Recent progress on complex Langevin simulations of QCD at finite density

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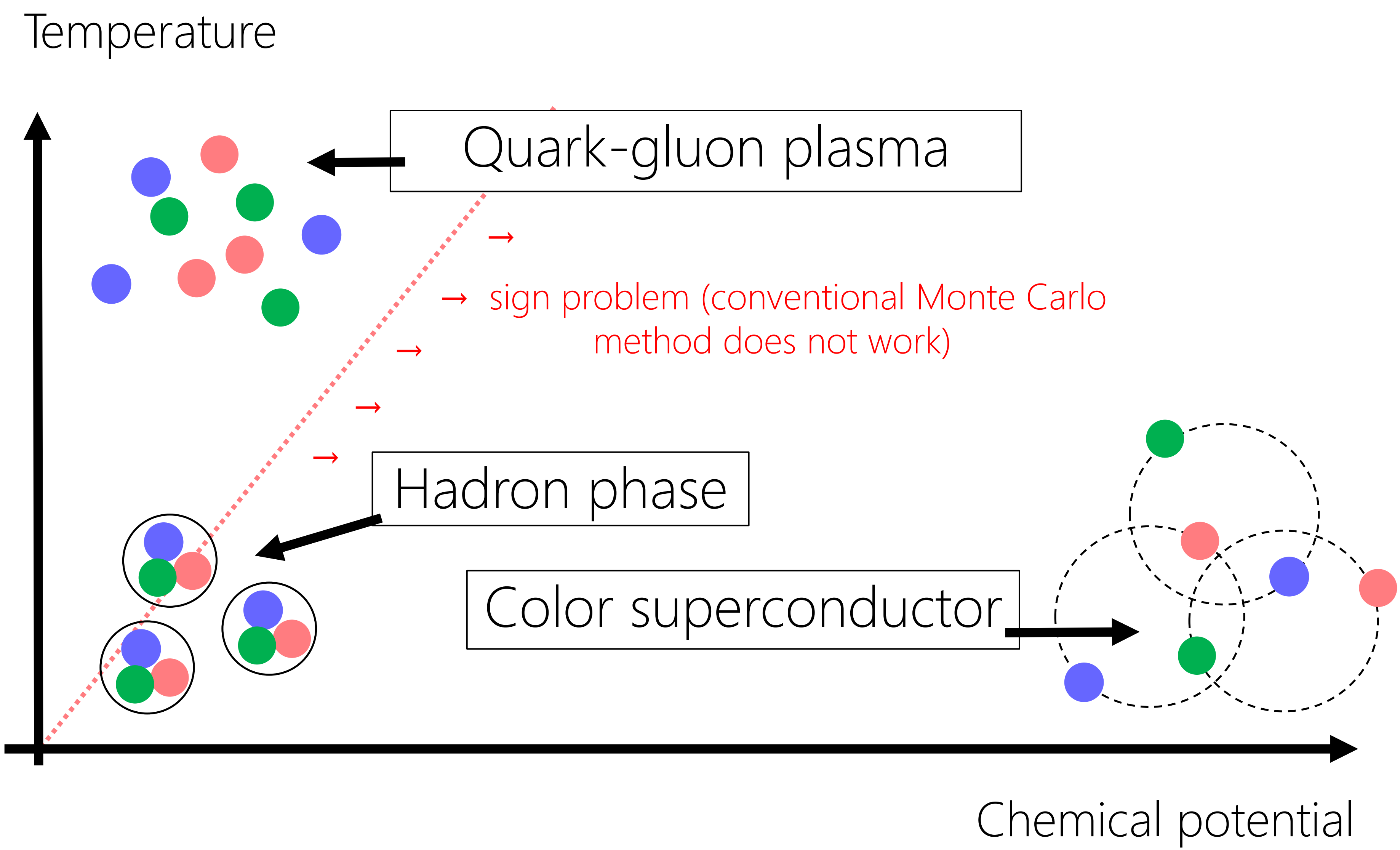
Outline

- Introduction
- Validity of the Complex Langevin method
- Color superconductivity on the lattice

QCD phase diagram



QCD action and the sign problem

$$Z = \int \prod_{x,\nu} dU_{x,\nu} \, \det M(U;\mu) e^{-S_{\mathrm{g}}[U]}$$

Plaquette action

$$\det M^{\dagger}(\mu) = \det M(-\mu)$$

The fermion matrix is complex when $\mu \neq 0$

Langevin method (stochastic quantization)

Let us consider the fictitious time evolution described by

$$\frac{d\phi}{dt} = -\frac{\partial S_{\mathrm{eff}}(\phi)}{\partial \phi} + \eta$$

White noise

Equilibrium distribution of the field ϕ is the Boltzmann weight thanks to Fokker-Planck eq.

$$P_{\mathrm{eq}}(\phi) \propto e^{-S_{\mathrm{eff}}(\phi)}$$

$$\langle O(\phi) \rangle = \lim_{s\to\infty} \frac{1}{s} \int_{t_0}^{t_0+s} dt \langle O(\phi^{(\eta)}(t)) \rangle_\eta$$

Parisi, Wu, Sci. Sinica 24 (1981) 483

Complex Langevin method (CLM)

Let us consider the fictitious time evolution described by

$$\frac{d\phi}{dt} = -\frac{\partial S_{\mathrm{eff}}(\phi)}{\partial \phi} + \eta$$

White noise

for the complex action. The field variable is also extended to complex field $\phi = \phi_{\mathrm{R}} + i\phi_{\mathrm{I}}$.

$$P_{\mathrm{eq}}(\phi_{\mathrm{R}}, \phi_{\mathrm{I}}) \propto \text{some complicated form}$$

$$\langle O(\phi) \rangle \stackrel{?}{=} \lim_{s\to\infty} \frac{1}{s} \int_{t_0}^{t_0+s} dt \langle O(\phi^{(\eta)}(t)) \rangle_{\eta}$$

Parisi, Phys. Lett. 131B (1983) 393, Klauder PRA 29 (1984) 2036

Justification of the CLM

If $P_{\rm eq}$ or $\frac{\partial S_{\rm eff}}{\partial \phi}$ has "good" properties,

$$\int \mathcal{D}\phi_{\rm R} \mathcal{D}\phi_{\rm I} O(\phi_{\rm R} + i\phi_{\rm I}) P_{\rm eq}(\phi_{\rm R}, \phi_{\rm I}) = \frac{1}{Z} \int \mathcal{D}\phi O(\phi) e^{-S_{\rm eff}(\phi)}$$

Obtained by complex Langevin — Original path integral

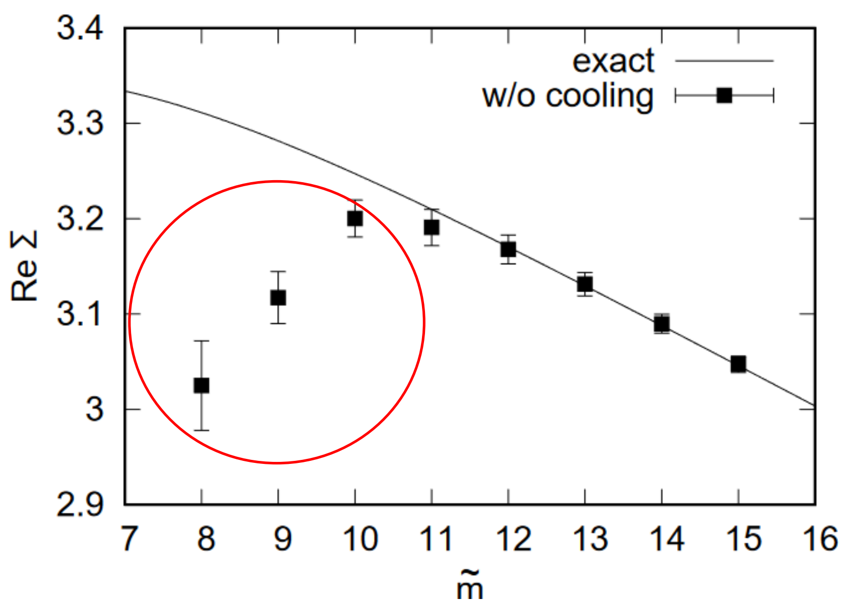
Aarts, Seiler, Stamatescu, PRD 81 (2010) 054608

