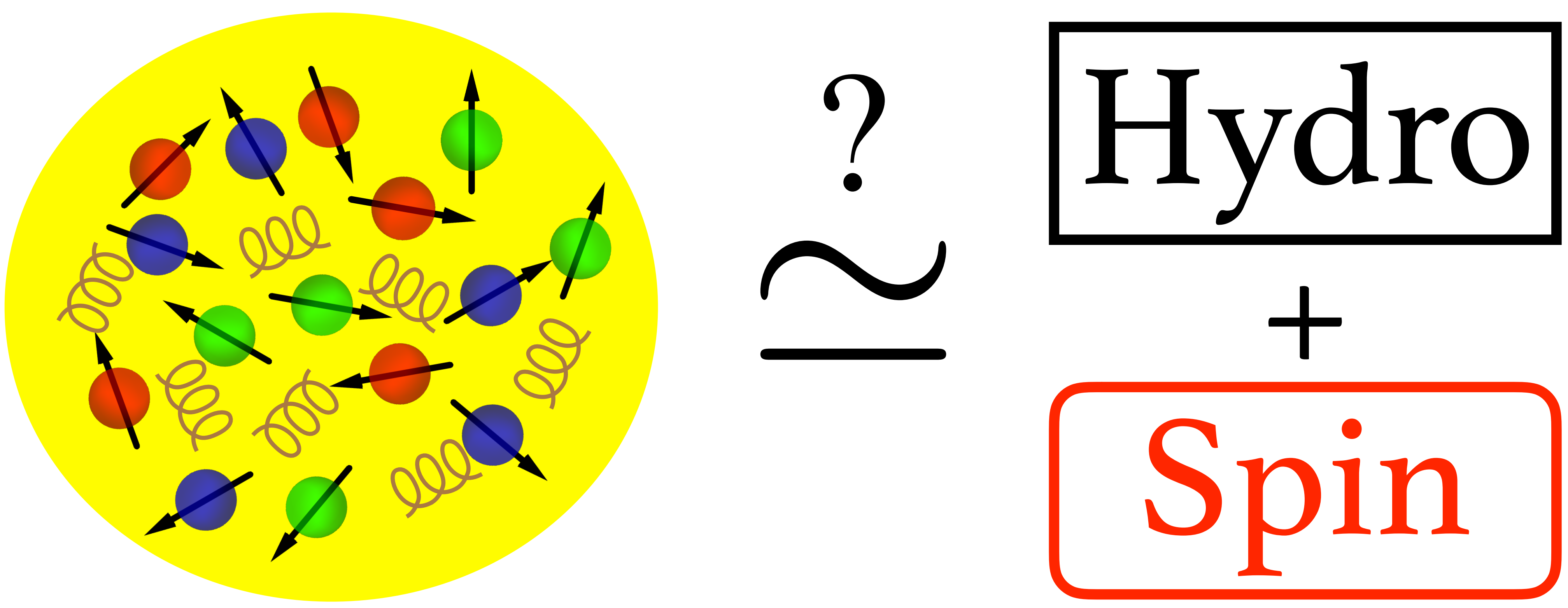
Relativistic spin hydrodynamics with torsion and linear response theory for spin relaxation



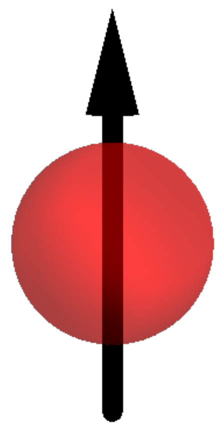
Masaru Hongo (Univ. of Illinois at Chicago)

2021/11/02, QCD theory Seminars

MH-Huang-Kaminski-Stephanov-Yee, arXiv:2107.14231 (to be published in JHEP)

Spin in Hydro?

◆ Spin as a quantum number

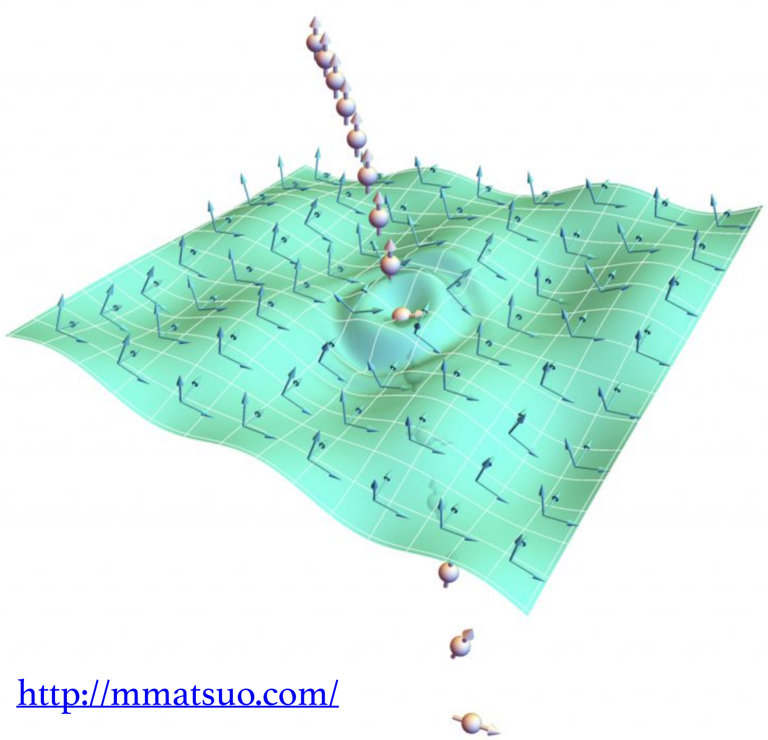


Spin ≠ good quantum # in ~~no~~nrelativistic theory

→ Transport phenomena of spin ?

◆ Where and Why ?

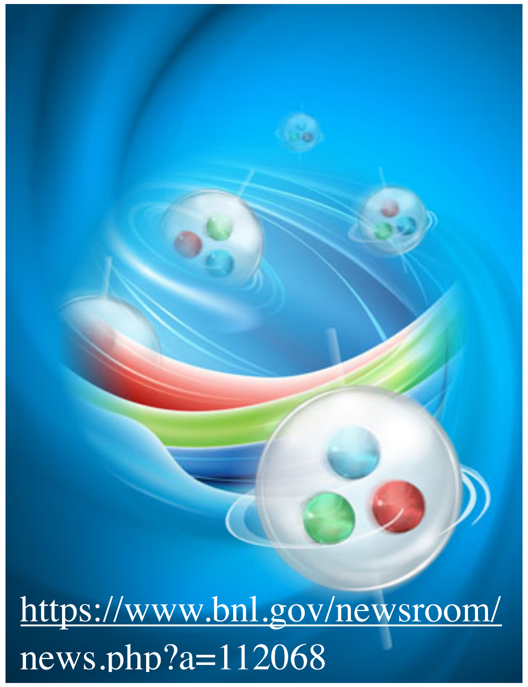
Spintronics



http://mmatsuo.com/

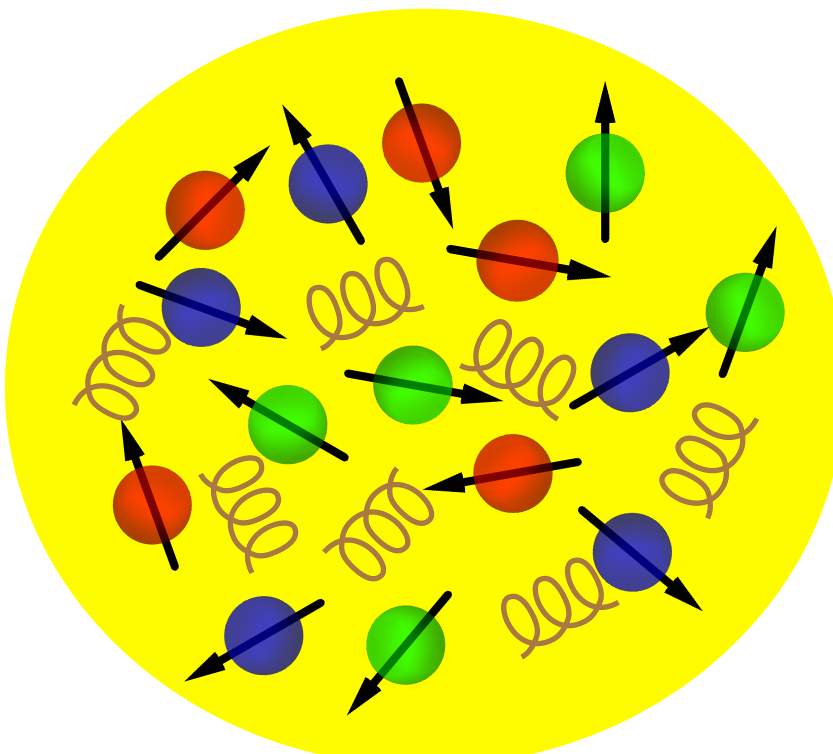
Possibility of
QGP spintronics!?

Heavy-ion collision



One-page Summary

Extending hydrodynamics to include spin



$\simeq$ **Hydro** + **Spin**

Similar to the one discussed in Hattori-MH-Huang-Matsuo-Taya: PLB795,100 (2019)

