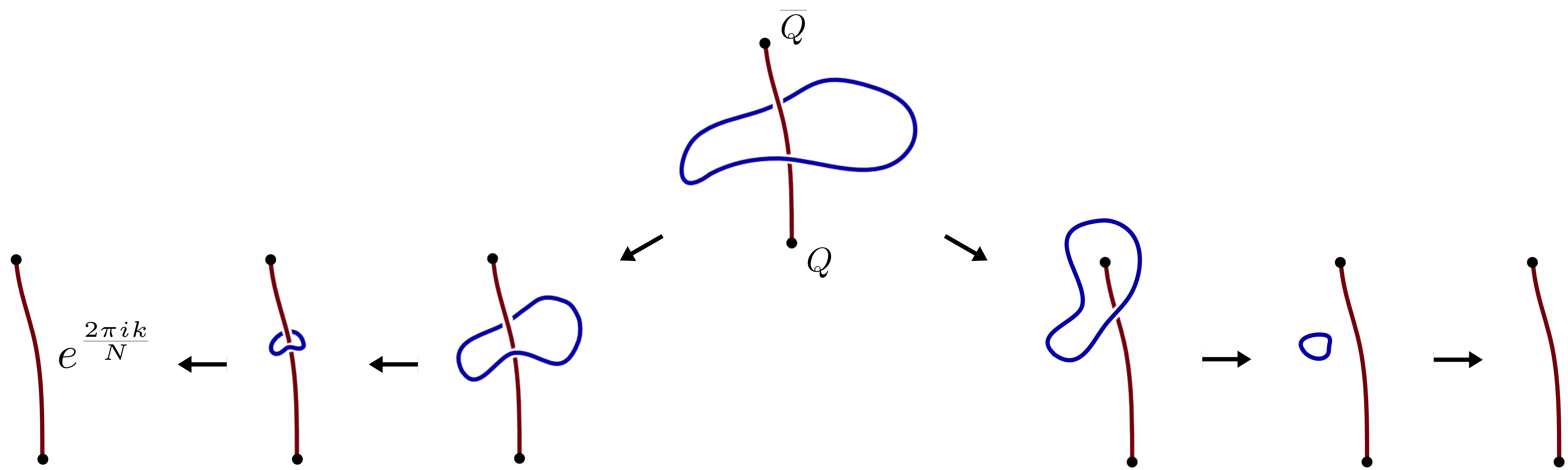
Upshot: there are no non-trivial topological co-dimension-2 operators in large N QCD with an action on Wilson loops.

Endability Issue

Rudelius, Shao 2020

Large N QCD has open Wilson lines: $\left\langle \frac{1}{N} \mathrm{tr}\, \bar{Q}(x) e^{i \int_x^{x'} a} Q(x') \right\rangle = \mathcal{O}(1)$

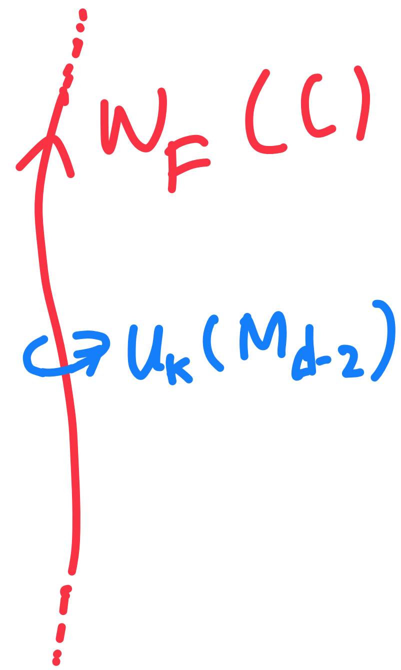
- Suppose it has topological $U_k(M_{d-2})$ operators. Then



- This is inconsistent, so $U_k(M_{d-2})$ cannot be topological operators in large N QCD.

Closed versus open Wilson lines

- Given the assumption that $U_k(M_{d-2})$ is topological, its action on a Wilson line on a curve C can be calculated by "shrinking":



- Data can be obtained from an infinitesimal neighborhood of C - no info on whether C is open or closed!
- So failure of topological property on open Wilson lines implies failure for closed Wilson loops.

Quark loops at large N

- If quark loop contributions are universally suppressed at large N, how could correlation functions of $U_k(M_{d-2})$ be different in QCD versus pure YM theory?

- We'll see that quark loops **aren't** universally suppressed!