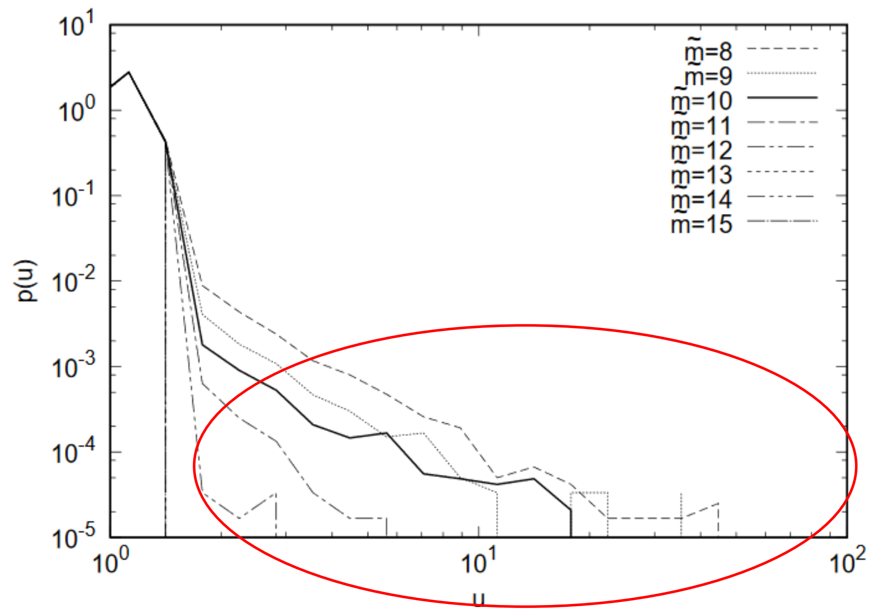
Aarts, James, Seiler, Stamatescu, EPJ C71 (2011) 1756

Nagata, Nishimura, Shimasaki, PRD 92 (2015) 011501, PTEP 2016 013B01

Condition for justification

Demonstration in chiral random matrix model





[Nagata, Nishimura, Shimasaki, JHEP05(2018)004]

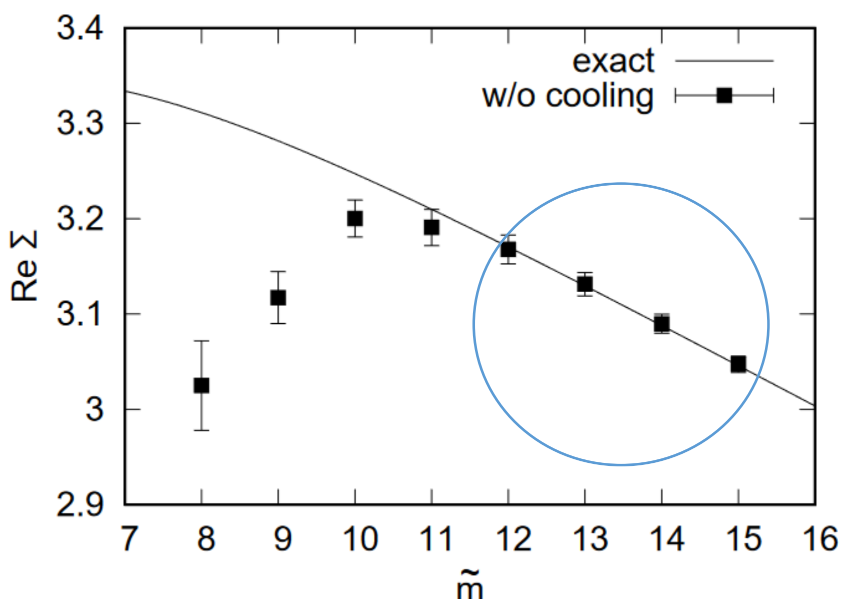
Probability distribution of the drift term $\dfrac{\partial S_{\rm eff}}{\partial \phi}$ shows power-law

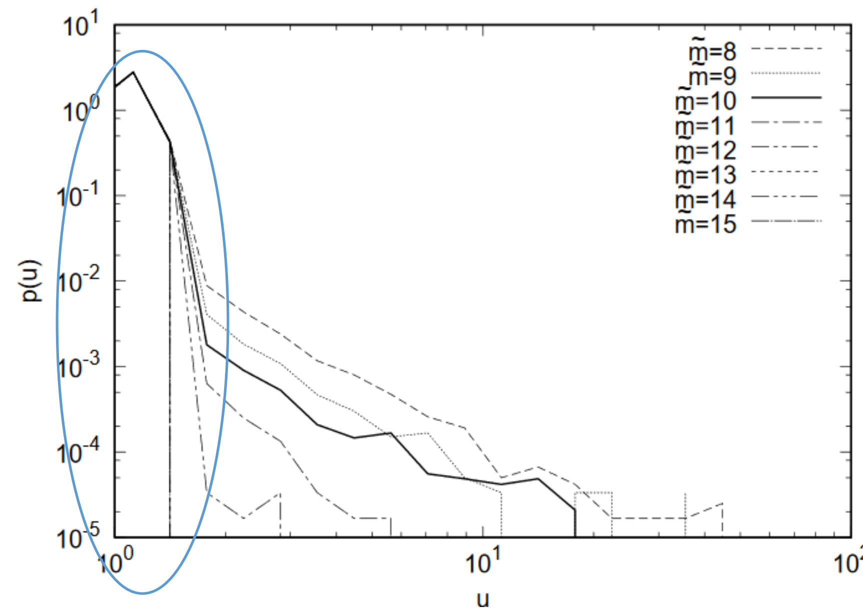
→ CLM fails

Nagata, Nishimura, Shimasaki, PRD 92 (2015) 011501, PTEP 2016 013B01

Condition for justification

Demonstration in chiral random matrix model





[Nagata, Nishimura, Shimasaki, JHEP05(2018)004]

Probability distribution of the drift term shows $\frac{\partial S_{\rm eff}}{\partial \phi}$ exponential fall-off

→ CLM gives correct answer

Validity of CLM for the finite density QCD

Complex Langevin equation for QCD

$$\mathcal{U}^{(\eta)}_{x,\nu}(t+\epsilon) = \exp\left[i\left(-\epsilon v_{x,\nu}(\mathcal{U}^{(\eta)}(t)) + \sqrt{\epsilon}\eta_{x,\nu}(t)\right)\right]\mathcal{U}^{(\eta)}_{x,\nu}(t)$$

SL(3,C) link variable

The drift term consists of two parts:

$$v = v_{\text{gauge}} + v_{\text{fermi}}$$

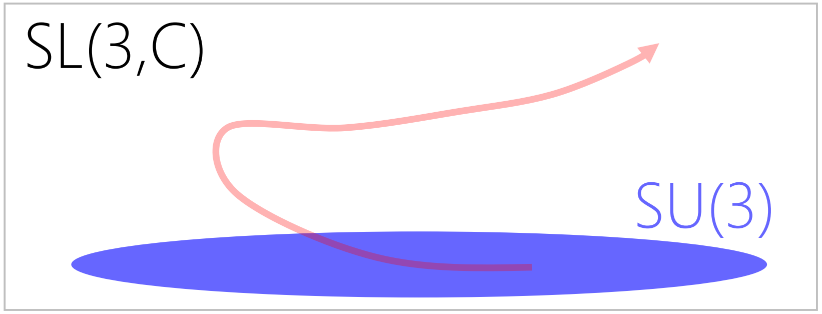
$$v_{\text{gauge}} = \sum_a \lambda_a \, \frac{d}{d\alpha} S_{\text{g}}[e^{i\alpha\lambda_a}\mathcal{U}]\Big|_{\alpha=0},$$

$$v_{\text{fermi}} = \sum_a \lambda_a \, \frac{d}{d\alpha}\left(-\log\det M(e^{i\alpha\lambda_a}\mathcal{U})\right)\Big|_{\alpha=0}.$$

Origins of power-law behavior

If $P(v_{\text{gauge}})$ shows power-law behavior

→ Excursion problem



Suppressed by gauge cooling
(Seiler, Sexty, Stamatescu (2013))

If $P(v_{\text{fermi}})$ shows power-law behavior

→ Singular drift problem

$$v_{\text{fermi}} \propto \frac{1}{\det M}$$

When the CLM is valid?