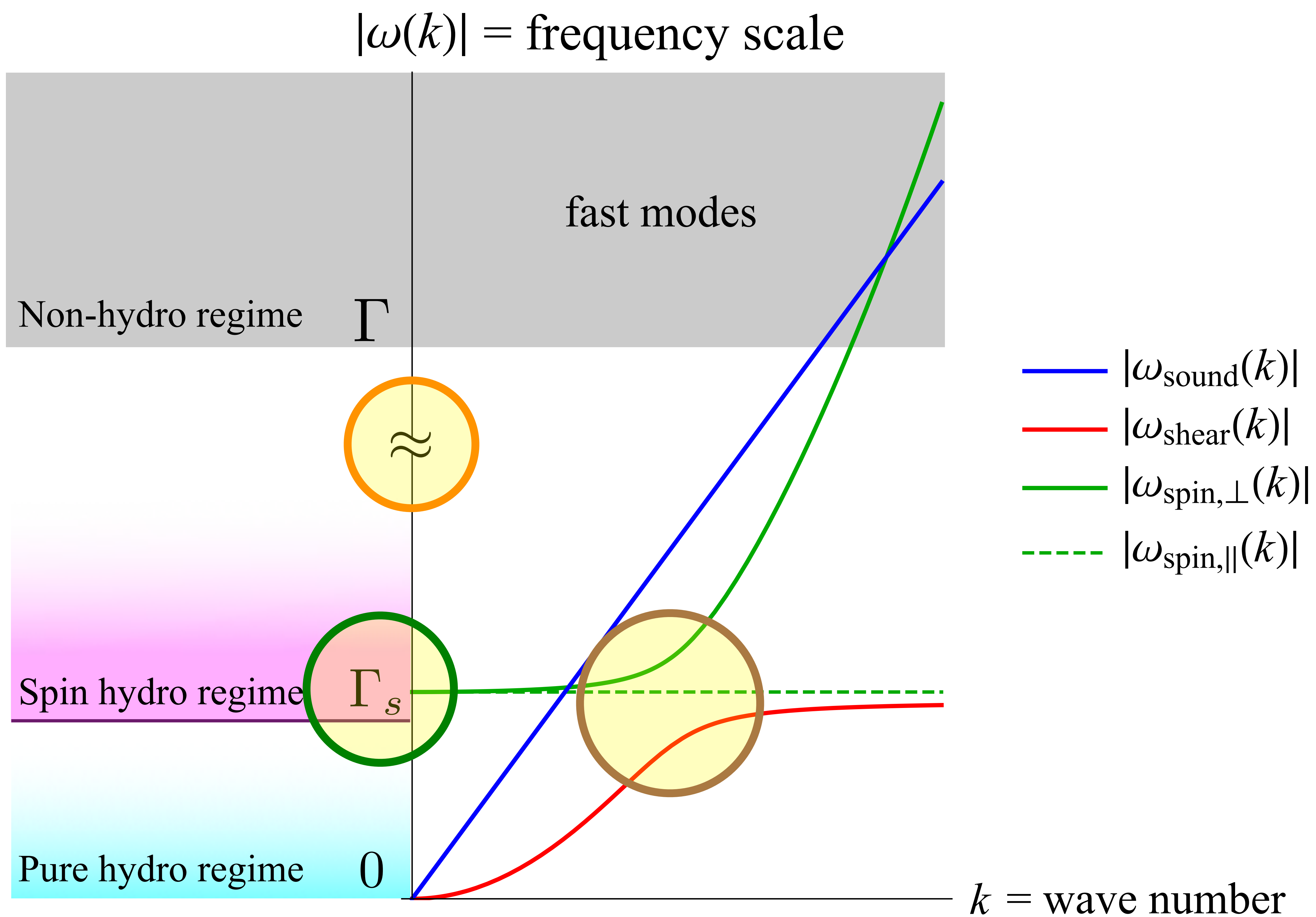
Three main messages from our new paper:

(1) Spin hydrodynamic equations in a **torsionful** geometry

(2) **Mode mixing** between shear and spin modes

(3) **Green-Kubo formula** for a rotational viscosity

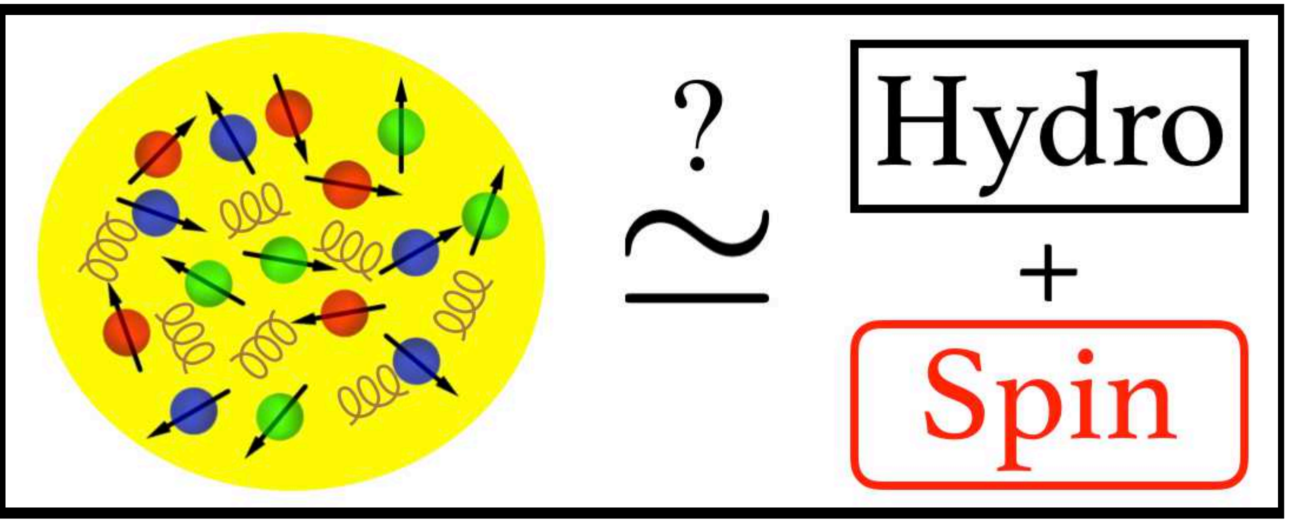
Sketch of our result



Outline

Motivation:

Hydrodynamics of
a relativistic spinful fluid?



Approach:

Semi-phenomenology based on local thermodynamics

Result:

(1) Spin hydrodynamic equations in a torsionful geometry

(2) Mode mixing between shear and spin modes

(3) Green-Kubo formula for a rotational viscosity

Semi-phenomenological derivation of hydrodynamic equation

What is hydrodynamics?

The oldest but state-of-the-art phenomenological field theory

Pascal's law

Hydro*dynamics*

Euler equations
(Perfect fluid)

Navier-Stokes equations
(Viscous fluid)

1600 1700 1800 1900

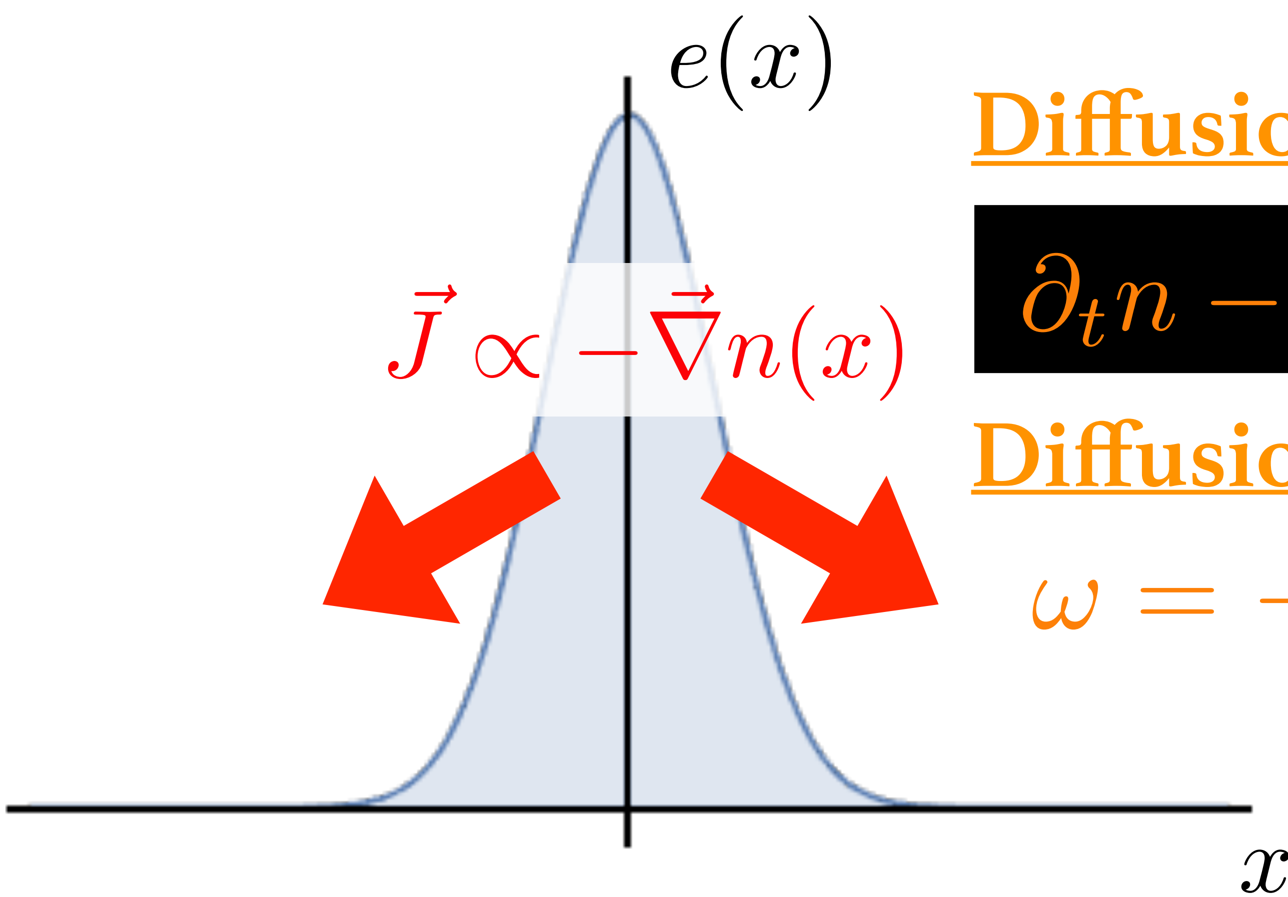
Prototype: Charge diffusion

◆ **Bulding blocks of hydrodynamic equation**

(1) Conservation law: $\partial_t n + \vec{\nabla} \cdot \vec{J} = 0$

(2) Constitutive relation: $\vec{J} = -T\kappa_n \vec{\nabla}(\beta\mu) \simeq -D\vec{\nabla} n$

(3) Physical properties: Values of $\kappa_n, \ \chi_n \quad (D = \kappa_n / \chi_n)$

