Counting Nambu-Goldstone modes of higher-form symmetries

Phys. Rev. Lett. 126, 071601 (2021)

Yuji Hirono

apctp

In collaboration with:
Yoshimasa Hidaka (KEK), Ryo Yokokura (KEK)

Spontaneous symmetry breaking → low-energy d.o.f

- Examples of Nambu-Goldstone (NG) modes
  - pions in QCD
  - magnons
  - superfluid phonons (He^4, He^3, atomic BEC)
  - lattice phonons
  - Kelvon (vibration of a vortex string)
  - photons
  - ...

Number of NG modes

- (Internal) symmetry breaking $G \to H$
- Lorentz invariance

$$N_{\mathrm{NG}} = N_{\mathrm{BG}}$$

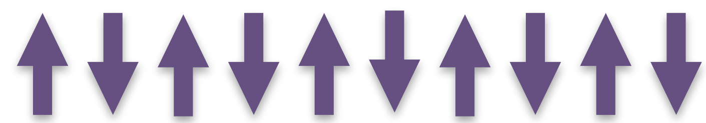
# of gapless NG modes   # of broken generators

- Dispersion relation: $\omega = ck$ $k = |\mathbf{k}|$

Magnets:Heisenberg model $H = J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j$

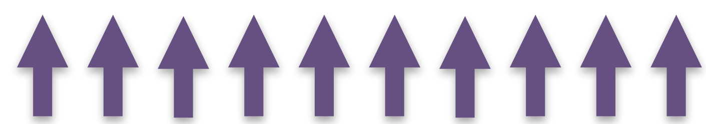
- $J > 0$: Anti-ferromagnetism

$N_{\mathrm{NG}} = 2 \qquad \omega \propto k$



- $J < 0$: Ferromagnetism

$N_{\mathrm{NG}} = 1 \qquad \omega \propto k^2$



Symmetry breaking pattern is common $SO(3) \to SO(2)$

$N_{\mathrm{BG}} = 2$

Thermodynamic properties: heat capacity

- When $\omega \sim k^n$

heat capacity behaves as

$$C \sim T^{\frac{d}{n}}$$

d : spatial dimension

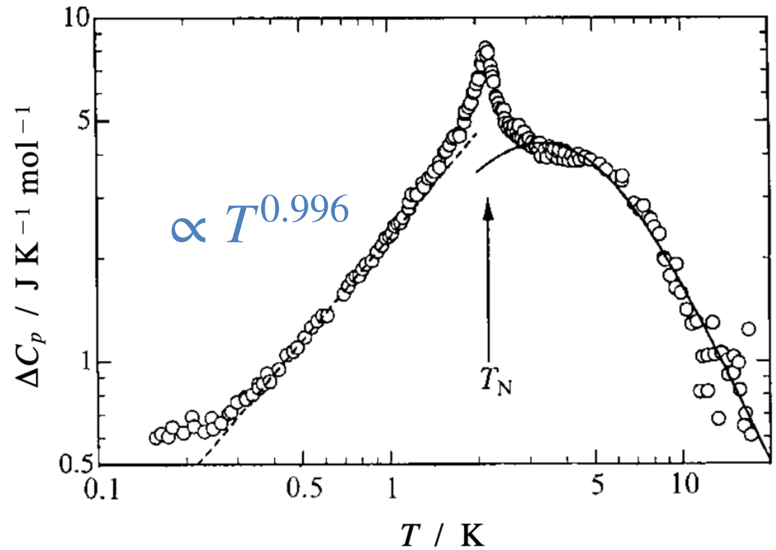


Fig.9 Excess heat capacities of MnCu(obbz)·$5H_2O$ around the antiferromagnetic phase transition temperature T_N. Solid line indicates the theoretical heat capacity curve estimated by the high-temperature series expansion for $S=2$ one-dimensional ferromagnetic Heisenberg model with $J/k_B = 0.75$ K. The broken straight line shows the heat capacity due to the spin-wave excitation.[38]

[K. Asano, et. al., Bull. Chem. Soc. Jpn. 72, 1749 (1999).]

What's the difference?