= When the probability distribution of the drift term shows exponential fall off ?

- Is it correlated to the severeness of the sign problem ?
  In particular, is it possible to simulate $\mu/T > 1$ region ?

- Depend on lattice setups ?

When the CLM is valid?

= When the probability distribution of the drift term shows exponential fall off ?

- Is it correlated to the severeness of the sign problem ?
  In particular, is it possible to simulate $\mu/T > 1$ region ?

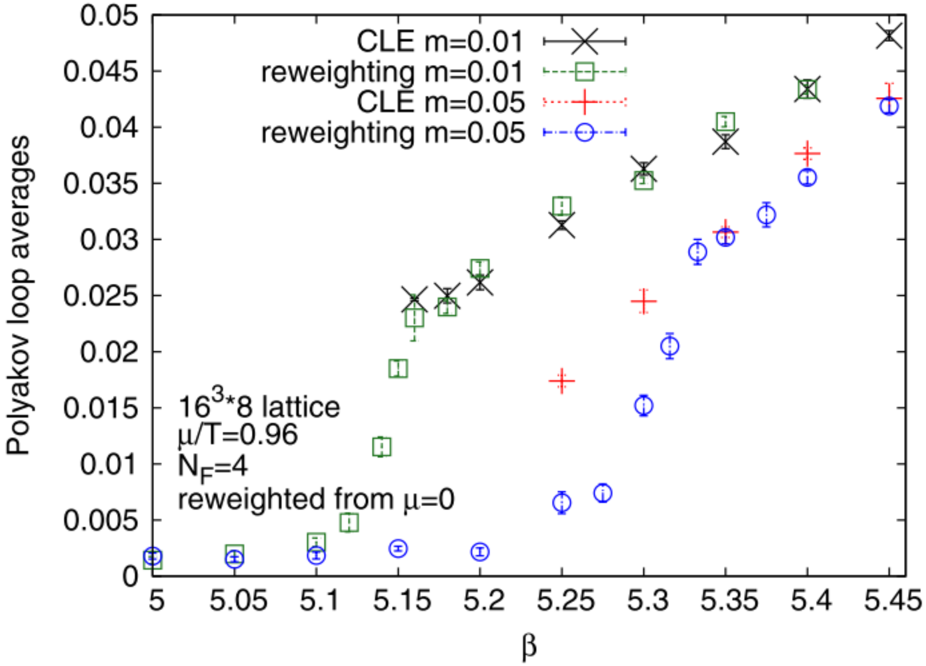
  - $\mu/T > 1$ is possible. Validity below T_c is an open problem.

- Depend on lattice setups ?

  - Large β (fine lattice), large T, small aspect ratio are favorable

Previous work

Fodor, Katz, Sexty, Török, PRD92 094516 (2015)



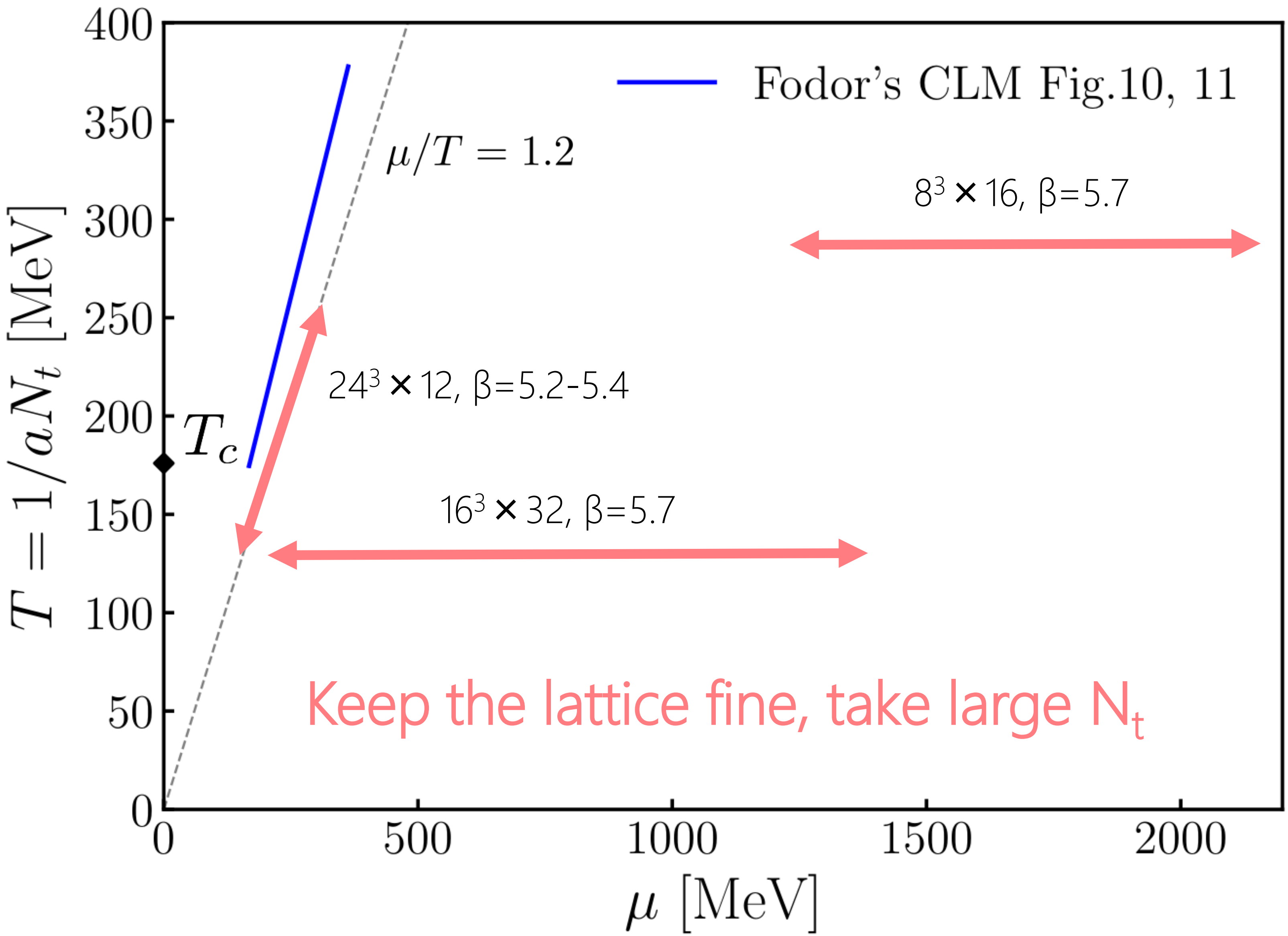
- $16^3 \times 8$ lattice
- N_f=4 Staggered fermion
- $\mu/T \sim 1$

They study the validity of the CLM by comparing the CLM and reweighting.