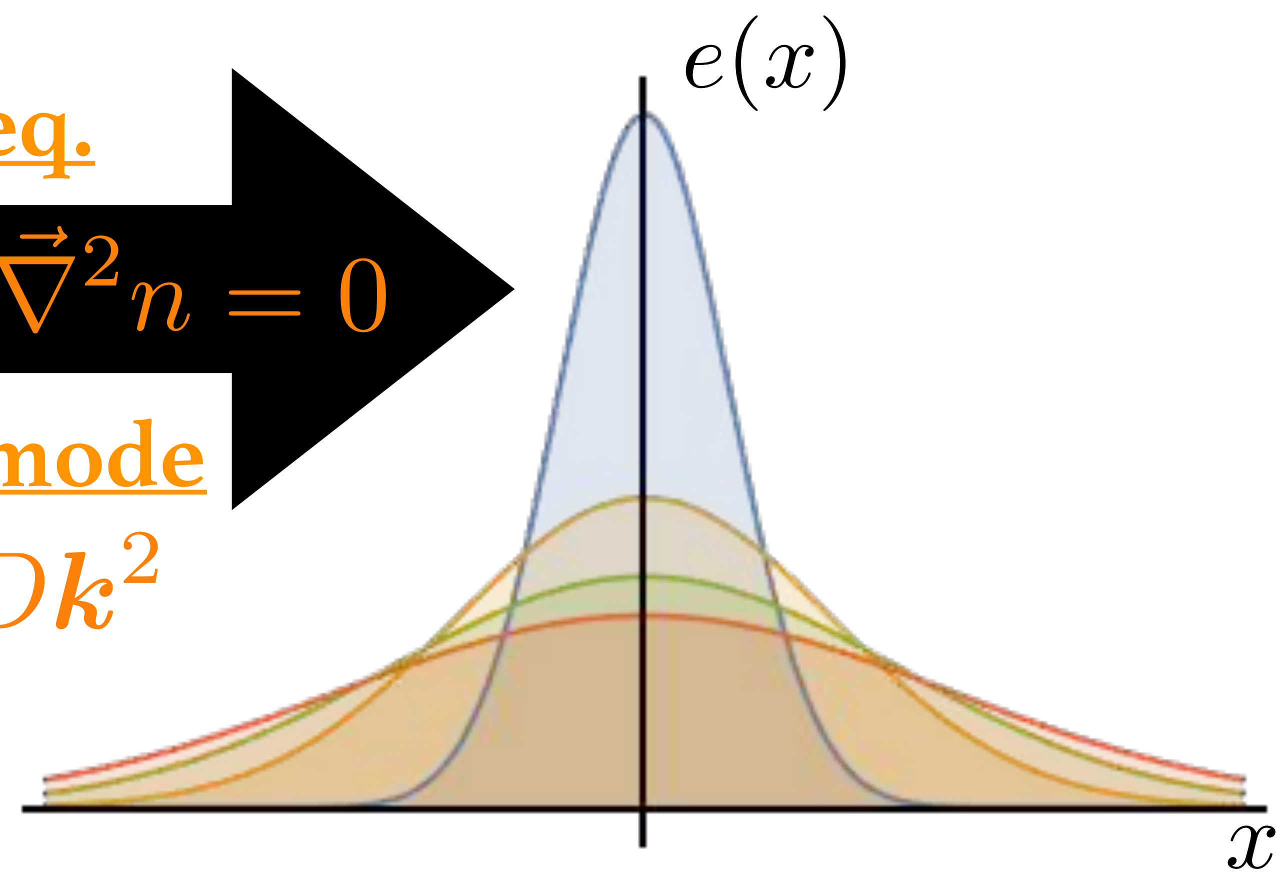


Diffusion eq.

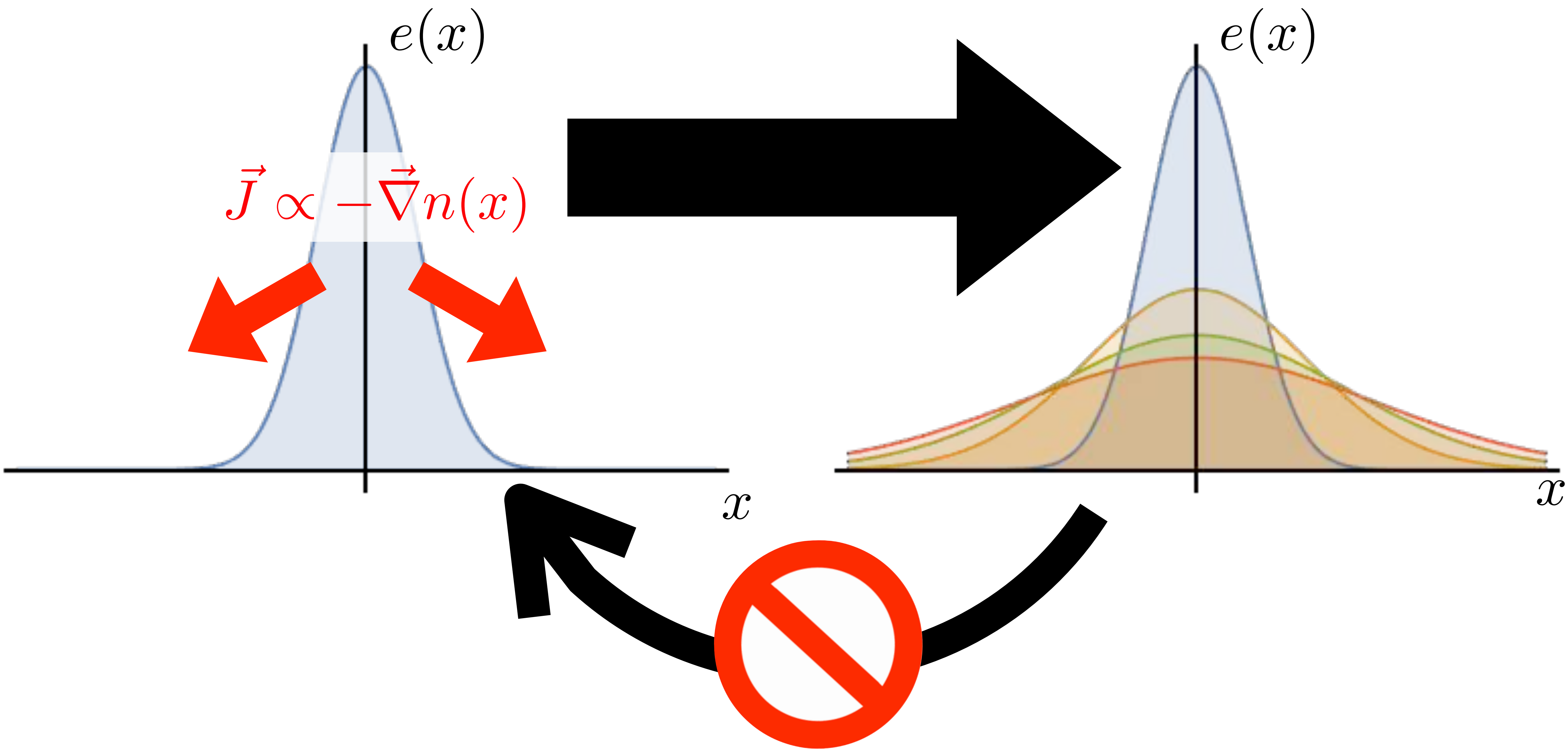
$$\partial_t n - D\vec{\nabla}^2 n = 0$$

Diffusion mode

$$\omega = -\mathrm{i}D\boldsymbol{k}^2$$



Irreversiblity of diffusion



No-go for time-reversal process!

Thermodynamic concepts, especially, 2nd law, should be there!

Phenomenological derivation

Step 1. Determine dynamical d.o.m (& its equation of motion)

Charge density: $n(x)$ EoM: $\partial_t n + \vec{\nabla} \cdot \vec{J} = 0$

Step 2. Introduce entropy & conjugate variable

Entropy density: $s(n)$ $\xrightarrow{T\mathrm{d}s = -\mu \mathrm{d}n}$ Chemical pot.: $\beta\mu \equiv -\frac{\partial s}{\partial n}$

Step 3. Write down all possible terms with finite derivatives

Current: $\vec{J} = 0 - T\kappa_n \vec{\nabla}(\beta\mu) + O(\vec{\nabla}^2) = -T\kappa_n \vec{\nabla} \frac{\partial s}{\partial n} + O(\vec{\nabla}^2)$

Step 4. Restrict terms to be compatible with local 2nd law

$\exists\, s^\mu$ such that $\partial_t s + \vec{\nabla} \cdot \vec{s} \geq 0$ $\Rightarrow$ $\kappa_n \geq 0$ with $\vec{s} = \beta\mu\vec{J}$

First way to determine κ_n

◆ **Linearized constitutive relation**

$$\vec{J} = -T\kappa_n \vec{\nabla}(\beta\mu) \simeq -D\vec{\nabla}n \quad \text{with} \quad D \equiv \frac{\kappa_n}{\chi_n}$$

➡ **Diffusion equation:** $\partial_t n - D\vec{\nabla}^2 n = 0$

➡ **Dispersion relation:** $\omega(\boldsymbol{k}) = -\mathrm{i}D\boldsymbol{k}^2$

◆ **Green's function interpretation of the result**

$$\widetilde{G}_R^{nn}(\omega, \boldsymbol{k}) = \frac{\mathrm{i}\chi_n D\boldsymbol{k}^2}{\omega + \mathrm{i}D\boldsymbol{k}^2} \qquad \left(\chi_n = \lim_{\boldsymbol{k}\to 0} \widetilde{G}_R^{nn}(\omega = 0, \boldsymbol{k})\right)$$

➡ **Charge density correlator enables us to obtain *D***

Second way to determine κ_n

◆ Constitutive relation under external electric field

$$\vec{J} = -T\kappa_n \vec{\nabla}(\beta\mu) \;\Rightarrow\; \vec{J} = -T\kappa_n\left[\vec{\nabla}(\beta\mu) - \beta\vec{E}\right]$$

Matching condition

This hydrodynamic constitutive relation should match with the field-theoretical expectation value of the current!

◆ First-order perturbation w.r.t. the external gauge field

$$\langle \hat{J}^i(x)\rangle = \int \mathrm{d}t\mathrm{d}^3x G_R^{J^i J^j}(x-x')A_j(x') \simeq \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G_R^{J^i J^j} E_j(x)$$

Green-Kubo formula: $\kappa_n = \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G_R^{J^x J^x}(\omega, \boldsymbol{k}=0)$

Semi-phenomenology

◆ Bulding blocks of hydrodynamic equation

(1) Conservation law: $\partial_t n + \vec{\nabla} \cdot \vec{J} = 0$

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(1) Conservation law — Ward-Takahashi identity resulting from symmetry of systems

(2) Constitutive relation — Phenomenological analysis based on local thermodynamics laws

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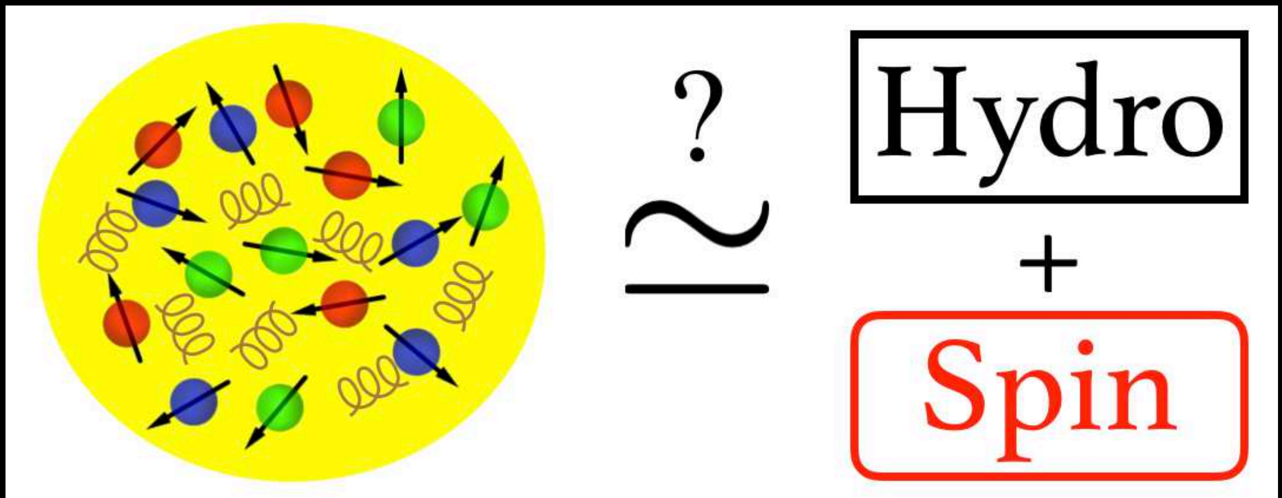