- Anti-ferro: $\langle \hat{s}_z \rangle = 0$
- Ferro: $\langle \hat{s}_z \rangle = \langle i[\hat{s}_x, \hat{s}_y] \rangle \neq 0$

- $\langle [Q_\alpha, Q_\beta] \rangle = 0$ for $^\forall \alpha, \beta \;\rightarrow\; N_{\mathrm{NG}} = N_{\mathrm{BG}}$

[Schafer-Son-Stephanov-Toublan-Verbaarschot '01]

Number of NG modes

- Non-relativistic systems

$$N_{\mathrm{NG}} = N_{\mathrm{BG}} - \frac{1}{2}\mathrm{rank}\,\langle[Q_\alpha, Q_\beta]\rangle$$

# of gapless NG modes

# of broken generators

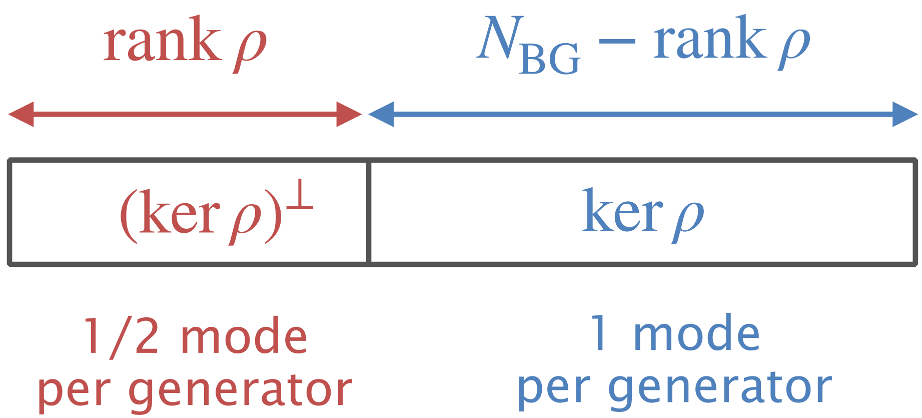
[Watanabe-Brauner '11]
[Watanabe-Murayama'12] [Hidaka'12]

- Dispersion relation
  - Type A: $\omega \sim k$
  - Type B: $\omega \sim k^2$

Derivation by effective field theories

- Up to 2nd order in fields & derivatives,

$$\mathscr{L} = \rho_{AB}\,\pi^A\dot{\pi}^B + \frac{1}{2}G_{AB}\,\dot{\pi}^A\dot{\pi}^B - \frac{1}{2}\bar{G}_{AB}\,\partial_i\pi^A\partial^i\pi^B$$



$$N_{\mathrm{NG}} = \frac{1}{2}\cdot \mathrm{rank}\,\rho + 1\cdot(N_{\mathrm{BG}} - \mathrm{rank}\,\rho) = N_{\mathrm{BG}} - \frac{1}{2}\mathrm{rank}\,\rho$$

Higher-form symmetries

[Gaiotto-Kapustin-Seiberg-Willett '15]

- Charged objects are **extended**
  - "p-form symmetry" → charged obj. is p-dimensional

0-form symmetry   1-form symmetry

- $U(1)^{[p]}$ symmetry: $W(C_p) \to e^{i\alpha} W(C_p)$

Higher-form symmetries

[Gaiotto-Kapustin-Seiberg-Willett '15]

- Classification of phases
  - Certain topological orders are understood as a consequence of broken higher-form symmetries
- Nambu-Goldstone theorem
- 't Hooft anomalies
  - Constraints on the low-energy physics
- …

1-form symmetries of Maxwell theory

Wilson loop $$W(C) \equiv \exp i \int_C a = \exp i \int_C a_\mu dx^\mu$$

Symmetry transformation $$W(C) \mapsto e^{i\alpha} W(C)$$

$$a \mapsto a + \lambda \qquad d\lambda = 0$$

Lagrangian $\mathscr{L} = -\dfrac{1}{2e^2} da \wedge \star da$ is invariant under this tr.

Noether charge $$Q_e(S) = \int_S \mathbf{E} \cdot d\mathbf{S}$$

1-form symmetries of Maxwell theory

Symmetry generator $$U^e_{e^{i\alpha}}(S) = e^{i\alpha Q_e(S)} = e^{i\alpha \int_S \mathbf{E}\cdot d\mathbf{S}}$$