Conclusion here:

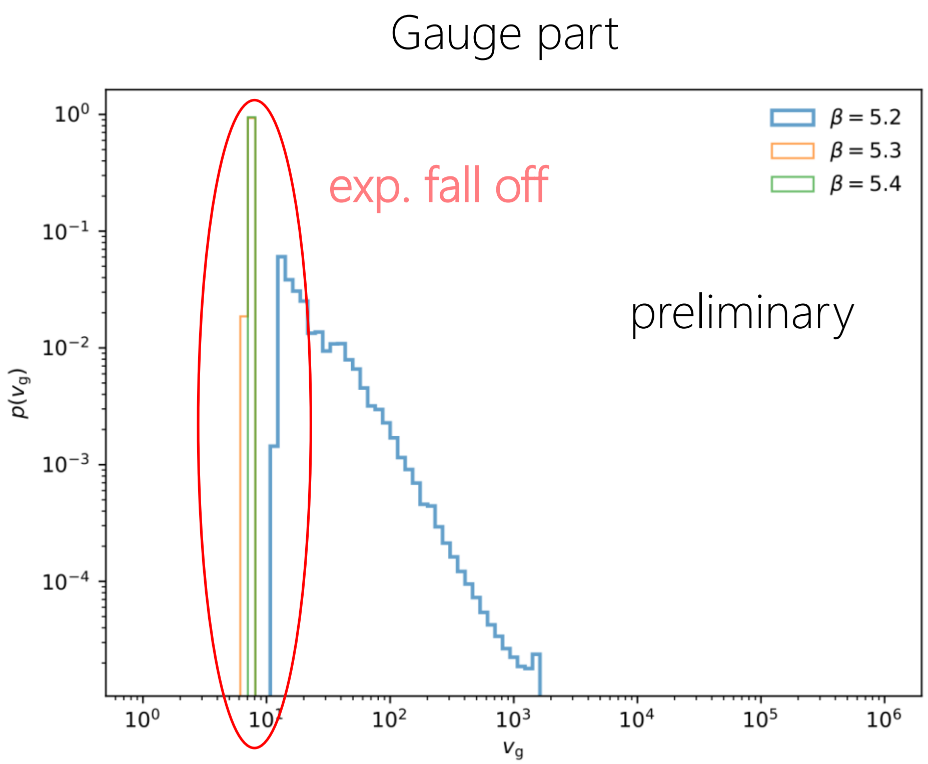
- For too small β (large lattice spacing), gauge cooling is not effective (~ excursion problem)
- Cannot explore deconfinement transition

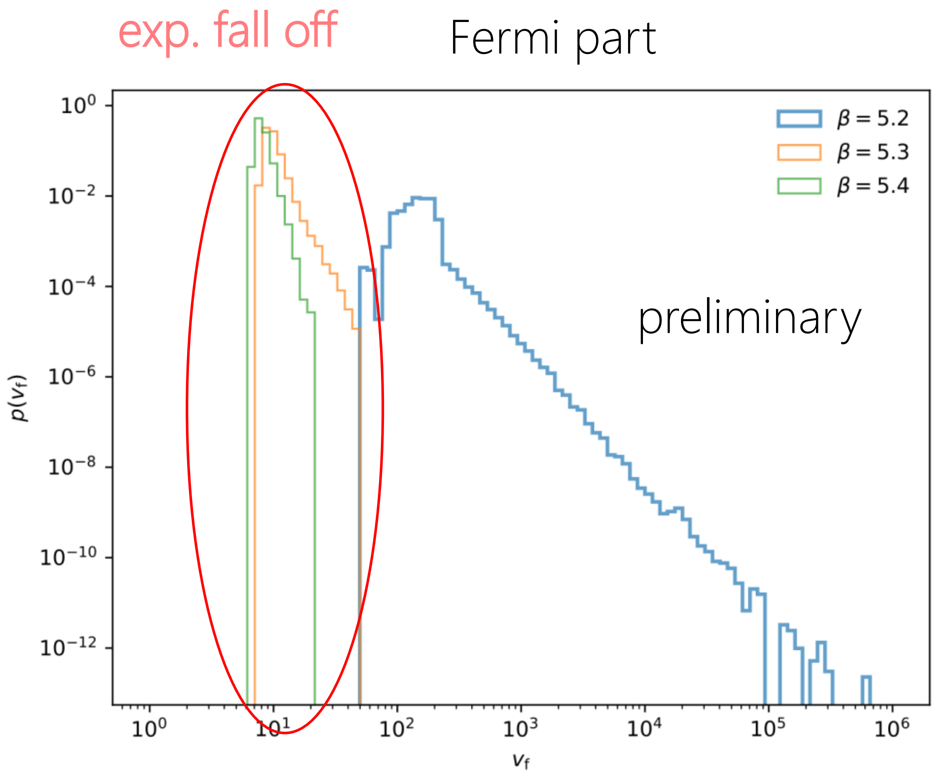
Our works



Fodor's CLM Fig.10, 11
$\mu/T = 1.2$
$8^3 \times 16$, β=5.7
$24^3 \times 12$, β=5.2-5.4
$16^3 \times 32$, β=5.7
T_c
Keep the lattice fine, take large N_t
$T = 1/aN_t$ [MeV]
μ [MeV]
0
50
100
150
200
250
300
350
400
0
500
1000
1500
2000

N_f=4 staggered, $24^3 \times 12$, μ/T=1.2



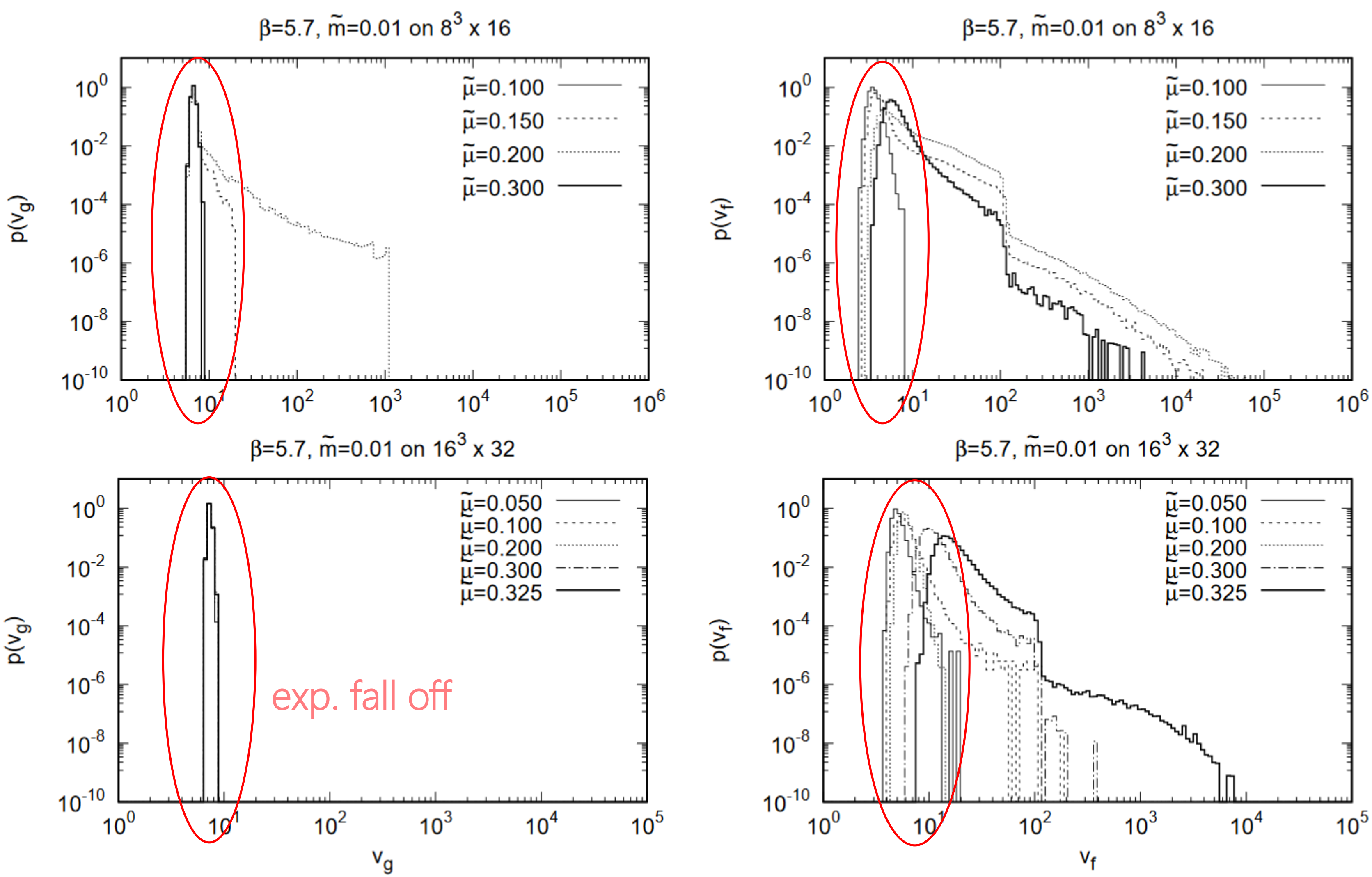


$$v_{\text{fermi}} \propto \frac{1}{\det M}$$

- Exp. behavior at β=5.3, 5.4, power law behavior at β=5.2
- Singular drift problem also occurs at β=5.2
- Since the fermion part of the drift term is related to the Dirac spectrum through the generalized Banks-Casher relation, chiral symmetry breaking may affect this behavior. On the other hand, the current simulation is away from the chiral limit (m_π >520 MeV). This interpretation is not trivial.