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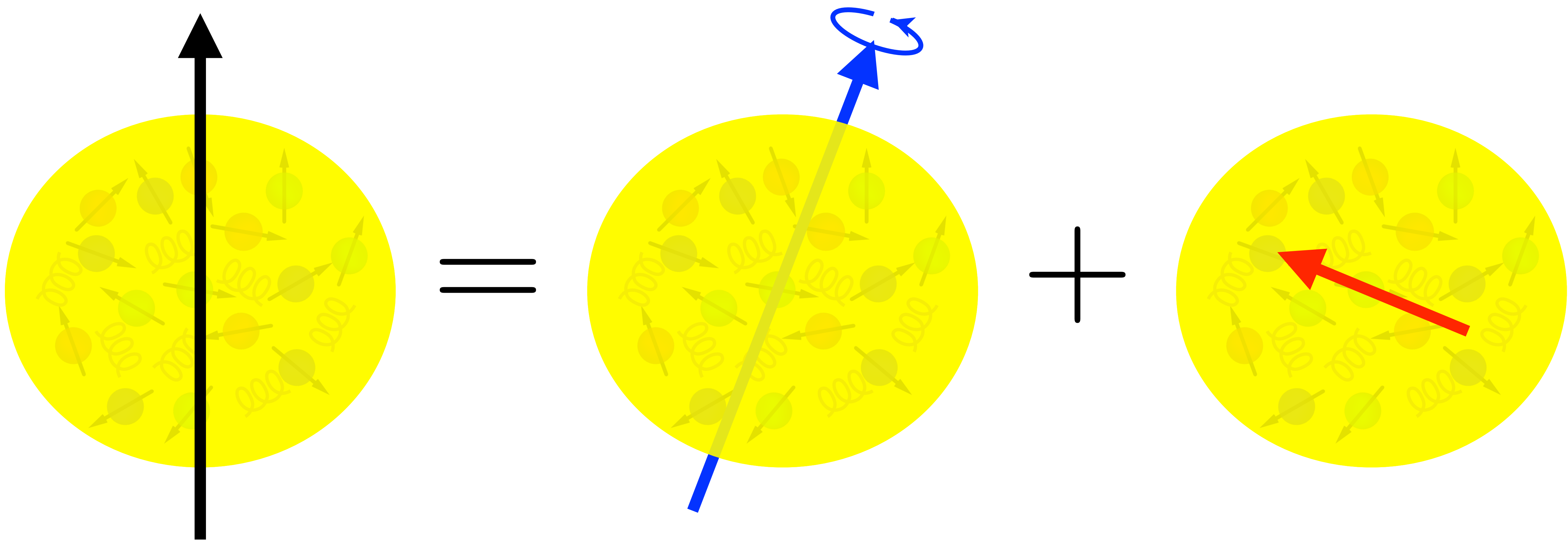
Angular momentum conservation

◆ **What we expect for the angular momentum:**

Conservation law: $\partial_\mu \Theta^{\mu\nu} = 0,\ \partial_\mu J^{\mu\nu\rho} = 0$

Decomposition: $J^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho}$

Total AM **Orbital AM** **Spin AM**



How we define spin current?

Noether current often have unacceptable property!

➡ Gauge currents are often more useful!!

◆ **Gauge current**

Introduce background gauge fields A_μ coupled to symmetry

Gauge current : $J^\mu(x) \equiv \dfrac{\delta \mathcal{S}[\varphi; A_\mu]}{\delta A_\mu(x)}$ $\quad$ (action : $\mathcal{S}[\varphi; A_\mu]$)

Symmetry of QCD = Poincare & flavor symmetries

Background field = Vierbein ${e_\mu}^{\hat{a}}$, spin connection ${\omega_\mu}^{\hat{a}\hat{b}}$

and flavor gauge field A_μ

Torsionful background

◆ **Subtle issue**

When there is no torsion (or $\Gamma^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu}$),
the spin connection is completely fixed by the vierbein!

➡ To make the spin connection independent bkg.,
we need to consider **a torsionful curved spacetime**!

$$T^{\hat{a}}{}_{\mu\nu} = \partial_\mu e_\nu{}^{\hat{a}} - \partial_\nu e_\mu{}^{\hat{a}} + \omega_\mu{}^{\hat{a}}{}_{\hat{b}} e_\nu{}^{\hat{b}} - \omega_\nu{}^{\hat{a}}{}_{\hat{b}} e_\mu{}^{\hat{b}} \neq 0$$

◆ **Definition of EM tensor and spin current**

$$\Theta^{\mu}{}_{\hat{a}}(x) \equiv \frac{1}{e(x)} \frac{\delta \mathcal{S}_{\mathrm{QCD}}}{\delta e_\mu{}^{\hat{a}}(x)}\bigg|_{\omega,\, A}, \quad \Sigma^{\mu}{}_{\hat{a}\hat{b}}(x) \equiv -\frac{2}{e(x)} \frac{\delta \mathcal{S}_{\mathrm{QCD}}}{\delta \omega_\mu{}^{\hat{a}\hat{b}}(x)}\bigg|_{e,\, A}$$

Spin current of QCD

$$\mathcal{L}_{\rm QCD} \equiv -\frac{1}{2}\bar{q}\left(\gamma^{\hat{a}} e_{\hat{a}}{}^{\mu}\overrightarrow{D}_{\mu} - \overleftarrow{D}_{\mu} e_{\hat{a}}{}^{\mu}\gamma^{\hat{a}}\right)q - \bar{q}Mq - \frac{1}{2}\,\mathrm{tr}\left(g^{\mu\nu}g^{\alpha\beta}G_{\mu\alpha}G_{\nu\beta}\right)$$

◆ **EM tensor and spin current of QCD**

$$\Theta^{\mu}{}_{\hat{a}} = \frac{1}{2}\bar{q}(\gamma^{\mu}\overrightarrow{D}_{\hat{a}} - \overleftarrow{D}_{\hat{a}}\gamma^{\mu})q + 2\,\mathrm{tr}\,(G^{\mu\rho}G_{\hat{a}\rho}) + \mathcal{L}_{\rm QCD}\, e_{\hat{a}}{}^{\mu},$$

$$\Sigma^{\mu}{}_{\hat{a}\hat{b}} = -\frac{\mathrm{i}}{2}\bar{q}e^{\mu}{}_{\hat{c}}\{\gamma^{\hat{c}}, \Sigma_{\hat{a}\hat{b}}\}q,$$

Healthy operators satisfying
- (1) Hermiticity
- (2) Gauge invariance

Besides, $\Sigma^{\mu}{}_{\hat{a}\hat{b}}$ is
- composed of only fermion spin
- totally anti-symmetric w.r.t 3 indices

→ only 3 spin densities as d.o.f.

Ward-Takahashi identity

Poincare invariance → (1) Diffeomorphism; (2) Local Lorentz invariance

◆ EM conservation and spin eom as WT identities

$$(D_\mu - \mathcal{G}_\mu)\Theta^\mu{}_{\hat a} = -\Theta^\mu{}_{\hat b} T^{\hat b}{}_{\mu \hat a} + \frac{1}{2}\Sigma^\mu{}_{\hat b}{}^{\hat c}\mathcal{R}^{\hat b}{}_{\hat c \mu \hat a} \quad : 4 \text{ eom}$$

$$(D_\mu - \mathcal{G}_\mu)\Sigma^\mu{}_{\hat a \hat b} = -(\Theta_{\hat a \hat b} - \Theta_{\hat b \hat a}), \quad : 3 \text{ eom} + 3 \text{ constraints}$$

$$\left(\begin{array}{l} \mathcal{R}^{\hat a}{}_{\hat b \mu\nu} \equiv \partial_\mu \omega_\nu{}^{\hat a}{}_{\hat b} - \partial_\nu \omega_\mu{}^{\hat a}{}_{\hat b} + \omega_\mu{}^{\hat a}{}_{\hat c}\omega_\nu{}^{\hat c}{}_{\hat b} - \omega_\nu{}^{\hat a}{}_{\hat c}\omega_\mu{}^{\hat c}{}_{\hat b}, \\ \mathcal{G}_\mu \equiv T^\nu{}_{\nu\mu} \ \text{ with } \ T^{\hat a}{}_{\mu\nu} = \partial_\mu e_\nu{}^{\hat a} - \partial_\nu e_\mu{}^{\hat a} + \omega_\mu{}^{\hat a}{}_{\hat b} e_\nu{}^{\hat b} - \omega_\nu{}^{\hat a}{}_{\hat b} e_\mu{}^{\hat b} \end{array} \right)$$

In flat limit, WT id. reproduces

$$\left\{ \begin{array}{l} \partial_\mu \Theta^{\mu\nu} = 0, \ \partial_\mu J^{\mu\nu\rho} = 0 \\ J^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} + \Sigma^{\mu\nu\rho} \end{array} \right.$$

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Step 4. Restrict terms to be compatible with local 2nd law

$\exists\, s^\mu$ such that $\partial_t s + \vec{\nabla} \cdot \vec{s} \geq 0$ $\Rightarrow$ $\kappa_n \geq 0$ with $\vec{s} = \beta\mu\vec{J}$

Step1-2. Identify d.o.f.

◆ **Equation of motion**

$$(D_\mu - \mathcal{G}_\mu)\Theta^\mu{}_{\hat{a}} = -\Theta^\mu{}_{\hat{b}} T^{\hat{b}}{}_{\mu\hat{a}} + \frac{1}{2}\Sigma^\mu{}_{\hat{b}}{}^{\hat{c}}\mathcal{R}^{\hat{b}}{}_{\hat{c}\mu\hat{a}}$$

$$(D_\mu - \mathcal{G}_\mu)\Sigma^\mu{}_{\hat{a}\hat{b}} = -(\Theta_{\hat{a}\hat{b}} - \Theta_{\hat{b}\hat{a}}),$$

- **7 dynamical variables:** $\epsilon, u^{\hat{a}}, \sigma_{\hat{a}\hat{b}}$ (or $\sigma^{\hat{a}}$) $\quad (\sigma^{\hat{a}} u_{\hat{a}} = 0 = \sigma_{\hat{a}\hat{b}} u^{\hat{a}})$

- **Entropy density:** $s(x) = s(\epsilon, \sigma_{\hat{a}\hat{b}})$ with $T\mathrm{d}s = \mathrm{d}\epsilon - \frac{1}{2}\mu^{\hat{a}\hat{b}}\mathrm{d}\sigma_{\hat{a}\hat{b}}$

- **Conjugate variables:** $\beta \equiv \frac{\partial s}{\partial \epsilon}, \quad \beta\mu^{\hat{a}\hat{b}} \equiv -2\frac{\partial s}{\partial \sigma_{\hat{a}\hat{b}}}$

- **Power counting scheme:**

$$O(\partial^0) = \{\beta,\, u^{\hat{a}},\, e_\mu{}^{\hat{a}}\} \text{ and } O(\partial^1) = \{\mu^{\hat{a}\hat{b}},\, \sigma_{\hat{a}\hat{b}},\, \omega_\mu{}^{\hat{a}\hat{b}}\}$$