$$U^e_{e^{i\alpha}}(S)W(C)(U^e_{e^{i\alpha}}(S))^{-1} = e^{i\alpha I(S,C)}W(C)$$

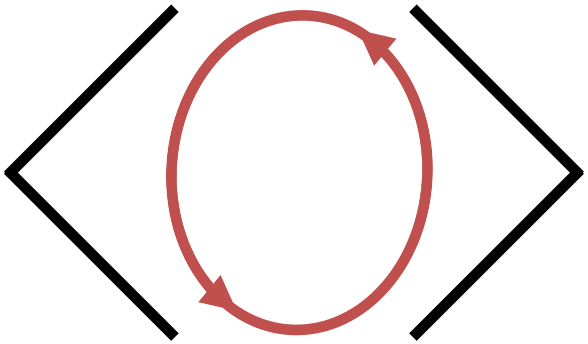
by using $$[E_i(t,\mathbf{x}), a_j(t,\mathbf{y})] = -i\delta_{ij}\delta(\mathbf{x}-\mathbf{y})$$

The generator is topological:

$$Q_e(S) - Q_e(S') = \int_{S\cup(-S')} \mathbf{E}\cdot d\mathbf{S} = \int_V d^3x \nabla\cdot\mathbf{E} = 0$$

$$\partial V = S \cup (-S')$$

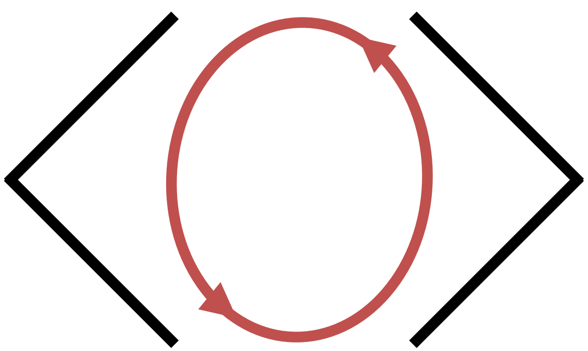
Spontaneous symmetry breaking



$$\simeq \exp\left(-\kappa\, \text{perimeter}[C]\right)$$

$$\simeq 1$$

Broken



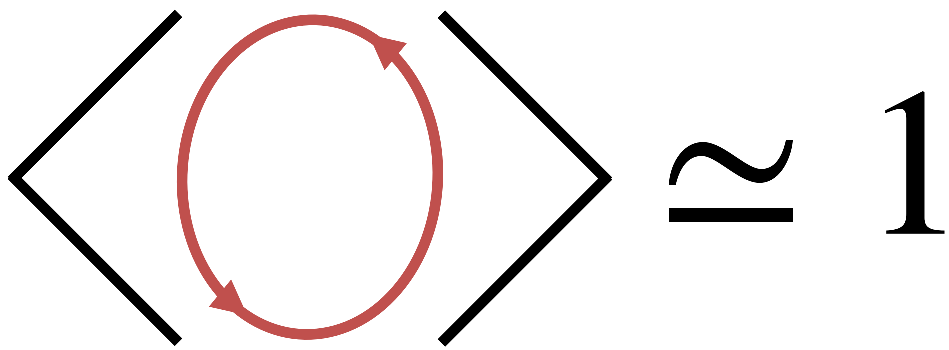
$$\simeq \exp\left(-\kappa\, \text{area}[C]\right)$$

$$\simeq 0$$

Unbroken

NG modes for higher-form symmetries

- Spontaneous breaking of a continuous HFS → gapless excitations



Perimeter law
or Coulomb law

- Breaking of $U(1)_e^{[1]}$ 1-form symmetry → photons

- Questions
  - How many NG modes?
  - Does $\langle [Q_\alpha, Q_\beta] \rangle$ affect the number of NG modes?
    - cf. Non-commuting 0-form and 1-form charges

[Sogabe-Yamamoto'19]

Generalized formula to count NG modes

$$N_{\mathrm{NG}} = \sum_{A} {}_{D-2}C_{p_A} - \frac{1}{2}\mathrm{rank}\,\langle [Q_\alpha, Q_\beta] \rangle$$

[Hidaka-Hirono-Yokokura '21]

- SSB of internal symmetries (that can include higher-form)
- D-dim Minkowski space $\mathbb{R}^{1,D-1}$
  - Lorentz symmetry doesn't have to be there
  - Translational symmetry is intact
- "A" is the label of a generator for a p_A-form symmetry

Generalized formula to count NG modes

- Symmetry generators Q_α should be enumerated taking into account how to place them
  - For 0-form symmetries, $Q(V)$ is $(D-1)$-dimensional
  - For a p-form symmetry, the generator is $(D-p-1)$-dim.
  - Ex) 1-form-symmetry in $(3+1)$-dim. $\rightarrow$ gen. is 2-dim.