- At large N, the interesting $U_k(M_{d-2})$ operators have $k \sim N$.
  - $\left\langle \frac{1}{d_R} W_R(C) \right\rangle = 0$ for $n_R \sim N$, so $n_R \sim 1$ is most interesting.
  - $U_1(M_{d-2})$ acts trivially on $W_R(C)$ with N-ality $n_R \sim 1$

$$U_1(\Sigma_{d-2}) W_R(C) = e^{\frac{2\pi i}{N} n_R}\, W_R(C) = W_R(C) + \mathcal{O}\left(\frac{1}{N}\right)$$

What we might expect

- In QFTs with exact $\mathbb{Z}_N$ 1-form symmetry on $\mathbb{R}^d$, the fusion rules force

$$\langle U_k(M_{d-2})\rangle = e^{2\pi \ell k/N}, \ell \in \mathbb{Z}$$

- In pure YM on $\mathbb{R}^d$, $\ell = 0$.
- Consider 2d QCD on $\mathbb{R}^2$ and suppose the U_k's are all topological up to 1/N corrections, and obey fusion rules:

$$\langle U_k(x)\rangle = 1 + v_k/N + \cdots$$
$$\langle U_k(x)U_{k'}(0)\rangle = 1 + f_{k,k'}(x)/N + \cdots$$

- NB: in $d = 2$ the U_k's are local operators.

Quark loop non-suppression

- We are assuming that

$$\langle U_k(x)\rangle = 1 + v_k/N + \cdots$$
$$\langle U_k(x)U_{k'}(0)\rangle = 1 + f_{k,k'}(x)/N + \cdots$$

- The fusion rules would imply

$$\langle U_k(x)\rangle \sim \langle U_1(x+\epsilon_1)U_1(x+\epsilon_2)\cdots U_1(x+\epsilon_k)\rangle \sim (1+v_1/N)^k$$

- If $k \sim 1$, no contradiction. But if $k = \bar{k}N$, then

$$\left(1+\frac{v_1}{N}\right)^{\bar{k}N} \sim e^{\bar{k}v_1} \neq 1$$

- Contradiction for $k \sim N$.

Obstructions to 1-form symmetry

In a sense, we're done: there's no 1-form symmetry in large N QCD, for two reasons:

- Existence of open Wilson lines in large N QCD.
- Large N quark loop suppression doesn't apply to the would-be symmetry generators.

Next, let's see how the failure of 1-form symmetry happens in a concrete example. Discussion will get more technical!

Punchline: either you can have selection rules without a symmetry (in the sense of existence of co-dim-n topological symmetry generators), **or** we need a further generalization of symmetry to understand (approximate) confinement in QCD.

2d scalar QCD

- Let's explore 2d scalar SU(N) QCD, on the lattice, using the hopping (= large mass) expansion.

$$Z = \prod_{\ell} \int du_\ell \prod_x \int d\phi_x d\phi_x^\dagger \prod_p e^{-s_{\rm YM}(u_p)} \prod_\ell e^{-s_{\rm H}(\phi_x^\dagger u_\ell \phi_{x'})} \prod_x e^{-m^2 \phi_x^\dagger \phi_x}$$

p = plaquettes ℓ = links   x = sites

- Rich enough to share the key features of real QCD, but simple enough to study explicitly.

2d scalar QCD

$$Z = \prod_\ell \int du_\ell \prod_x \int d\phi_x d\phi_x^\dagger \prod_p e^{-s_{\rm YM}(u_p)} \prod_\ell e^{-s_{\rm H}(\phi_x^\dagger u_\ell \phi_{x'})} \prod_x e^{-m^2 \phi_x^\dagger \phi_x}$$

- In 2d there is a maximally-nice gauge action called the 'heat kernel' action:

$$e^{-s_{\rm YM}(u_p)} = \sum_\alpha d_\alpha \chi_\alpha(u_p)\, e^{-g^2 c_\alpha A_p}$$

Migdal 1975,
Menotti+Onofri, 1981

- For pure YM get continuum results even on coarse lattices
- The hopping term (scalar kinetic term) is

$$s_{\rm H}(\phi_x^\dagger u_\ell \phi_{x'}) = -\kappa\, \phi_x^\dagger u_\ell \phi_{x'} + {\rm h.c.}$$

Hopping expansion

- For small κ, matter field integral gives a sum over all possible Wilson loop insertions $W_F(\Gamma)$, representing quark world-lines. Schematically:

$$\langle \mathcal{O}_{\rm glue} \rangle = \langle \mathcal{O}_{\rm glue} \rangle_0 + \sum_{\text{loops } \Gamma} \left(\frac{\kappa}{m^2} \right)^{L_\Gamma} \langle \mathcal{O}_{\rm glue} \, W_F[\Gamma] \rangle_0 + \cdots$$

- Physically, $\kappa/m^2 \sim 1/(m^2 a^2)$. Large mass expansion!
- For us the two interesting $\mathcal{O}_{\rm glue}$ operators are Wilson loops $W_F(C)$ and $U_k(x)$'s, the generators of the $\mathbb{Z}_N$ 1-form symmetry of the 2d YM theory.