Step3-4. Local second law

◆ **Tensor decomposition**

$$\Theta^{\mu}{}_{\hat{a}} = \epsilon u^{\mu} u_{\hat{a}} + p\Delta^{\mu}_{\hat{a}} + u^{\mu}\delta q_{\hat{a}} - \delta q^{\mu} u_{\hat{a}} + \delta\Theta^{\mu}{}_{\hat{a}},$$

$$\Sigma^{\mu}{}_{\hat{a}\hat{b}} = \varepsilon^{\mu}{}_{\hat{a}\hat{b}\hat{c}}(\sigma^{\hat{c}} + \delta\sigma u^{\hat{c}}),$$

$$(\, \Delta^{\mu}_{\hat{a}} \equiv e_{\hat{a}}{}^{\mu} + u^{\mu} u_{\hat{a}} \quad \text{satisfying} \quad \Delta^{\mu}_{\hat{a}} u^{\hat{a}} = 0 = \Delta^{\mu}_{\hat{a}} u_{\mu} \,)$$

Using eom, we find the entropy current ($s^{\mu} \equiv s u^{\mu} + \delta s^{\mu}$) satisfies

$$\begin{aligned}(\nabla_{\mu} - \mathcal{G}_{\mu})s^{\mu} =& \big[s - \beta(\epsilon + p)\big](\nabla_{\mu} - \mathcal{G}_{\mu})u^{\mu} + (\nabla_{\mu} - \mathcal{G}_{\mu})\delta s^{\mu} \\ &- (D_{\mu}\beta^{\hat{a}} - T^{\hat{a}}{}_{\mu\hat{b}}\beta^{\hat{b}} - \beta\mu_{\mu}{}^{\hat{a}})\delta\Theta^{\mu}{}_{\hat{a}}\big|_{(a)} \\ &- (D_{\mu}\beta^{\hat{a}} - T^{\hat{a}}{}_{\mu\hat{b}}\beta^{\hat{b}})\delta\Theta^{\mu}{}_{\hat{a}}\big|_{(s)} + O(\partial^2)\end{aligned}$$

Step3-4. Local **second** law

Require **the local second law:** $(\nabla_\mu - \mathcal{G}_\mu)s^\mu \geq 0$ for $\forall \beta,\ u^\mu,\ \mu^{\hat{a}\hat{b}}$

$$
\begin{aligned}
(\nabla_\mu - \mathcal{G}_\mu)s^\mu = & \big[s - \beta(\epsilon + p)\big](\nabla_\mu - \mathcal{G}_\mu)u^\mu + (\nabla_\mu - \mathcal{G}_\mu)\delta s^\mu \\
& - (D_\mu \beta^{\hat{a}} - T^{\hat{a}}{}_{\mu\hat{b}}\beta^{\hat{b}} - \beta\mu_\mu{}^{\hat{a}})\delta\Theta^\mu{}_{\hat{a}}\big|_{(a)} \\
& - (D_\mu \beta^{\hat{a}} - T^{\hat{a}}{}_{\mu\hat{b}}\beta^{\hat{b}})\delta\Theta^\mu{}_{\hat{a}}\big|_{(s)} + O(\partial^2)
\end{aligned}
$$

$$
Ts = \epsilon + p\,, \quad \delta s^\mu = O(\partial^2) = \delta\sigma \ \ \text{and} \ \ \eta, \zeta, \eta_s \geq 0
$$

$$
\delta\Theta^\mu{}_{\hat{a}}\big|_{(s)} = -T\eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(D_\nu \beta^{\hat{b}} - T^{\hat{b}}{}_{\nu\hat{c}}\beta^{\hat{c}})
$$

$$
\delta\Theta^\mu{}_{\hat{a}}\big|_{(a)} = -T(\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(D_\nu \beta^{\hat{b}} - T^{\hat{b}}{}_{\nu\hat{c}}\beta^{\hat{c}} - \beta\mu_\nu{}^{\hat{b}})
$$

$$
\text{with} \quad
\begin{cases}
\eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}} = 2\eta\left(\dfrac{1}{2}(\Delta^{\mu\nu}\Delta_{\hat{a}\hat{b}} + \Delta^\mu_{\hat{b}}\Delta^\nu_{\hat{a}}) - \dfrac{1}{3}\Delta^\mu_{\hat{a}}\Delta^\nu_{\hat{b}}\right) + \zeta\Delta^\mu_{\hat{a}}\Delta^\nu_{\hat{b}}, \\
(\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}} = \dfrac{1}{2}\eta_s(\Delta^{\mu\nu}\Delta_{\hat{a}\hat{b}} - \Delta^\mu_{\hat{b}}\Delta^\nu_{\hat{a}}).
\end{cases}
$$

Result

◆Equation of motion

$$(D_\mu - \mathcal{G}_\mu)\Theta^\mu{}_{\hat{a}} = -\Theta^\mu{}_{\hat{b}} T^{\hat{b}}{}_{\mu\hat{a}} + \frac{1}{2}\Sigma^\mu{}_{\hat{b}}{}^{\hat{c}} \mathcal{R}^{\hat{b}}{}_{\hat{c}\mu\hat{a}}, \quad (D_\mu - \mathcal{G}_\mu)\Sigma^\mu{}_{\hat{a}\hat{b}} = -(\Theta_{\hat{a}\hat{b}} - \Theta_{\hat{b}\hat{a}})$$

◆Constitutive relation

$$\Theta^\mu{}_{\hat{a}} = \epsilon u^\mu u_{\hat{a}} + p\Delta^\mu_{\hat{a}} - \eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(D_\nu u^{\hat{b}} - T^{\hat{b}}{}_{\nu\hat{c}} u^{\hat{c}}) - (\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(D_\nu u^{\hat{b}} - T^{\hat{b}}{}_{\nu\hat{c}} u^{\hat{c}} - \mu_\nu{}^{\hat{b}})$$

$$= \epsilon u^\mu u_{\hat{a}} + p\Delta^\mu_{\hat{a}} - \eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}} \mathring{D}_\nu u^{\hat{b}} - (\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(\mathring{D}_\nu u^{\hat{b}} - u^{\hat{c}} K_{\hat{c}\nu}{}^{\hat{b}} - \mu_\nu{}^{\hat{b}}),$$

$$\Sigma^\mu{}_{\hat{a}\hat{b}} = \varepsilon^\mu{}_{\hat{a}\hat{b}\hat{c}} \sigma^{\hat{c}},$$ with a contorsion tensor: $$K_\mu{}^{\hat{a}\hat{b}} \equiv \frac{1}{2} e^{\hat{a}\nu} e^{\hat{b}\rho}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\nu\mu})$$

◆Transport coefficient: η, ζ, η_s

$$\eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}} = 2\eta\left(\frac{1}{2}(\Delta^{\mu\nu}\Delta_{\hat{a}\hat{b}} + \Delta^\mu_{\hat{b}}\Delta^\nu_{\hat{a}}) - \frac{1}{3}\Delta^\mu_{\hat{a}}\Delta^\nu_{\hat{b}}\right) + \zeta\Delta^\mu_{\hat{a}}\Delta^\nu_{\hat{b}}$$

$$(\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}} = \frac{1}{2}\eta_s(\Delta^{\mu\nu}\Delta_{\hat{a}\hat{b}} - \Delta^\mu_{\hat{b}}\Delta^\nu_{\hat{a}}).$$

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✓(2) Constitutive relation — Phenomenological analysis based on local thermodynamics laws

(3) Physical properties — Matching the hydrodynamic result with the field-theoretical correlator

First way to compute η_s

Linear-mode analysis

on spin-hydro

Linearized spin-hydro

Perturbation on the top of global static thermal equilibrium:

$$\begin{cases} \epsilon(x) = \epsilon_0 + \delta\epsilon(x) \\ v^i(x) = 0 + \delta v^i(x) \\ \sigma^{\hat{a}}(x) = 0 + \delta\sigma^{\hat{a}}(x) \end{cases}$$

with the flat background

Pickup $O(\delta)$-terms only

◆ **Linearized spin-hydrodynamic equations:**

$$0 = \partial_0 \delta\epsilon + \partial_i \delta\pi^i,$$

$$0 = \partial_0 \delta\pi_i + c_s^2 \partial_i \delta\epsilon - \gamma_{\|} \partial_i \partial^j \delta\pi_j - (\gamma_\perp + \gamma_s)(\delta_i^j \boldsymbol{\nabla}^2 - \partial_i \partial^j)\delta\pi_j + \frac{1}{2}\Gamma_s \varepsilon_{0ijk} \partial^j \delta\sigma^k,$$

$$0 = \partial_0 \delta\sigma_i + \Gamma_s \delta\sigma_i + 2\gamma_s \varepsilon_{0ijk} \partial^j \delta\pi^k,$$

with a set of parameters:

$$\begin{cases} c_s^2 \equiv \dfrac{\partial p}{\partial \epsilon}, \quad \gamma_{\|} \equiv \dfrac{1}{\epsilon_0 + p_0}\left(\zeta + \dfrac{4}{3}\eta\right), \quad \gamma_\perp \equiv \dfrac{\eta}{\epsilon_0 + p_0}, \\ \chi_s \delta_{ij} \equiv \dfrac{\partial \sigma_i}{\partial \mu^j}, \quad \gamma_s \equiv \dfrac{\eta_s}{2(\epsilon_0 + p_0)}, \quad \Gamma_s \equiv \dfrac{2\eta_s}{\chi_s}. \end{cases}$$

Linear-mode analysis

Linearized eom can be solved by the use of Fourier tr.!

$$\delta\mathcal{O}(x) = e^{-\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}\delta\tilde{\mathcal{O}}(k)$$ ➡ **EoM:** $A(\omega,\boldsymbol{k})\delta\tilde{\mathcal{O}}(k) = 0$

$(A(\omega,\boldsymbol{k}) : 7\times 7\mathrm{matrix})$

Characteristic equation: $\det A(\omega,\boldsymbol{k}) = 0$

◆ Dispersion relation

$$\omega_{\mathrm{sound}}(\boldsymbol{k}) = \pm c_s|\boldsymbol{k}| - \frac{\mathrm{i}}{2}\gamma_{\parallel}\boldsymbol{k}^2 + O(\boldsymbol{k}^3),$$