$$U(e^{i\alpha}, S) = e^{i\alpha Q^{[1]}(S)}$$

Generalized formula to count NG modes

$$N_{\mathrm{NG}} = \sum_A {}_{D-2}C_{p_A} - \frac{1}{2}\mathrm{rank}\,\langle [Q_\alpha, Q_\beta] \rangle$$

$$\Downarrow$$

When all the symmetries are 0-form

$$N_{\mathrm{NG}} = N_{\mathrm{BG}} - \frac{1}{2}\mathrm{rank}\,\langle [Q_\alpha, Q_\beta] \rangle$$

Reproduce the previous result

[Watanabe-Brauner '11]

[Watanabe-Murayama'12] [Hidaka'12]

Generalized formula to count NG modes

$$N_{\mathrm{NG}} = \sum_A {}_{D-2}C_{p_A} - \frac{1}{2}\mathrm{rank}\,\langle[Q_\alpha, Q_\beta]\rangle$$

- Photons in $(3+1)$-dimensions
  - Broken symmetry: $U(1)_e^{[1]}$

$$N_{\mathrm{NG}} = {}_{D-2}C_p = 2 \qquad D = 4,\ \ p = 1$$

Ex) Photons under θ gradient

[Yamamoto '16]

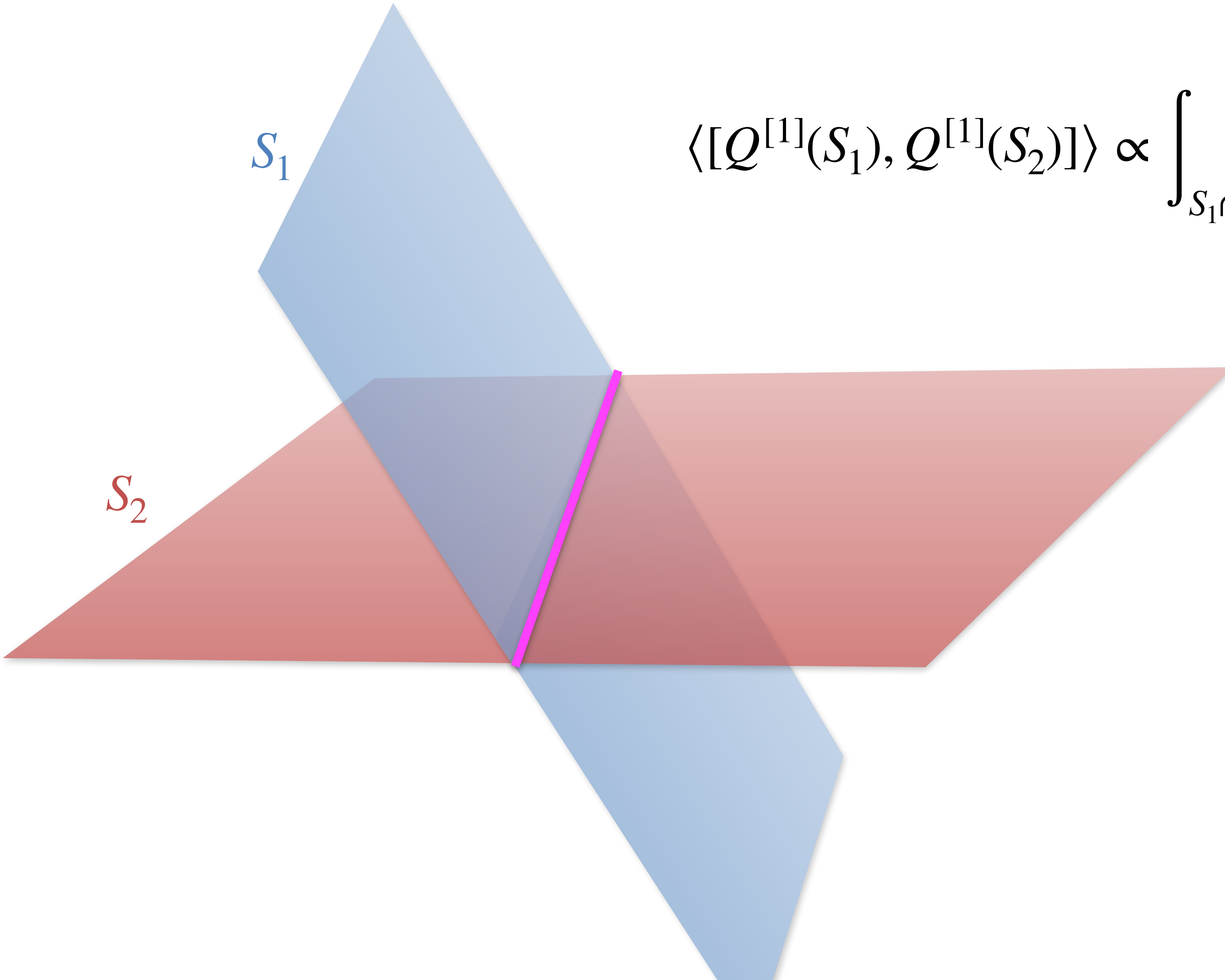
- Lagrangian

$$L = -\frac{1}{2e^2} f \wedge \star f - \frac{\theta}{2\pi} f \wedge f$$

- $d\theta = C = C_i \, dx^i \neq 0$
- Broken symmetry: $U(1)_e^{[1]}$
    - Conserved charge: $Q^{[1]}(S)$
- Charge commutators $\langle [Q^{[1]}(S_1), Q^{[1]}(S_2)] \rangle \propto \int_{S_1 \cap S_2} d\theta \neq 0$

Ex) Photons under θ gradient

$$\langle [Q^{[1]}(S_1), Q^{[1]}(S_2)] \rangle \propto \int_{S_1 \cap S_2} d\theta \neq 0$$



Ex) Photons under θ gradient

- Independent generators: $\{Q^{[1]}(S_x), Q^{[1]}(S_y), Q^{[1]}(S_z)\}$

  S_i is a 2-dim surface perpendicular to i-axis

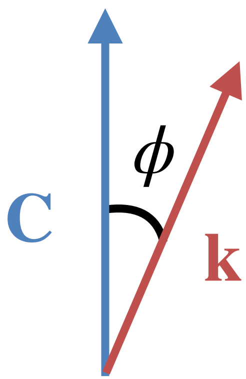
- Charge commutators

$$M_{ij} = \langle\, [Q^{[1]}(S_i), Q^{[1]}(S_j)]\, \rangle \propto \epsilon_{ijk} C_k$$

- Number of gapless NG modes

