Skyrmions in a magnetic field and π^0 domain wall formation in dense nuclear matter

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We elucidate magnetic effects in the Skyrmion system to probe into the dense nuclear matter. We find a deformed π^0 dipole structure of a Skyrmion and quantify nontrivial rearrangements of the confining pressure distribution. We confirm an isospin-dependent baryon spectrum from the anomaly-induced action. We then extend our scope to stacked Skyrme Crystal layers to scrutinize phases of magnetized nuclear matter. We observe a quantized magnetic flux and identify a phase transition from a crystalline state to a π^0 domain wall corresponding to a topological transmutation from $\pi_3(S^3)$ to $\pi_1(S^1)$. We establish the phase diagram, which could be explored also in analogous systems with two-component Bose-Einstein condensates.

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Skyrmion: Model

Lagrangian of $\Sigma \in SU(2)$ (w/ $N_f = 2, N_c = 3$):

$$\mathcal{L} = \mathrm{Tr}\left(-\frac{f_\pi^2}{4} L_\mu L^\mu + \frac{f_\pi^2 m_\pi^2}{2}(\Sigma - 1) + \frac{1}{32a^2}[L_\alpha, L_\beta][L^\alpha, L^\beta]\right) + \mathcal{L}_{\mathrm{WZW}}$$

$SU(2)_{L/R}$ currents, covariant derivative, charge matrix:

$$L_\mu \equiv \Sigma^\dagger D_\mu \Sigma,\ \ R_\mu \equiv \Sigma D_\mu \Sigma^\dagger; \quad D_\mu \Sigma \equiv \partial_\mu \Sigma - i A_\mu [Q, \Sigma],\ \ Q = \frac{1}{6} + \frac{\tau^3}{2}$$

Gauged Wess-Zumino-Witten term:

$$\mathcal{L}_{\mathrm{WZW}} = \frac{1}{2}\int d^4x A_\mu J_B^\mu :\ \ J_B^\mu = \frac{\varepsilon^{\mu\nu\alpha\beta}}{24\pi^2}\mathrm{Tr}\left(L_\nu L_\alpha L_\beta + \frac{3i}{2} F_{\alpha\beta} Q (L_\nu - R_\nu)\right)$$

Skyrmion: Setup

Magnetic field: homogeneous and downward

$$\boldsymbol{B} = -B\hat{z} \ \ (B > 0)\,, \ \text{i.e.,} \ A_0 = 0 \ \text{and} \ \boldsymbol{A} = \frac{1}{2} B\boldsymbol{r} \times \hat{z}$$

Symmetry: axial component of $SU(2)_V \times SU(2)_{rot}$

$$(\Sigma\,(r \to \infty) \to +1) \quad \Sigma\left(\boldsymbol{r}e^{\times\alpha\hat{z}}\right) = e^{i\alpha Q}\Sigma\,(\boldsymbol{r})\,e^{-i\alpha Q}$$

Parametrization: $\Sigma = i(\tau_1\Pi^1 + \tau_2\Pi^2 + \tau_3\Pi^3) + \Pi^4$

$$\Pi_1 = \sin f \ \sin g \ \cos\varphi \qquad \Pi_3 = \sin f \ \cos g$$

$$\Pi_2 = \sin f \ \sin g \ \sin\varphi \qquad \Pi_4 = \cos f$$

$$\text{To be solved:} \quad f = f\,(r,\theta)\,, \quad g = g\,(r,\theta)$$

Skyrmion: EOM

Energy functional $M = 2\pi \int_0^\infty dr \int_0^\pi d\theta r^2 \sin\theta \, T^{00}$

$$T^{00} = \frac{f_\pi^2}{2}\left\{|\nabla f|^2 + \sin^2 f \left[|\nabla g|^2 + \Upsilon^2 \sin^2 g\right]\right\} \quad \left(\text{w/ } \Upsilon \equiv \frac{1}{r\sin\theta} - \frac{Br\sin\theta}{2}\right)$$
$$+ \frac{1}{2a^2}\sin^2 f \left\{|\nabla f \times \nabla g|^2 + \Upsilon^2 \sin^2 g \left(|\nabla f|^2 + \sin^2 f \, |\nabla g|^2\right)\right\}$$