N_f=4 staggered, $8^3\times16$ and $16^3\times32$

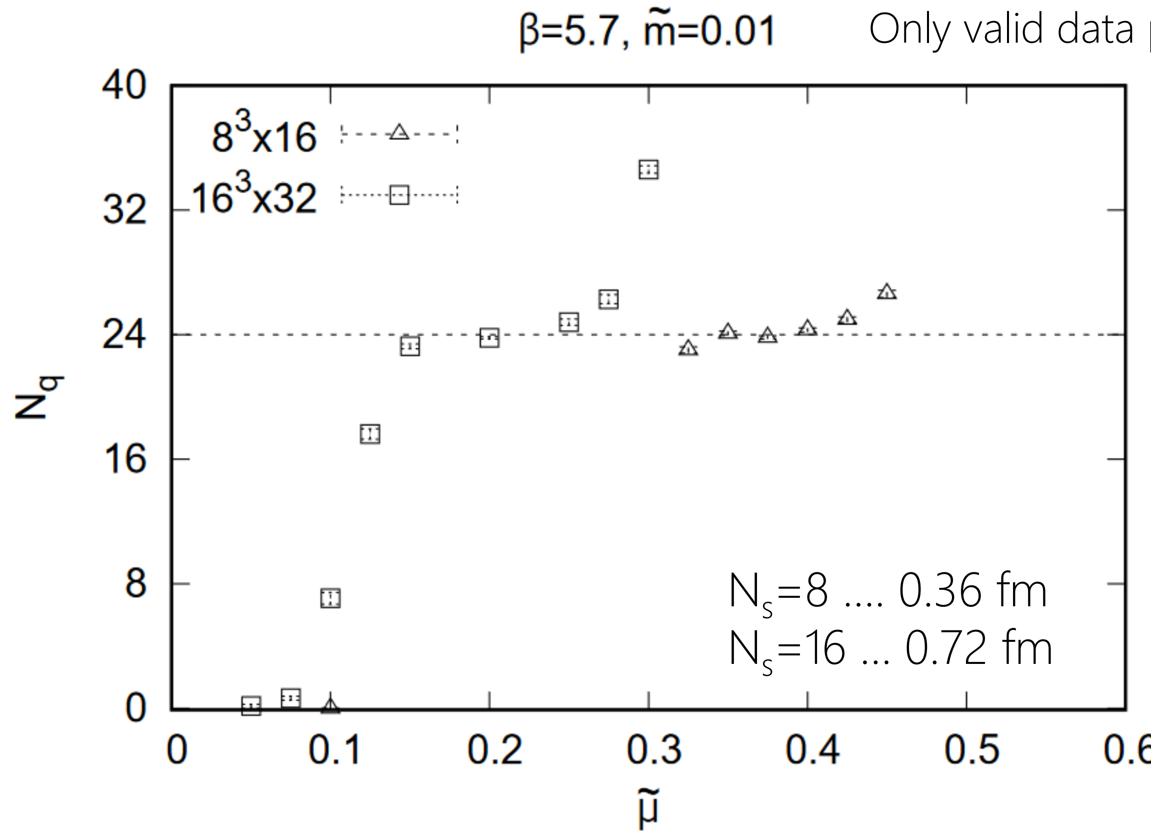
Gauge part | Fermi part



Ito, Matsufuru, Namekawa, Nishimura, Shimasaki, Tsuchiya, ST (2020)

Quark number vs chemical potential

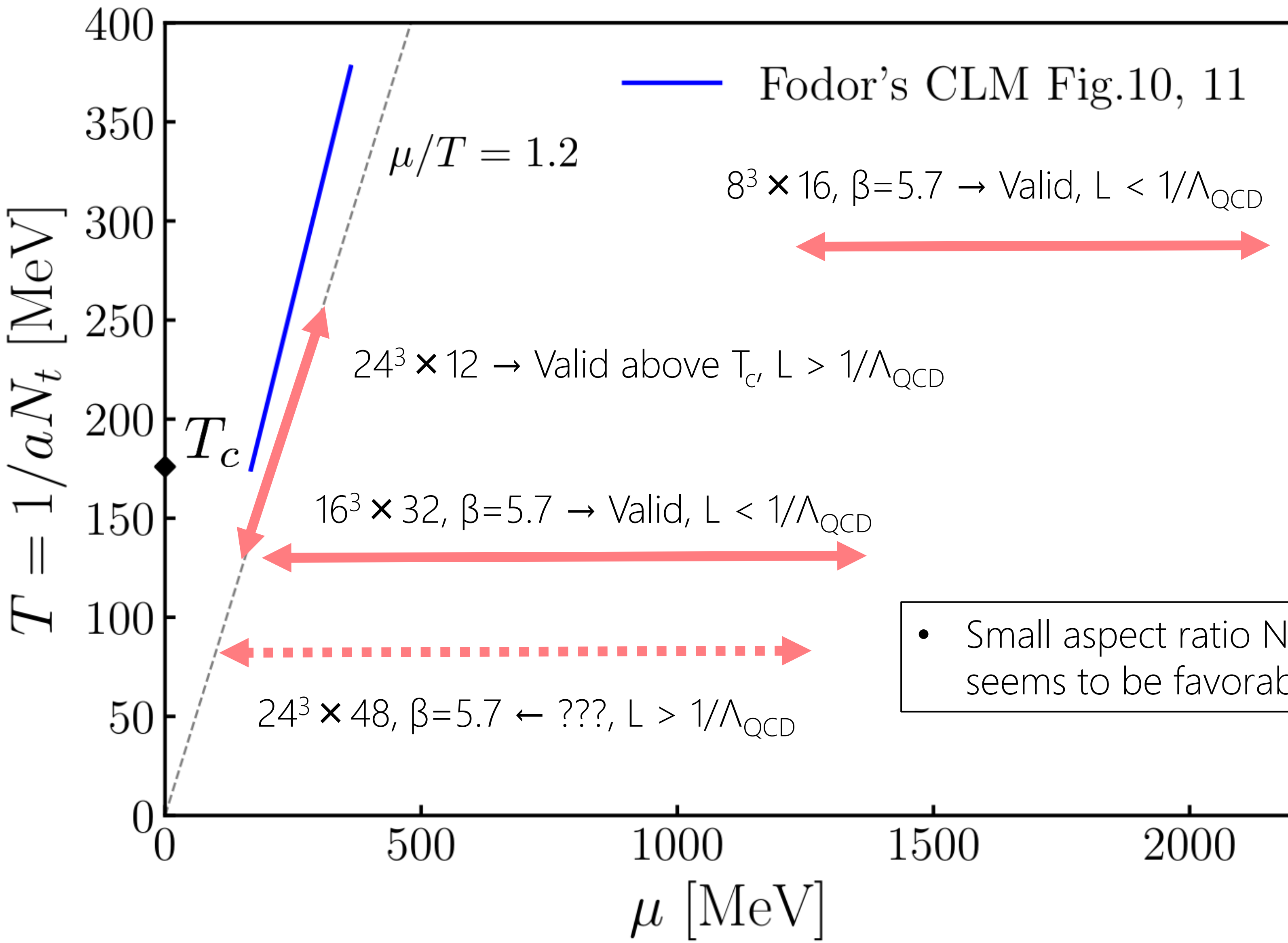
Only valid data points are plotted



- The energy of quarks is discretized as $E_{\vec{n}} = \sqrt{(2\pi\vec{n}/L)^2 + m_{\rm eff}^2}$
- The height of the plateau = the degeneracy of the first energy level: 4(flavor) × 3(color) × 2(spin) = 24