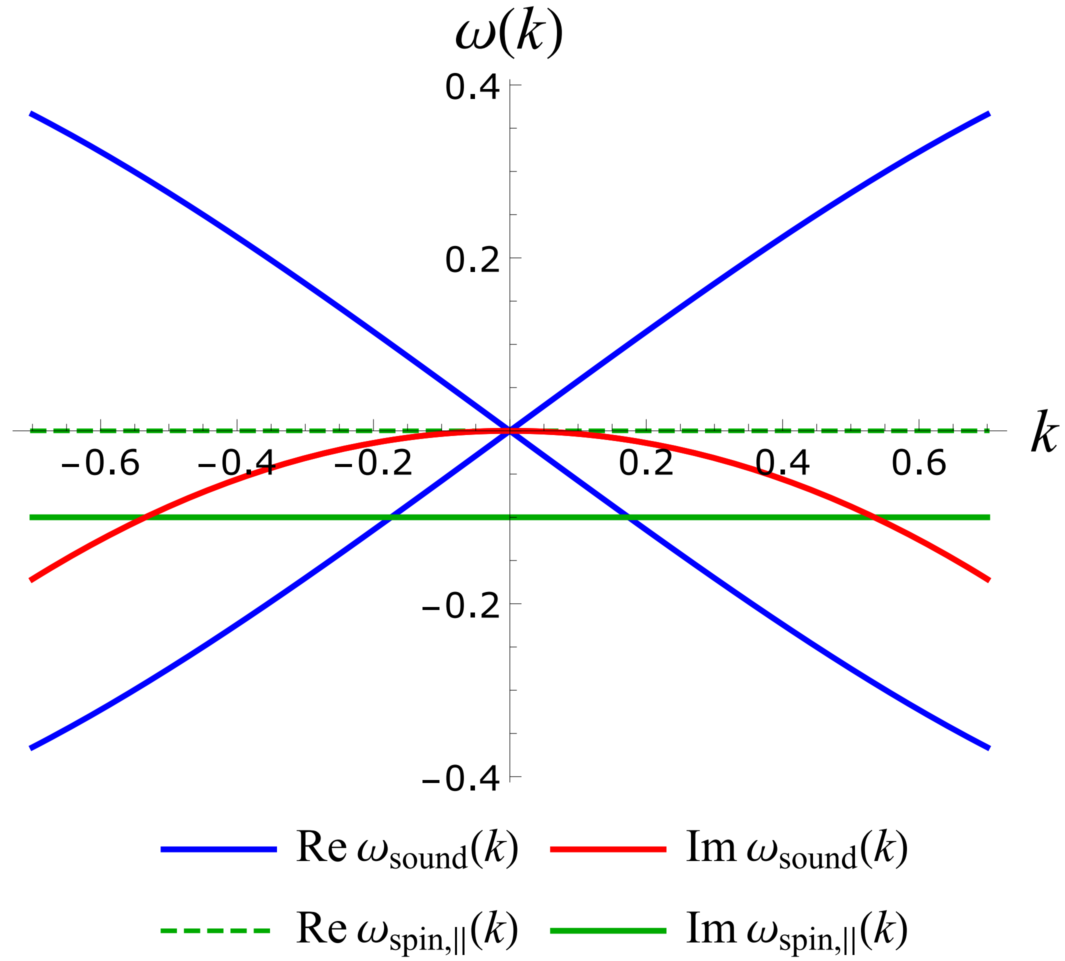
$$\omega_{\mathrm{spin},\parallel}(\boldsymbol{k}) = -\mathrm{i}\Gamma_s$$

$$\omega_{\mathrm{shear}}(\boldsymbol{k}) = -\frac{\mathrm{i}\Gamma_s + \mathrm{i}(\gamma_\perp+\gamma_s)\boldsymbol{k}^2 - \mathrm{i}\sqrt{\Gamma_s^2 - 2\Gamma_s(\gamma_\perp-\gamma_s)\boldsymbol{k}^2 + (\gamma_\perp+\gamma_s)^2\boldsymbol{k}^4}}{2}$$

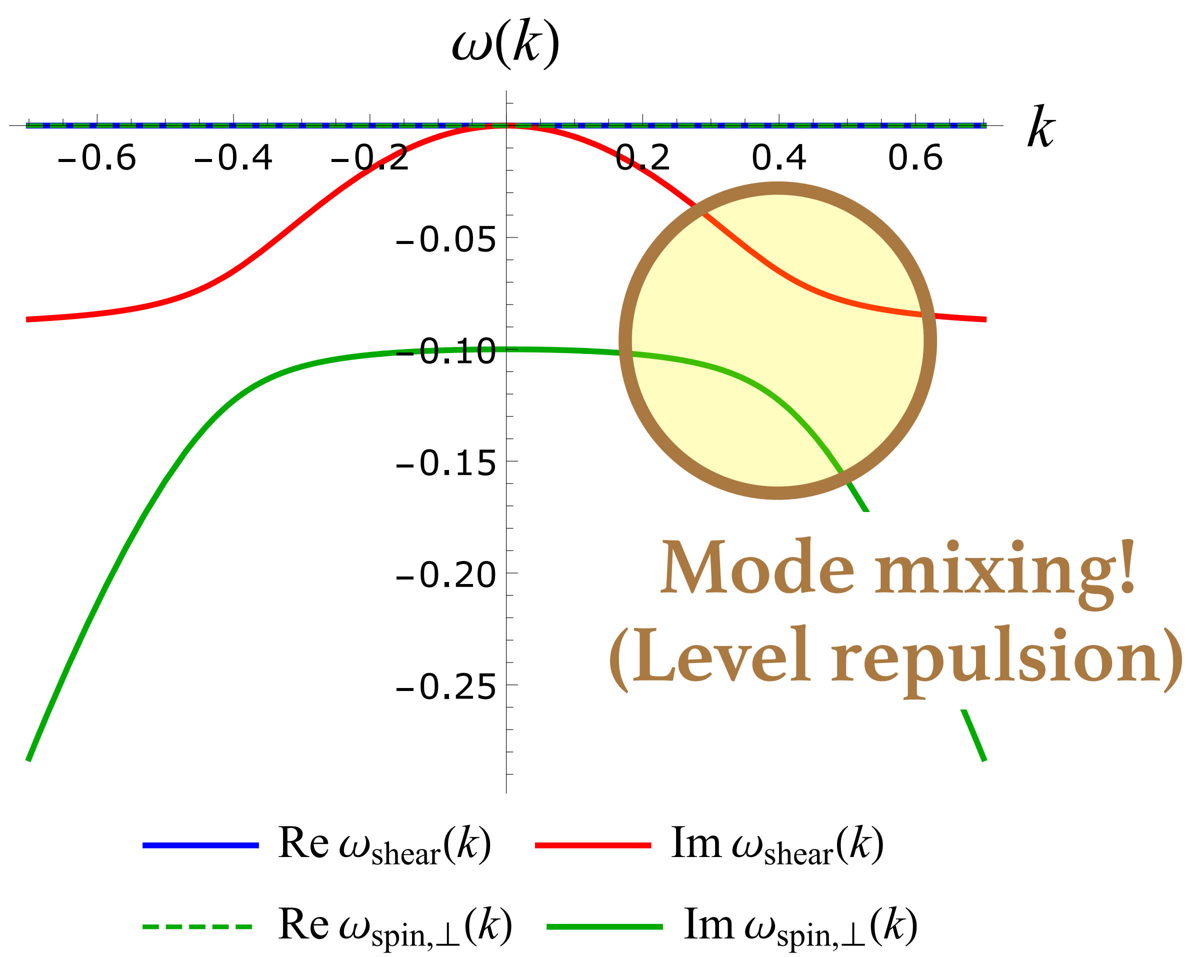
$$\omega_{\mathrm{spin},\perp}(\boldsymbol{k}) = -\frac{\mathrm{i}\Gamma_s + \mathrm{i}(\gamma_\perp+\gamma_s)\boldsymbol{k}^2 + \mathrm{i}\sqrt{\Gamma_s^2 - 2\Gamma_s(\gamma_\perp-\gamma_s)\boldsymbol{k}^2 + (\gamma_\perp+\gamma_s)^2\boldsymbol{k}^4}}{2}$$

Dispersion relation

(a) Longitudinal modes



(b) Transverse modes



$$\omega_{\text{shear}}(\boldsymbol{k}) = -\frac{\mathrm{i}\Gamma_s + \mathrm{i}(\gamma_\perp + \gamma_s)\boldsymbol{k}^2 - \mathrm{i}\sqrt{\Gamma_s^2 - 2\Gamma_s(\gamma_\perp - \gamma_s)\boldsymbol{k}^2 + (\gamma_\perp + \gamma_s)^2\boldsymbol{k}^4}}{2}$$

$$\omega_{\text{spin},\perp}(\boldsymbol{k}) = -\frac{\mathrm{i}\Gamma_s + \mathrm{i}(\gamma_\perp + \gamma_s)\boldsymbol{k}^2 + \mathrm{i}\sqrt{\Gamma_s^2 - 2\Gamma_s(\gamma_\perp - \gamma_s)\boldsymbol{k}^2 + (\gamma_\perp + \gamma_s)^2\boldsymbol{k}^4}}{2}$$

Spin-spin correlator

Taking $k = 0$, all spin modes has $\omega_{\mathrm{spin}}(k=0) = -\mathrm{i}\Gamma_s$

Spin densities shows a gapped relaxation dynamics with a characteristic time scale $\tau_s = \Gamma_s^{-1}$

◆ **Green's function interpretation of the result**

Spin-spin correlator: $\widetilde{G}_{\mathrm{R}}^{\sigma^i\sigma^j}(\omega, \boldsymbol{k}) = \dfrac{\mathrm{i}\chi_s\Gamma_s + \cdots}{\omega + \mathrm{i}\Gamma_s + O(\boldsymbol{k}^2)}\delta^{ij}$

(Definition of spin susceptibility: $\lim_{\boldsymbol{k}\to 0} \widetilde{G}_{\mathrm{R}}^{\sigma_i\sigma_j}(\omega = 0, \boldsymbol{k}) = \chi_s\delta^{ij}$)

Spin-spin correlator enables us to obtain $\Gamma_s \equiv \dfrac{2\eta_s}{\chi_s}$

Second way to compute η_s

Linear-reponse theory

w.r.t. background field

Linear-response theory

◆ Constitutive relation

$$\delta\Theta^{\mu}{}_{\hat{a}}\Big|_{(a)} = -(\eta_s)^{\mu}{}_{\hat{a}}{}^{\nu}{}_{\hat{b}}(\mathring{D}_{\nu}u^{\hat{b}} - u^{\hat{c}}K_{\hat{c}\nu}{}^{\hat{b}} - \mu_{\nu}{}^{\hat{b}})$$

Linear-response theory

◆ **Constitutive relation**

$$\delta\Theta^{\mu}{}_{\hat{a}}\big|_{(a)} = \begin{cases} -(\eta_s)^{\mu}{}_{\hat{a}}{}^{\nu}{}_{\hat{b}}(\mathring{D}_\nu u^{\hat{b}} - u^{\hat{c}}K_{\hat{c}\nu}{}^{\hat{b}} - \mu_\nu{}^{\hat{b}}) & \text{when} \quad \Gamma_s \ll \omega \ll \Gamma \\ 0 & \text{when} \quad \omega \ll \Gamma_s \end{cases}$$

By perturbing the system with a contorsion K, we obtain

$$\langle \delta\hat{\Theta}^{\mu}{}_{\hat{a}}(x)|_{(a)}\rangle \simeq -\frac{1}{2}\lim_{\Gamma_s \ll \omega \ll \Gamma}\lim_{\boldsymbol{k}\to 0}\widetilde{G}_{\mathrm{R}}^{\Theta^{\mu}{}_{\hat{a}}|_{(a)},\Sigma^{\hat{0}\nu}{}_{\hat{b}}}(\omega,\boldsymbol{k})K_{\hat{0}\nu}{}^{\hat{b}}(x)$$

Unusual but necessary limit!!