Wilson loop in pure YM

- At 0th order in the hopping expansion, Wilson loop expectation value is, of course, same as in pure YM:

$$\frac{1}{N}\langle W_{\mathrm{F}}(C)\rangle = e^{-\sigma A[C]}\left\{1+O(\kappa^4)\right\},\ \sigma = \tfrac{1}{2}\lambda + O(1/N)$$

- Area law behavior ⇔ 2d pure YM confines. ✔

Wilson loop in QCD

- Smallest possible loop on the lattice is built from 4 links, so first corrections comes at κ^4. Calculation gives:

$$\frac{1}{N}\langle W_{\rm F}(C)\rangle = \frac{1}{N}\langle W_{\rm F}(C)\rangle_0 \left\{ 1 + A[C]\left(\frac{\kappa}{m^2}\right)^4 \frac{2}{N}\sinh\left(\frac{\lambda}{2}\right) + O(\kappa^6)\right\}$$

- The κ^4 term is coming from a quark loop, and it's 1/N suppressed as expected. ✓

Perimeter law behavior

- Working to higher order, we find a perimeter-law piece:

$$\frac{1}{N}\langle W_{\mathrm{F}}(C)\rangle = e^{-\sigma A[C]} + \frac{1}{N}e^{-\mu L[C]} + \cdots\,,\ \ \mu = \log m^2/\kappa$$

- Perimeter term also comes from a 1/N suppressed quark loop
  - If $N \to \infty$ with loop geometry fixed, confinement! ✔
- 2d QCD contains all the necessary physics to explore our puzzles.

1-form symmetry generators $U_k(x)$

- Several equivalent definitions.
  - As a Gukov-Witten operator.
    - Delete x from spacetime, and pick the (conjugacy class of) the SU(N) holonomy g around x.
    - The choice $g = e^{2\pi i/N}\mathbb{1} = \omega\mathbb{1}$ defines the $U_k(x)$'s.
  - On the lattice, `thin center-vortex' definition is more convenient:

$$U_k(\tilde{x} = \star p) = \exp\left[s_{\mathrm{YM}}(u_p) - s_{\mathrm{YM}}(\omega^{-k}u_p)\right]$$

  - This is the definition we use.

Expectation value of $U_k(\tilde{x})$

- Let's write $\langle \mathcal{O} \rangle = \dfrac{\langle\langle \mathcal{O} \rangle\rangle}{Z}$. Then

$$\langle\langle U_k(\tilde{x}) \rangle\rangle = \langle\langle U_k(\tilde{x}) \rangle\rangle_0 + \left(\frac{\kappa}{m^2}\right)^4 \sum_p \langle\langle U_k(\tilde{x}) \mathrm{tr}(u_p + u_p^\dagger) \rangle\rangle_0 + \mathcal{O}(\kappa^6)$$

smallest possible "hopping" Wilson loop

Expectation value of $U_k(\tilde{x})$

- Let's write $\langle \mathcal{O} \rangle = \dfrac{\langle\!\langle \mathcal{O} \rangle\!\rangle}{Z}$. Then

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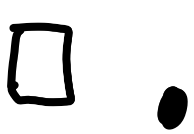
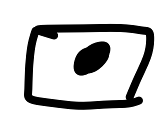
smallest possible "hopping" Wilson loop

$$\langle\!\langle U_k(\tilde{x}) \rangle\!\rangle = 1 + \left(\frac{\kappa}{m^2}\right)^4 \left[2(A-1)\, N\, e^{-g^2 c_{\mathrm{F}}} + N(\omega^k + \omega^{-k})\, e^{-g^2 c_{\mathrm{F}}} \right] + \mathcal{O}(\kappa^6)$$





$\langle W_F(C) \rangle_0 = N e^{-g^2 c_F A(C)}$

$c_F = \dfrac{N^2-1}{2N}$

$$Z = 1 + \left(\frac{\kappa}{m^2}\right)^4 2A\, N e^{-g^2 c_{\mathrm{F}}} + \cdots$$