Ito, Matsufuru, Namekawa, Nishimura, Shimasaki, Tsuchiya, ST (2020)

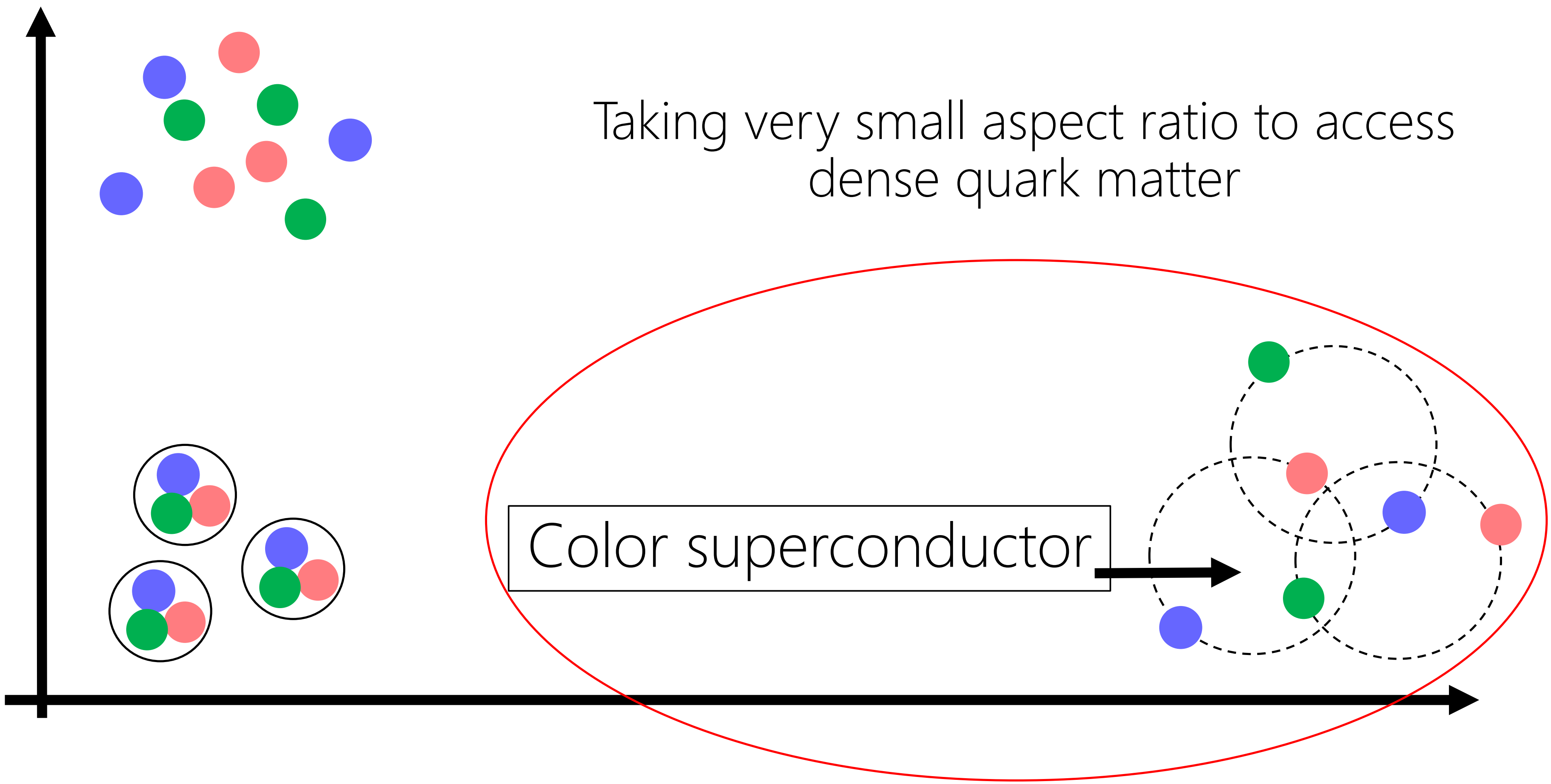
Short summary



- Small aspect ratio $N_s/N_t < 1$ seems to be favorable.

Color superconductivity on the lattice

Color superconductivity (CSC)

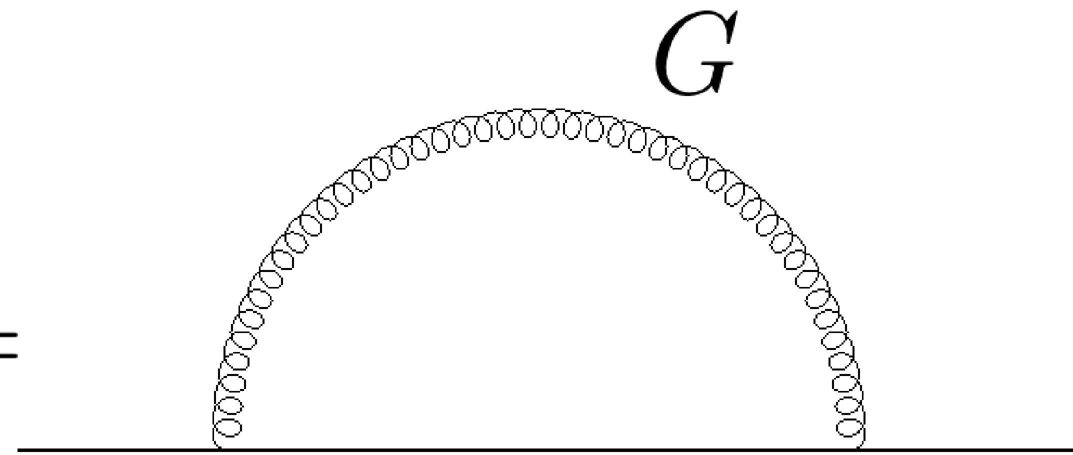


Barrois, NPB (1977), Frautschi (1978), Bailin, Love, Phys. Rept. (1984)
Alford, Rajagopal, Wilczek, PLB (1998), Rapp, Schäfer, Shuryak, Velkovsky, PRL (1998)