Static equation of motion (EOM): minimizing M

$$\delta M/\delta f = 0 \text{ and } \delta M/\delta g = 0 \quad \left(\text{w/ scaling: } \bar{r} \equiv r f_\pi a, \ \bar{B} \equiv \frac{B}{f_\pi^2 a^2} \text{ etc.}\right)$$

Boundary conditions: baryon number amounts to 1

$$f(\infty,\theta) = 0, \ f(0,\theta) = \pi, \ g(r,0) = 0, \ g(r,\pi) = \pi$$

Skyrmion: Profile

$$\Pi_4 = \cos f,\ \Pi_3 = \sin f \cos g,\ \Pi_2 = \sin f \sin g \sin\phi,\ \Pi_1 = \sin f \sin g \cos\phi$$

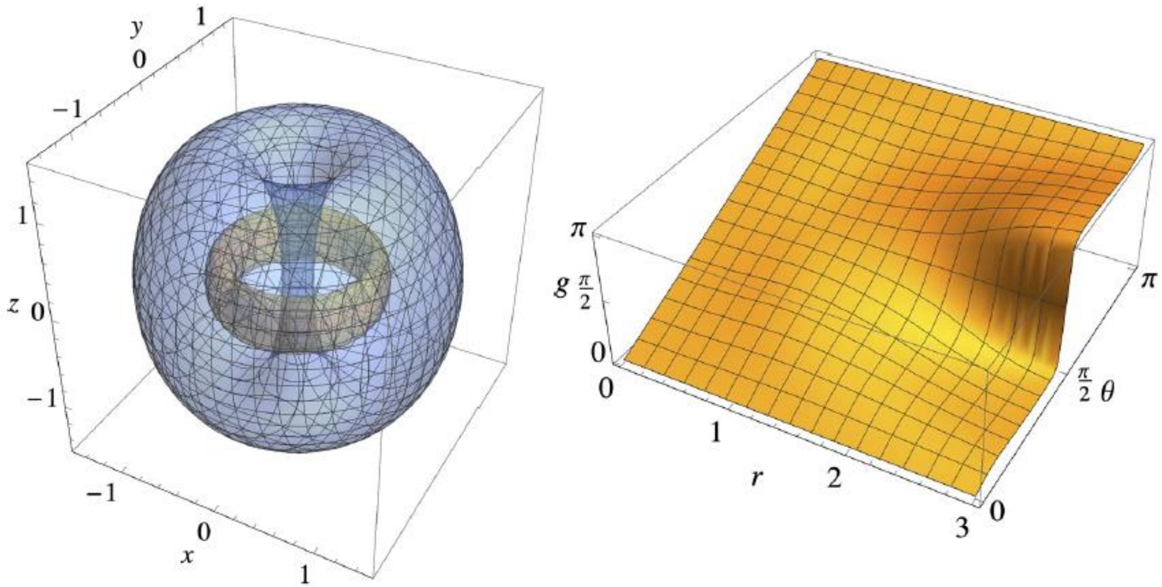
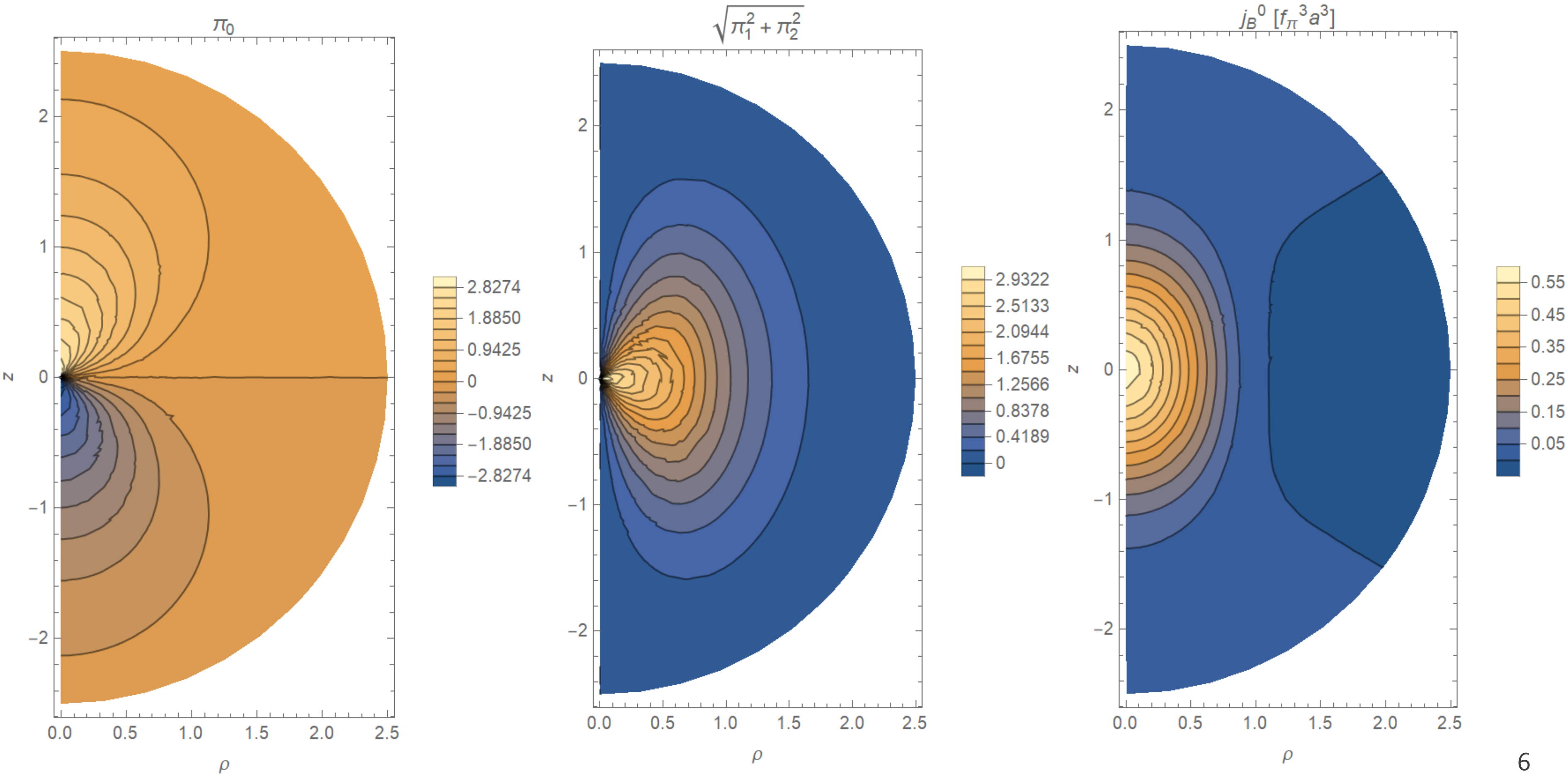