◆ **Green-Kubo formula for the rotational viscosity η_s**

$$\eta_s = -\lim_{\Gamma_s \ll \omega \ll \Gamma}\lim_{\boldsymbol{k}\to 0}\widetilde{G}_{\mathrm{R}}^{\Theta^{x}{}_{\hat{y}}|_{(a)},\Sigma^{\hat{0}x}{}_{\hat{y}}}(\omega,\boldsymbol{k})$$

Green-Kubo formula

WT identity: $-\,\mathrm{i}\omega\widetilde{\Sigma}^{\hat{0}x}{}_{\hat{y}}(\omega,\boldsymbol{k}) + \mathrm{i}k\widetilde{\Sigma}^{zx}{}_{\hat{y}}(\omega,\boldsymbol{k}) = -2\widetilde{\Theta}^{x}{}_{\hat{y}}\big|_{(a)}(\omega,\boldsymbol{k})$

◆ **Three expressions of Green-Kubo formula for η_s**

$$\eta_s = -\lim_{\Gamma_s \ll \omega \ll \Gamma} \lim_{\boldsymbol{k}\to 0} \widetilde{G}_{\mathrm{R}}^{\Theta^x{}_{\hat{y}}|_{(a)},\Sigma^{\hat{0}x}{}_{\hat{y}}}(\omega,\boldsymbol{k})$$

$$= 2\lim_{\Gamma_s \ll \omega \ll \Gamma} \lim_{\boldsymbol{k}\to 0} \frac{1}{\omega} \operatorname{Im} \widetilde{G}_{\mathrm{R}}^{\Theta^x{}_{\hat{y}}|_{(a)},\Theta^x{}_{\hat{y}}|_{(a)}}(\omega,\boldsymbol{k})$$

$$= \frac{1}{2}\lim_{\Gamma_s \ll \omega \ll \Gamma} \lim_{\boldsymbol{k}\to 0} \omega \operatorname{Im} \widetilde{G}_{\mathrm{R}}^{\Sigma^{\hat{0}x}{}_{\hat{y}},\Sigma^{\hat{0}x}{}_{\hat{y}}}(\omega,\boldsymbol{k})$$

Constrained limit indeed agrees with the result of the linear-mode analysis!

$$\omega\widetilde{G}_R^{\sigma^i\sigma^j}(\omega,\boldsymbol{k}=0) = \frac{\mathrm{i}\omega\chi_s\Gamma_s+\cdots}{\omega+\mathrm{i}\Gamma_s+\cdots}\delta^{ij} \xrightarrow{\Gamma_s\ll\omega\ll\Gamma} \frac{\mathrm{i}\omega\chi_s\Gamma_s}{\omega} = 2\mathrm{i}\eta_s$$

Semi-phenomenology

◆ Bulding blocks of hydrodynamic equation

(1) Conservation law: $\partial_t n + \vec{\nabla} \cdot \vec{J} = 0$

(2) Constitutive relation: $\vec{J} = -T\kappa_n \vec{\nabla}(\beta\mu) \simeq -D\vec{\nabla} n$

(3) Physical properties: Values of $\kappa_n, \ \chi_n \quad (D = \kappa_n / \chi_n)$

✓(1) Conservation law — Ward-Takahashi identity resulting from symmetry of systems

✓(2) Constitutive relation — Phenomenological analysis based on local thermodynamics laws

✓(3) Physical properties — Matching the hydrodynamic result with the field-theoretical correlator

Summary of our result

◆Equation of motion

$$(D_\mu - \mathcal{G}_\mu)\Theta^\mu{}_{\hat{a}} = -\Theta^\mu{}_{\hat{b}} T^{\hat{b}}{}_{\mu\hat{a}} + \frac{1}{2}\Sigma^\mu{}_{\hat{b}}{}^{\hat{c}}\mathcal{R}^{\hat{b}}{}_{\hat{c}\mu\hat{a}}, \quad (D_\mu - \mathcal{G}_\mu)\Sigma^\mu{}_{\hat{a}\hat{b}} = -(\Theta_{\hat{a}\hat{b}} - \Theta_{\hat{b}\hat{a}})$$

◆Constitutive relation

$$\Theta^\mu{}_{\hat{a}} = \epsilon u^\mu u_{\hat{a}} + p\Delta^\mu_{\hat{a}} - \eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(D_\nu u^{\hat{b}} - T^{\hat{b}}{}_{\nu\hat{c}}u^{\hat{c}}) - (\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(D_\nu u^{\hat{b}} - T^{\hat{b}}{}_{\nu\hat{c}}u^{\hat{c}} - \mu_\nu{}^{\hat{b}})$$

$$= \epsilon u^\mu u_{\hat{a}} + p\Delta^\mu_{\hat{a}} - \eta^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}\mathring{D}_\nu u^{\hat{b}} - (\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}}(\mathring{D}_\nu u^{\hat{b}} - u^{\hat{c}}K_{\hat{c}\nu}{}^{\hat{b}} - \mu_\nu{}^{\hat{b}}),$$

$\Sigma^\mu{}_{\hat{a}\hat{b}} = \varepsilon^\mu{}_{\hat{a}\hat{b}\hat{c}}\sigma^{\hat{c}}$, with a contorsion tensor: $K_\mu{}^{\hat{a}\hat{b}} \equiv \frac{1}{2}e^{\hat{a}\nu}e^{\hat{b}\rho}(T_{\mu\nu\rho} - T_{\nu\rho\mu} + T_{\rho\nu\mu})$

◆Green-Kubo-formula for rotational viscosity η_s

$$(\eta_s)^\mu{}_{\hat{a}}{}^\nu{}_{\hat{b}} = \frac{1}{2}\eta_s(\Delta^{\mu\nu}\Delta_{\hat{a}\hat{b}} - \Delta^\mu_{\hat{b}}\Delta^\nu_{\hat{a}})$$

$$\eta_s = 2\lim_{\Gamma_s \ll \omega \ll \Gamma}\lim_{\boldsymbol{k}\to 0}\frac{1}{\omega}\,\mathrm{Im}\,\widetilde{G}_{\mathrm{R}}^{\Theta^x{}_{\hat{y}}|_{(a)},\Theta^x{}_{\hat{y}}|_{(a)}}(\omega, \boldsymbol{k})$$

When scale separation occurs?

$$(\Gamma_s \ll \Gamma)$$

Phenomenological derivation

Step 1. Determine dynamical d.o.m (& its equation of motion)

Charge density: $n(x)$ EoM: $\partial_t n + \vec{\nabla} \cdot \vec{J} = 0$

Step 2. Introduce entropy & conjugate variable

Entropy density: $s(n)$ $\xrightarrow{T\mathrm{d}s = -\mu \mathrm{d}n}$ Chemical pot.: $\beta\mu \equiv -\dfrac{\partial s}{\partial n}$

Step 3. Write down all possible terms with finite derivatives

Current: $\vec{J} = 0 - T\kappa_n \vec{\nabla}(\beta\mu) + O(\vec{\nabla}^2) = -T\kappa_n \vec{\nabla}\dfrac{\partial s}{\partial n} + O(\vec{\nabla}^2)$

Step 4. Restrict terms to be compatible with local 2nd law

$\exists\, s^\mu$ such that $\partial_t s + \vec{\nabla} \cdot \vec{s} \geq 0$ $\Rightarrow$ $\kappa_n \geq 0$ with $\vec{s} = \beta\mu\vec{J}$

Phenomenological derivation

Step 1. Determine dynamical d.o.m (& its equation of motion)

$$\text{d.o.f.}: \{\Theta^{0\nu}, \phi\} \qquad \text{EoM: } \partial_\mu \Theta^{\mu\nu} = 0,\ \partial_\mu \phi = f_\mu$$

Step 2. Introduce entropy & conjugate variable

$$\text{Entropy: } s(\Theta^{0\nu}, \phi) \Rightarrow \beta_\nu \equiv \frac{\partial s}{\partial \Theta^{0\nu}}, \quad \pi \equiv \frac{\partial s}{\partial \phi}$$

Step 3. Write down all possible terms with finite derivatives

$$\Theta^{\mu\nu} = e u^\nu u^\nu + p\Delta^{\mu\nu} + \Theta^{\mu\nu}_{(1)},\ f_\mu = q u_\mu + f^{(1)}_\mu$$

Step 4. Restrict terms to be compatible with local 2nd law

$$\exists\, s^\mu \text{ s.t. } \partial_\mu s^\mu \geq 0 \Rightarrow q = -\gamma\pi = -\gamma \frac{\partial s}{\partial \phi}$$

This gives EoM in **Hydro+**!!

Spin hydro as Hydro+

[See Stephanov-Yin, PRD, **98**, 036006 (2018) , ...]

Hydro+ is a general framework describing both

◆ Hydrodynamic (gapless) mode

- Conserved charge densities: Normal hydrodynamics
- Nambu-Goldstone mode: Superfluid hydrodynamics

◆ Non-hydrodynamic (gapped) mode

- Critical fluctuation around $T \sim T_c$: Original Hydro+
- $SU(2)_A$ charge density in QCD: Chiral hydrodynamics

} **well-defined**

- Spin density: Spin hydrodynamics
- Stress tensor: Muller-Israel-Stewart theory
- $U(1)_A$ charge density in QCD: Chiral hydrodynamics

} **ill-defined**

➡ **There are well-defined and (possibly) ill-defined Hydro+!**

Caution from old paper

PHYSICAL REVIEW A VOLUME 6, NUMBER 6 DECEMBER 1972

Unified Hydrodynamic Theory for Crystals, Liquid Crystals, and Normal Fluids*

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(Received 31 May 1972)

Liquid crystal can have spin density!