Gap equation

Anomalous propagator

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$



2 × 2 self energy matrix in Nambu basis

$$S = (\mathbf{D} + \Sigma)^{-1}$$

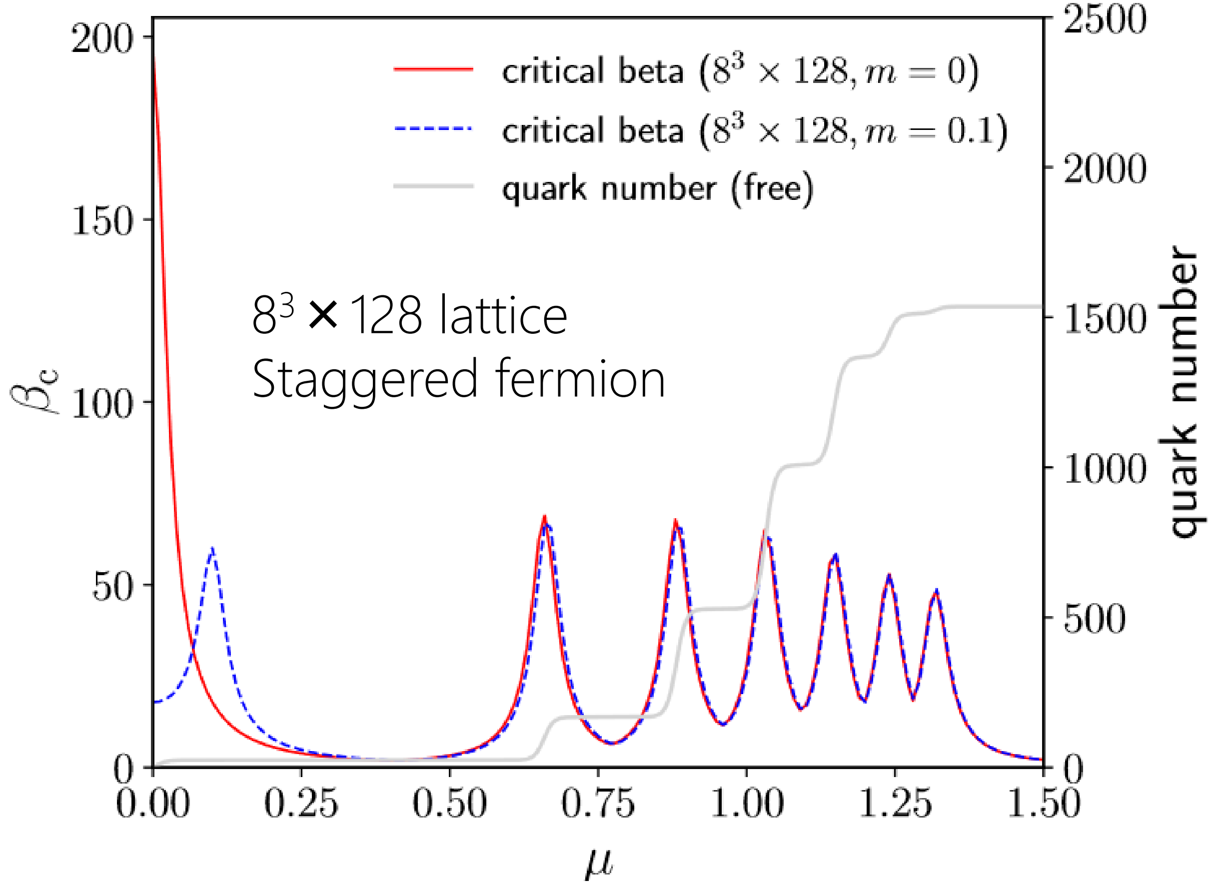
free propagator

$$\Sigma_{12(21)} \neq 0 \Rightarrow \text{CSC}$$

We estimate critical β from this condition **on a lattice** assuming $\beta = 6/g^2 >> 1$

Result of lattice perturbation theory

Yokota, Asano, Ito, Matsufuru, Namekawa, Nishimura, Tsuchiya ST, PoS lattice 2021



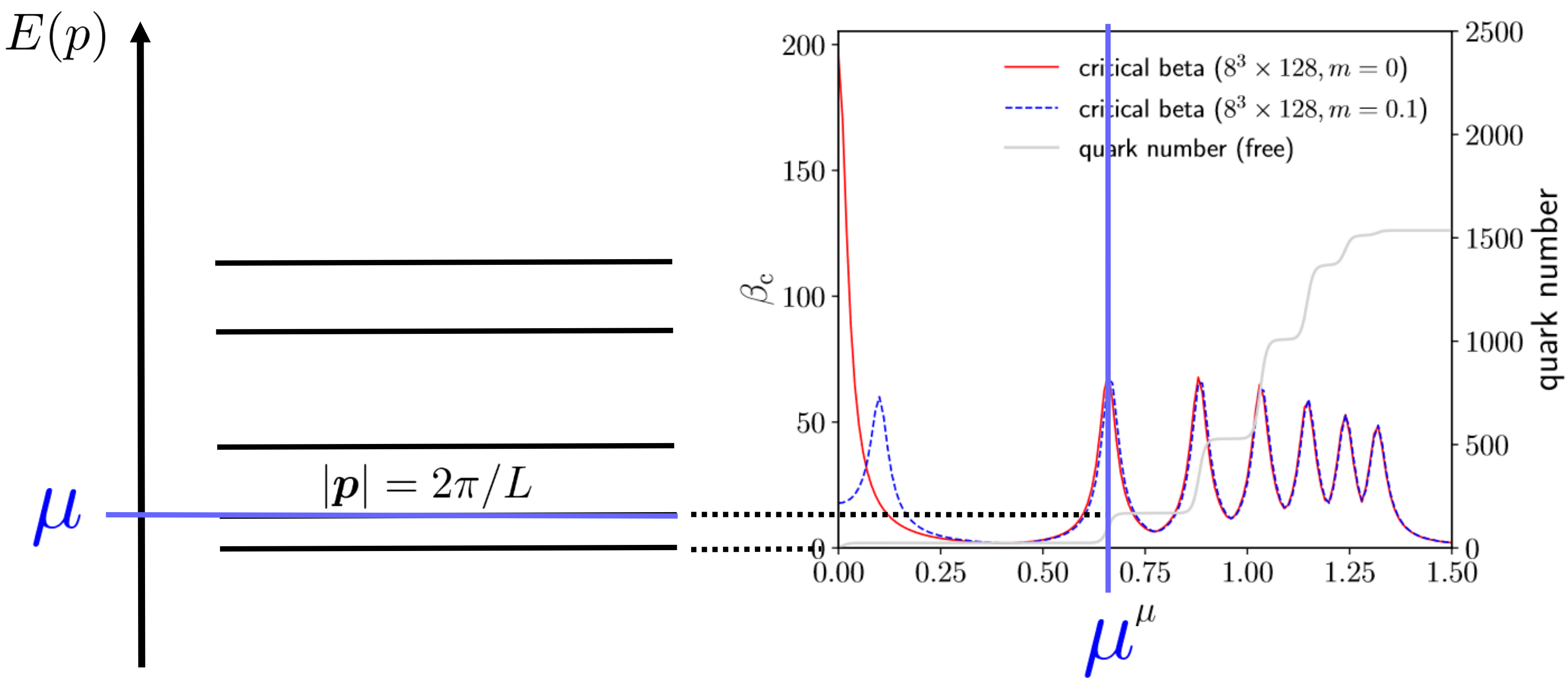
- Low-β region is the Color superconductivity (CSC) phase
- Similar result is obtained for the Wilson fermion

Peak structure

Peak positions correspond to μ at which the quark number changes.

→ At such μ, quarks exist on the Fermi surface.

→ Cooper pairs are easy to form



Order parameter of CSC (O_{CSC})

O_{CSC}

$$O_{\mathrm{CSC}} = -\sum_x \varphi_a^\dagger(x)\varphi_a(x)$$

Quark pair

$$\varphi_a(x) = \epsilon_{abc}\mathrm{tr}(C^{-1}\Psi_b^T(x)C\Psi_c(x))$$

4-flavor Dirac field consists of staggered fermion χ

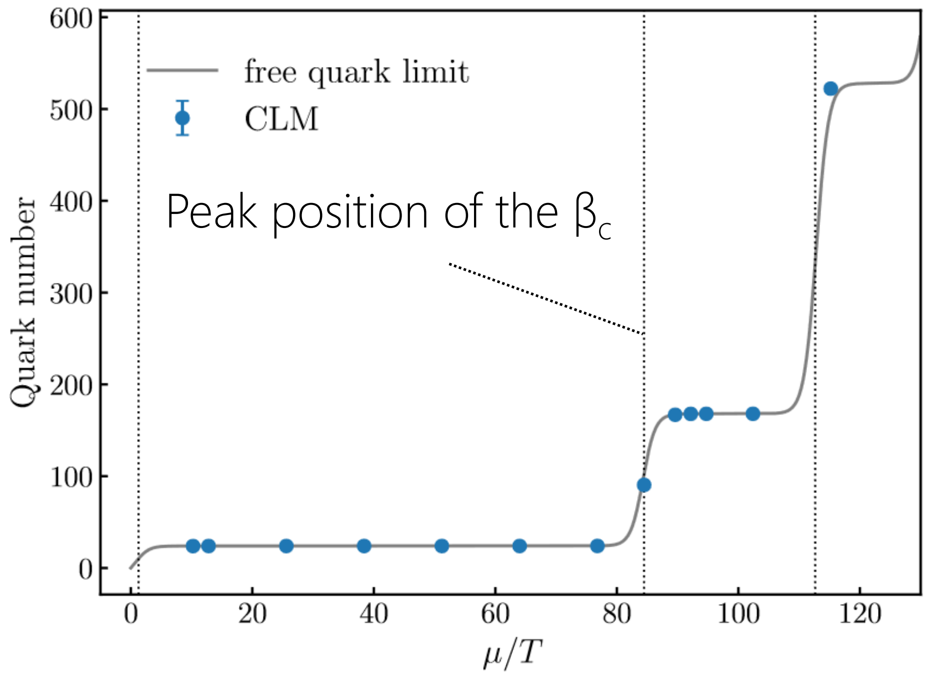
$$\Psi(x) = \sum_{A=0,1}(\gamma_1)^{A_1}(\gamma_2)^{A_2}(\gamma_3)^{A_3}(\gamma_4)^{A_4}\chi(x+A)$$