Expectation value of $U_k(\tilde{x})$

- Putting things together and working through $\mathcal{O}(\kappa^8)$, we get

$$\langle U_k(\tilde{x})\rangle = 1 - \left(\frac{\kappa}{m^2}\right)^4 2N\, e^{-g^2 c_{\mathrm{F}}} \left(1 - \cos\left(\frac{2\pi k}{N}\right)\right) + \frac{1}{2}\left[\left(\frac{\kappa}{m^2}\right)^4 2N\, e^{-g^2 c_{\mathrm{F}}} \left(1 - \cos\left(\frac{2\pi k}{N}\right)\right)\right]^2 + \cdots$$

- As is natural to guess from the result above, and one can prove in general, the result exponentiates:

$$\langle U_k(\tilde{x})\rangle = \exp\left[-\left(\frac{\kappa}{m^2}\right)^4 2N\, e^{-g^2 c_{\mathrm{F}}} \left(1 - \cos\left(\frac{2\pi k}{N}\right)\right) + O(\kappa^6)\right]$$

Expectation value of $U_k(\tilde{x})$

- We've learned that $\langle U_k(\tilde{x})\rangle \sim e^{-N}$ for $k \sim N$, within the hopping expansion.
- This means that for $k \sim N$, at large N we can write

$$\langle U_k(\tilde{x})\rangle = 0$$

- Meanwhile, in pure YM, $\langle U_k(\tilde{x})\rangle = e^{2\pi i \ell k/N} = 1$, and $1 - 0 = 1 \sim \mathcal{O}(1)$. ``Quark loops'' give $\mathcal{O}(1)$ correction!
- Finally, for $k \sim \sqrt{N}$, $\langle U_k(\tilde{x})\rangle \in (0,1)$.
- Not consistent with $U_k(\tilde{x})$ generating a $\mathbb{Z}_N$ symmetry.

Fate of $\mathbb{Z}_N$ 1-form symmetry in large N QCD

- Large N QCD doesn't have a $\mathbb{Z}_N$ 1-form symmetry.
  - Open Wilson line considerations are inconsistent with the existence of topological co-dim-2 operators.
  - One might have hoped that co-dim-2 operators would be topological thanks to quark loop suppression. But there's no reason to expect that in the cases of interest.
- We verified that the would-be 1-form symmetry generators fail to be topological in a solvable model.
  - Fundamental matter contributions are not suppressed!

Outlook

- Wilson loops in large N QCD obey the same selection rules as pure YM.
  - They are not explained by a $\mathbb{Z}_N$ 1-form symmetry.
- Is there some symmetry principle that could explain them?
  - If yes, seems to require some appropriate generalization of "generalized global symmetry". Is there one?
  - If no, we'd have to accept selection rules without symmetries. Seems very strange!
    - Are there examples apart from large N QCD?
- We think both options are entertaining…

Thank you for listening!

Backup: exponentiation of $\langle U_k(\tilde{x})\rangle$

- The heat kernel action on each plaquette can be written as

$$\sum_\alpha d_\alpha \chi_\alpha(u_p)\, e^{-g^2 c_\alpha} = \exp\left[\sum_\alpha \mathrm{Re}\frac{1}{g_\alpha^2}\chi_\alpha(u_p)\right]$$

- Inserting $U_k(\tilde{x})$ = changing weight of one plaquette $p = \star\,\tilde{x}$:

$$g_\alpha^2 \to e^{\frac{2\pi i k}{N} n_\alpha} g_\alpha^2$$

n_α = N-ality

- Clustering arguments then imply

$$\tilde{Z}(g^2, k) = e^{-A\tilde{F}(k)} = e^{-A(F+\frac{1}{A}f(k))} \Rightarrow \langle U_k(\tilde{x})\rangle = e^{-f(k)}$$

where f(k) has a nice 1/N expansion, akin to free energy F

Backup: rescaling

- We could try defining $V_k(x) \equiv \dfrac{U_k(x)}{|\langle U_k(x)\rangle|}$, which is forced to have a unit VEV both in YM and in large N QCD.
  - Immediate conflict with $\mathbb{Z}_N$ fusion rule is avoided.
- But at large N these operators are quite singular. Correlation functions do not satisfy cluster decomposition!
  - At least for $\kappa \ll 1$, $\langle V_k(x)^\dagger V_k(0)\rangle$ with $k \sim N$ diverges for any separation $r \sim N^0$, only decays once $r \gtrsim \sqrt{N}$.
- Can't interpret $V_k(x)$ as generators of a $\mathbb{Z}_N$ 1-form symmetry.