$$N_{\mathrm{NG}} = {}_{D-2}C_1 - \frac{1}{2}\mathrm{rank}\, M_{ij} = 2 - \frac{1}{2}\cdot 2 = 1$$

Ex) Photons under θ gradient



Dispersion relation

$$\omega^2(k) = \begin{cases} C^2 + (2 - \sin^2\phi)k^2 + O(k^4) & \text{gapped} \\ \sin^2\phi\, k^2 + \dfrac{\cos^4\phi}{C^2}k^4 + O(k^6) & \text{gapless} \end{cases}$$

$$C := |\mathbf{C}|$$

- $\omega \sim k^2$ when $\mathbf{C} \,||\, \mathbf{k}$
- $\omega \sim k$ for other angles
- No generic rule about dispersion relations for higher-form symmetries?

Derivation: effective field theories

- EFT of NG modes
  - Write down terms consistent with the symmetries
  - Derivative expansion
- Higher-form symmetries?

Coset construction

- NG mode: coordinate of G/H $\qquad \xi(x) = e^{i\pi^A(x)X_A}$

$$\{X_A, S_a\}$$

Broken generators $\qquad$ Unbroken generators

- Since $g\xi(\pi) \in G$,

$$g\xi(\pi) = \xi(\pi')h(\theta(\pi, g)) \qquad h(\theta) = e^{i\theta^a(x)S_a}$$

- The coset variable is transformed under $g \in G$ as

$$\xi(\pi) \mapsto \xi(\pi') = g\,\xi(\pi)\,h^{-1}(\theta(\pi, g))$$

Coset construction

- Maurer-Cartan (MC) form

$$\omega(x) = -\,i\xi^{-1}(x)\,d\xi(x)$$
$$= \omega_\perp + \omega_{||}$$

$\propto$ Broken generators $\propto$ unbroken generators

- Transformation under $g \in G$:

$$\omega_\perp \mapsto h\,\omega_\perp h^{-1}$$
$$\omega_{||} \mapsto h\,\omega_{||} h^{-1} - ihdh^{-1}$$

where $h = h(\pi, g)$

Coset construction

- For Lorentz-invariant systems,

$$\mathscr{L} = -F^2 \,\mathrm{tr}\left[\,\omega_\perp \wedge \star\, \omega_\perp\right] = -F^2 \,\mathrm{tr}\left[\,\omega_{\perp\mu}\,\omega_\perp^\mu\right]$$