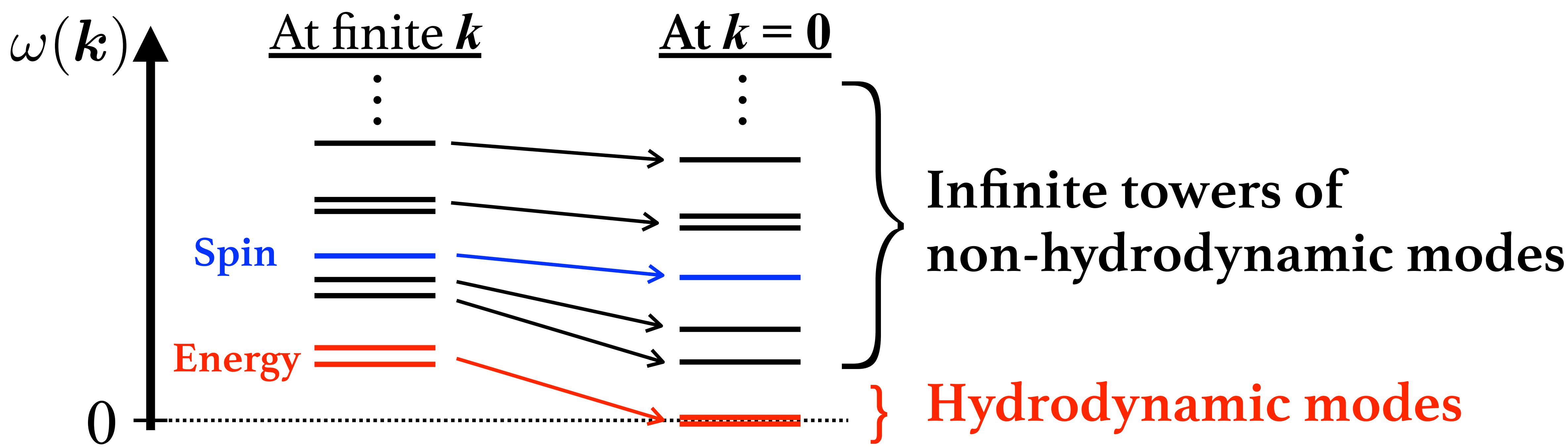
A unified hydrodynamic theory is presented that is appropriate for crystals; smectic, cholesteric, and nematic liquid crystals; glasses; and normal fluids. In the theory, the increased spatial degeneracy as the system progresses from crystalline and mesomorphic phases to the isotropic fluid phase is marked by successive reductions in the number of first-order elastic constants and in the number of transport coefficients. Distinction between local lattice dilations and local mass changes, and recognition of processes like vacancy diffusion that this difference makes possible, are crucial for understanding the connection between theories in different phases. Formulas are derived that give the number of hydrodynamic modes and the frequencies, lifetimes, and intensities of these modes in all of the above systems. In the nematic and cholesteric phases, the results agree with some found previously. In more complex systems, they are new. An attempt is made to explain the differences between the present hydrodynamic theory and other phenomenological proposals.

Caution from old paper

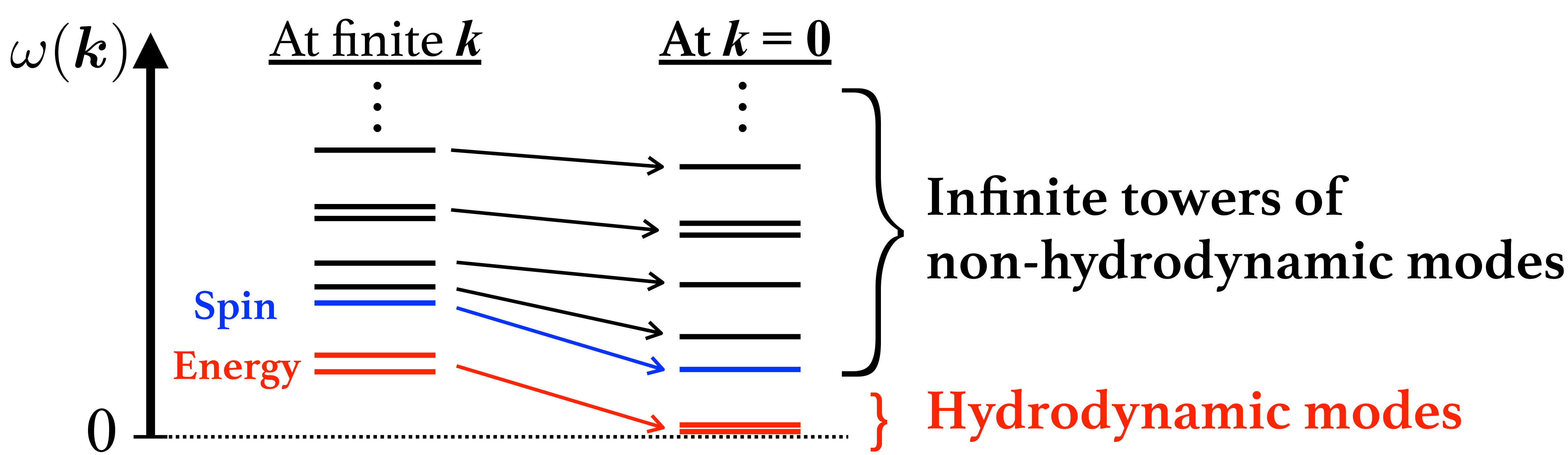
[17] In the hydrodynamic regime for nematics, the "extension" of H. W. Hwang, Phys. Rev. Letters 26, 1525 (1971), is equivalent to FLMPS. Outside of the hydrodynamic regime, the terms he keeps in addition are *ad hoc* and incomplete and there is no reason to think experiments would necessarily give the line shapes they predict *even if* the experiments could be performed. They are just the "irrelevant transport coefficients" which should be discarded as discussed in Ref. 11. Some readers may object to our use of the word irrelevant, since under certain circumstances nonhydrodynamic modes are slow and measurable, e.g., near phase transitions. We agree but point out in response that the same arguments apply in such cases to other variables that have been omitted (e.g., to the magnitude of the order parameter as well as its direction).

Spin hydro is ill-defined

◆ Scenario 1 (Bad: Spin hydro = Hydro++++?)

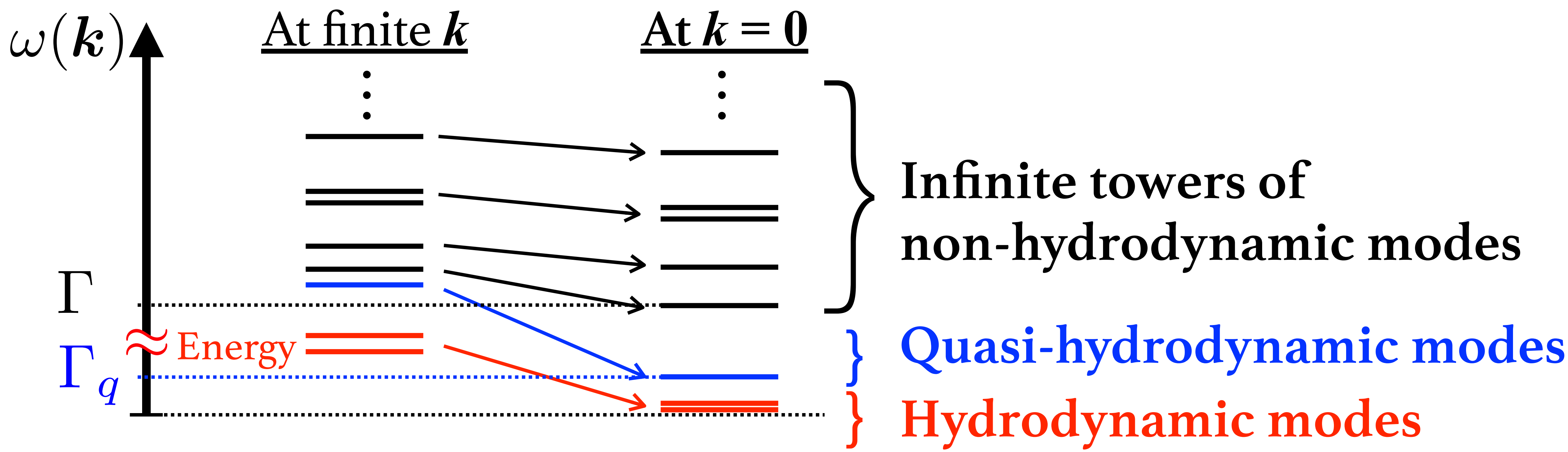


◆ Scenario 2 (Better but still not good: Spin hydro = Hydro+?)



Well-defined HYDRO+

◆ When Hydro+ is well-defined



If $\Gamma_q \ll \Gamma$ is satisfied, Hydro+ becomes well-defined!!

This generally happens when

emergent symmetry appears by tuning parameters (T, m, …)!

- Critical fluid: Scale symmetry emerges at $T = T_c$
- SU(2)$_A$ chiral fluid: SU(2)$_A$ symmetry emerges at $m_q = 0$

HQ-spin hydro is well-defined

[38] If for some reason the coupling between "spin" and orbital angular momentums vanishes, or can be neglected, a separate conservation for "spin" angular momentum will follow from the microscopic Hamiltonian. This is actually the case for a number of models employed to describe magnetic problems.

When we consider **heavy quark limit:** $m_Q \to \infty$,

emergent heavy quark symmetry appears!

◆ Heavy quark spin hydrodynamics

Heavy quark spin damping rate is suppressed by 1/*m*$_Q$,

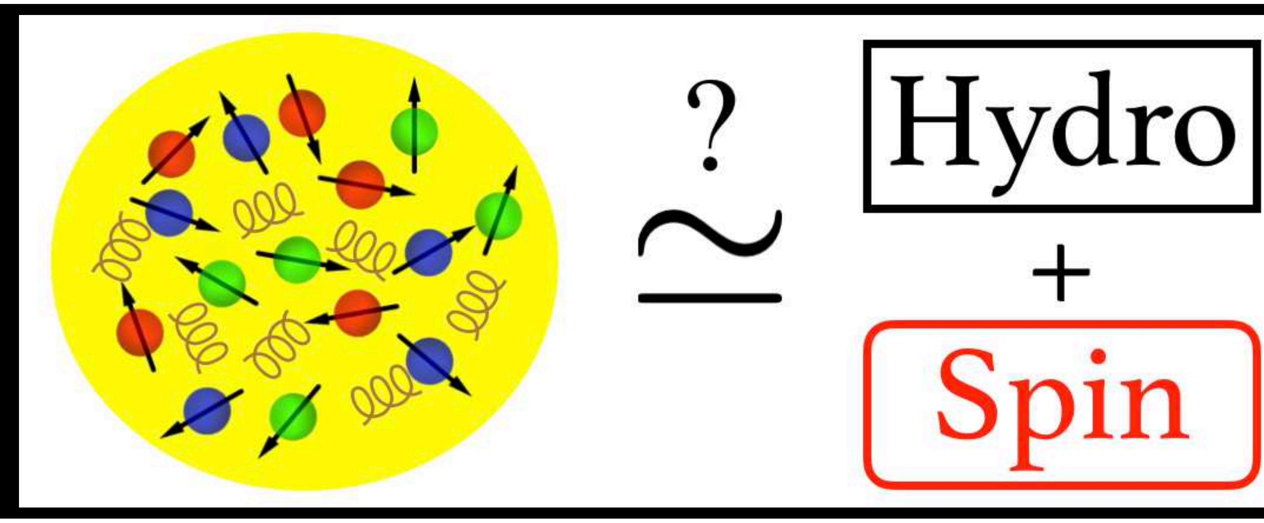
so that HQ-spin hydro is well-defined Hydro+!

(But I do not know whether there is enough # of heavy quarks...)

Summary

Motivation:

Hydrodynamics of
a relativistic spinful fluid?



Approach:

Semi-phenomenology based on local thermodynamics

Result:

(1) Spin hydrodynamic equations in a torsionful geometry

(2) Mode mixing between shear and spin modes

(3) Green-Kubo formula for a rotational viscosity

Sketch of our result