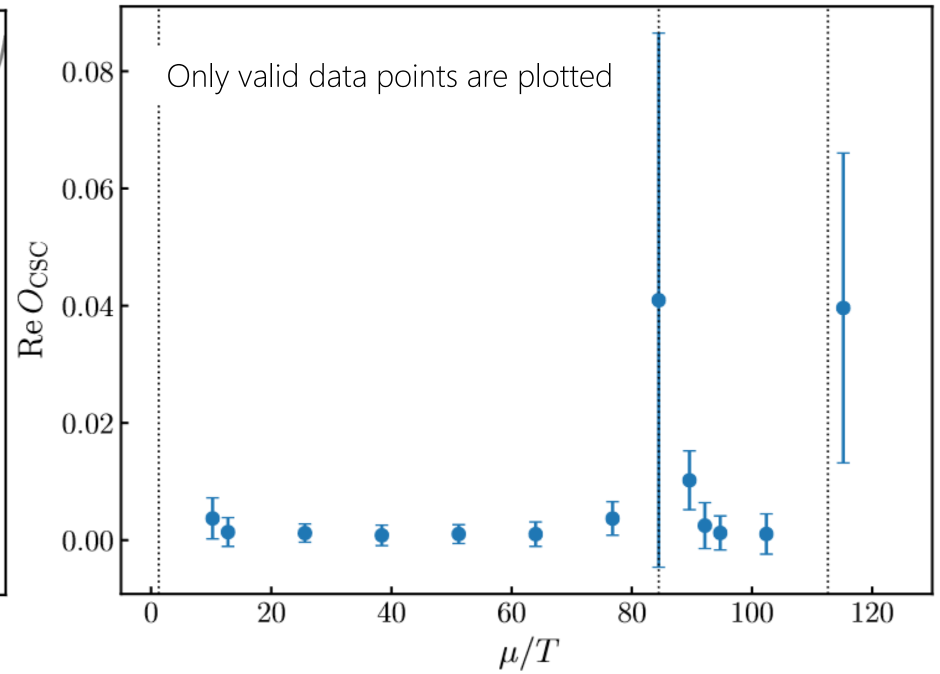
The trace is calculated by the U(1)-noisy estimator (*)

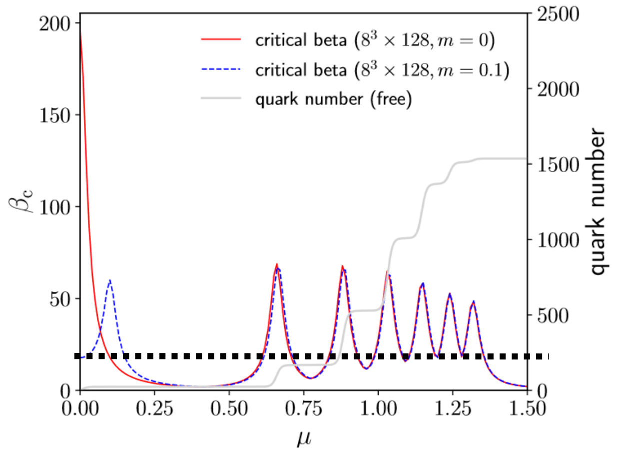
(*) Standard gaussian noisy estimator cannot be applied to calculate 4-point functions.

CLM study (β=20, staggered fermion)

ST, Asano, Ito, Matsufuru, Namekawa, Nishimura, Tsuchiya Yokota, PoS lattice 2021



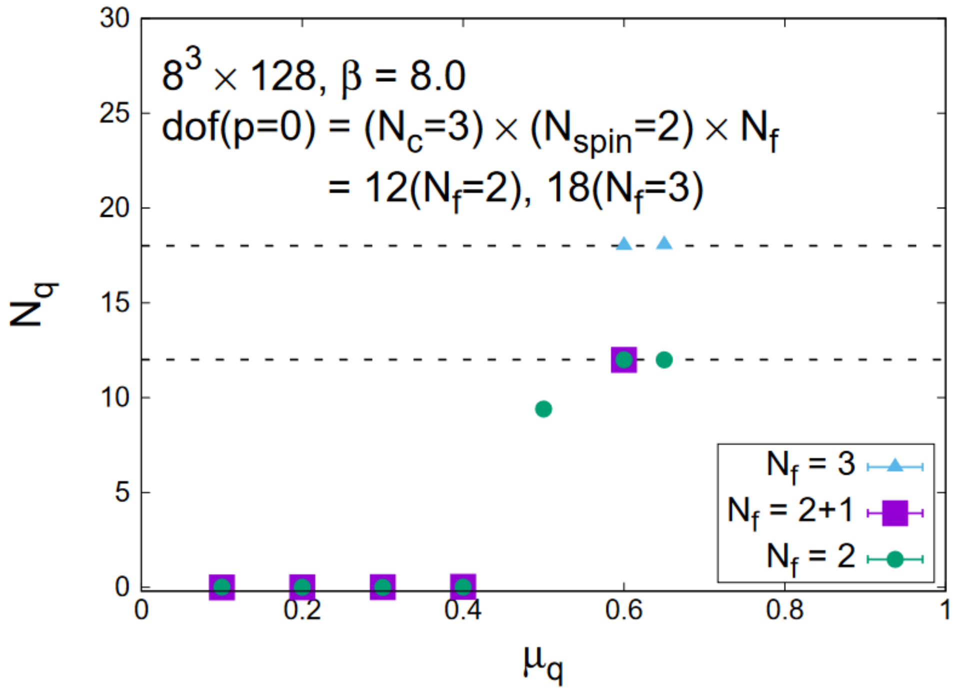




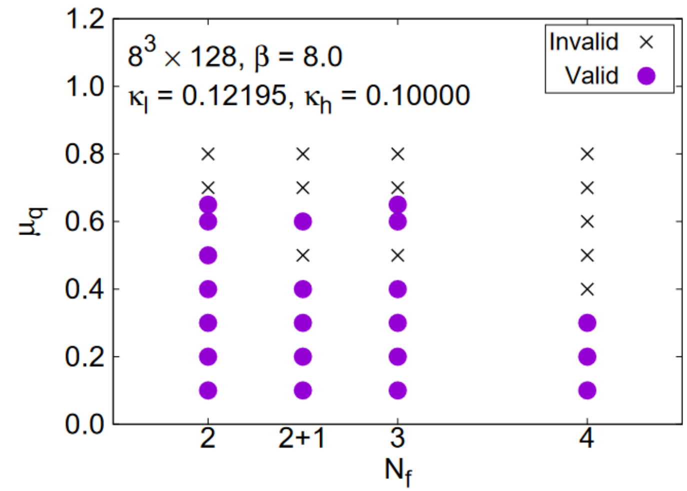
- Fermi sphere is developing
- Order parameter fluctuates violently around the peak position of the critical β. (Part of error is coming from the noisy estimator)

CLM study (β=8, Wilson fermion)

Namekawa, Asano, Ito, Matsufuru, Nishimura, Tsuchiya, ST, Yokota, PoS lattice 2021



Flavor dependence



- Similar result to the staggered case
- For a realistic flavor choice, broader validity region is found than N_f =4 case.

Summary

- Validity region of the CLM for finite density QCD
  - For $24^3 \times 12$ lattice, the singular drift problem is occurred below T_c.
  - For $16^3 \times 32$ lattice, CLM is valid up to μ~1.4GeV.
- Color superconductivity
  - A CSC is predicted for a lattice with small aspect ratio.
  - Cooper pair formation is suggested at which the quark number jumps
  - CLM may be valid around the CSC phase.
- Work in progress
  - CLM on $24^3 \times 48$ lattice to study low temperature region
  - Measuring the order parameter of CSC