- If we write $\omega_\perp = e^A{}_B(\pi)\, d\pi^B X_A$

$$\mathscr{L} = -\frac{1}{2} G_{AB}(\pi)\, d\pi^A \wedge \star d\pi^B$$

$G_{AB} = F^2\, e_{CA}(\pi) e^C{}_B(\pi)$ is a G-invariant metric over G/H

Coset construction

- For non-relativistic systems, we can have

$$F_t^2 \,\mathrm{tr}\left[(\omega_{\perp 0})^2\right] - F_s^2 \,\mathrm{tr}\left[\omega_{\perp i}\,\omega_\perp^i\right]$$

- There can also be:

$$\mathrm{tr}\,\left[\omega_{\perp 0}\,X_A\right], \quad \mathrm{tr}\,\left[\omega_{||0}\,S_a\right]$$

$$\mathrm{tr}\,\left[\omega_{\perp 0}X_A\right] \mapsto \mathrm{tr}\,\left[\omega_{\perp 0}\,h^{-1}X_A h\right]$$

Invariant if $[X_A, S_a] = 0$ for $^\forall S_a$

$$\mathrm{tr}\,\left[\omega_{||0}S_a\right] \mapsto \mathrm{tr}\,\left[\omega_{||0}\,h^{-1}S_a h\right] - i\,\mathrm{tr}\left[S_a h \partial_0 h^{-1}\right]$$

Invariant if $[S_a, S_b] = 0$ for $^\forall S_b$

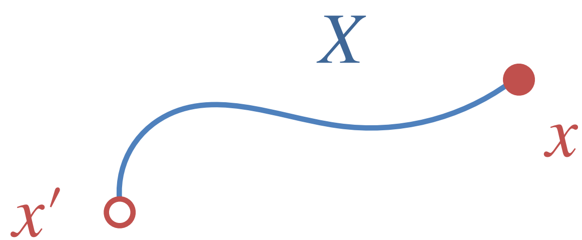
Extension to higher-form symmetries

$U(1)^{[p]}$ breaking $\to$ coset variable $\qquad W(C_p) = \exp i \int_{C_p} a^{(p)}$

Redundancy: $\qquad a^{(p)} \mapsto a^{(p)} + d\theta^{(p-1)}$

Let's write MC form as

$$\omega = -i\xi^\dagger(x)d\xi(x) \quad \Longrightarrow \quad \xi^\dagger(x')\xi(x) = \mathrm{P}e^{i\int_X \omega}$$

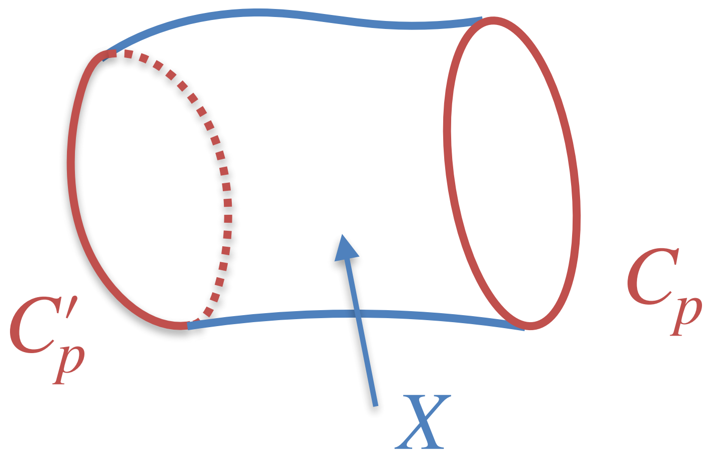


$\partial X = \{x\} \cup \{-x'\}$

Independent on the choice of X $\quad \Longrightarrow \quad$ MC equation $\quad d\omega + i\,\omega \wedge \omega = 0$

Extension to higher-form symmetries

$$\xi^\dagger(x')\xi(x) = \mathrm{P}e^{i\int_X \omega} \quad \Longrightarrow \quad W^\dagger(C'_p)W(C_p) = e^{i\int_X f^{(p+1)}}$$



$\partial X = \{C_p\} \cup \{-C'_p\}$

Independent of the choice of X $\quad \Longrightarrow \quad df^{(p+1)} = 0$

Extension to higher-form symmetries

Coset variable $W(C_p) = \exp i \int_{C_p} a^{(p)}$

Redundancy $a^{(p)} \mapsto a^{(p)} + d\theta^{(p-1)}$

Generalized MC form $W^\dagger(C_p')W(C_p) = e^{i\int_X f^{(p+1)}}$

$f^{(p+1)} = da^{(p)}$ and $df^{(p+1)} = 0$

$$\mathscr{L} = -\frac{1}{2e^2} f^{(p+1)} \wedge \star f^{(p+1)}$$

$f^{(p+1)}$ is nothing but the field strength in Maxwell theory

MC eq. ↔ Bianchi identity ↔ magnetic symmetry

Effective Lagrangian

- For systems without Lorentz invariance, terms with one derivative is possible:

$$\Omega_{AB} \wedge a_A \wedge da_B \qquad \text{where } \Omega_{AB} \text{ is a } (D - p_A - p_B - 1)\text{-form}$$

- In the case of 0-form symmetries, $\Omega_{AB} \propto dx \wedge dy \wedge dz$ for $D = 4$

- Those terms lead to $\langle [Q_A^{[p_A]}(V_A), Q_B^{[p_B]}(V_B)] \rangle \propto \int_{V_A \cap V_B} \Omega_{AB}$

$V_A : (D - p_A - 1)$-dim. subspace, $V_B : (D - p_B - 1)$-dim. subspace

Conclusions

- Generalized formula for counting NG modes for higher-form symmetries,

$$N_{\mathrm{NG}} = \sum_{A} {}_{D-2}C_{p_A} - \frac{1}{2}\mathrm{rank}\,\langle [Q_\alpha, Q_\beta] \rangle$$

- Not sure if there is a generic rule on dispersion relations
- EFT for broken higher-form symmetries
  - Generalization of MC form
  - MC eq. $\leftrightarrow$ Bianchi identity $\leftrightarrow$ magnetic symmetry

Backup slides

Ex) Neutral pion + photon + external **B**

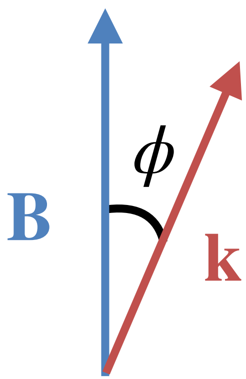
- Kinetic term $$L_{\text{kin}} = -\frac{1}{2e^2} f \wedge \star f - \frac{1}{2} v^2 d\pi \wedge \star d\pi$$
- mixing $$L_{\text{mix}} = c\,\pi (f + B) \wedge (f + B) \supset 2c\,\pi\, f \wedge B$$
- Charge commutators $$\langle\, [Q_A^{[0]}(V), Q_e^{[1]}(S)]\,\rangle \propto \int_{V \cap S} B$$
- $Q_\alpha = \{Q_A^{[0]}(V), Q_e^{[1]}(S_x), Q_e^{[1]}(S_y), Q_e^{[1]}(S_z)\}$

[Sogabe-Yamamoto'19]

$$\text{rank}\,\langle [Q_\alpha, Q_\beta] \rangle = \text{rank} \begin{pmatrix} 0 & \mathbf{B}^T \\ -\mathbf{B} & \mathbf{0}_{3\times 3} \end{pmatrix} = 2$$

- # of gapless NG modes: $N_{\text{NG}} = (1 + 2) - \frac{1}{2} \cdot 2 = 2$

Ex) Neutral pion + photon + external **B**



Dispersion relation

$$\omega^2(k) = \begin{cases} \alpha^2 + (1+\sin^2\phi)k^2 + O(k^4) & \text{gapped} \\ \cos^2\phi\, k^2 + \dfrac{\sin^4\phi}{\alpha^2} k^4 + O(k^6) & \text{gapless} \\ k^2 & \text{gapless} \end{cases}$$

$$\alpha := \frac{c\,|\mathbf{B}|}{v}$$

- $\omega \sim k^2$ when $\mathbf{B} \perp \mathbf{k}$
- $\omega \sim k$ otherwise