FIG. 1. (Left) Graphical representation of $\pi_3(\mathrm{SU}(2))$ with $B = 3.00 f_\pi^2 a^2$; $\Pi_1^2 + \Pi_2^2 = 0.90$ on the inner torus (orange) and $\Pi_3^2 + \Pi_4^2 = 0.90$ on the outer torus (blue). (Right) $g(r,\theta)$ with $B = 3.00 f_\pi^2 a^2$. (x, y, z and r are of unit $f_\pi^{-1} a^{-1}$.)

Skyrmion: Pion

$$\Sigma = \exp\left(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}\right):\ \ \pi^1 = f\sin g\cos\phi,\ \ \pi^2 = f\sin g\sin\phi,\ \ \pi^0 = f\cos g$$



Quantization: Nucleon

Lagrangian:

$$\Sigma(t) = e^{i\alpha(t)Q}\Sigma e^{-i\alpha(t)Q} \Longrightarrow \mathcal{L} = -M - \Phi\dot{\alpha} + \frac{1}{2}\Gamma\dot{\alpha}^2$$

Hamiltonian:

$$\beta \equiv \frac{\delta\mathcal{L}}{\delta\dot{\alpha}}, \quad H \equiv \beta\dot{\alpha} - \mathcal{L} = M + \frac{1}{2\Gamma}(\beta + \Phi)^2$$

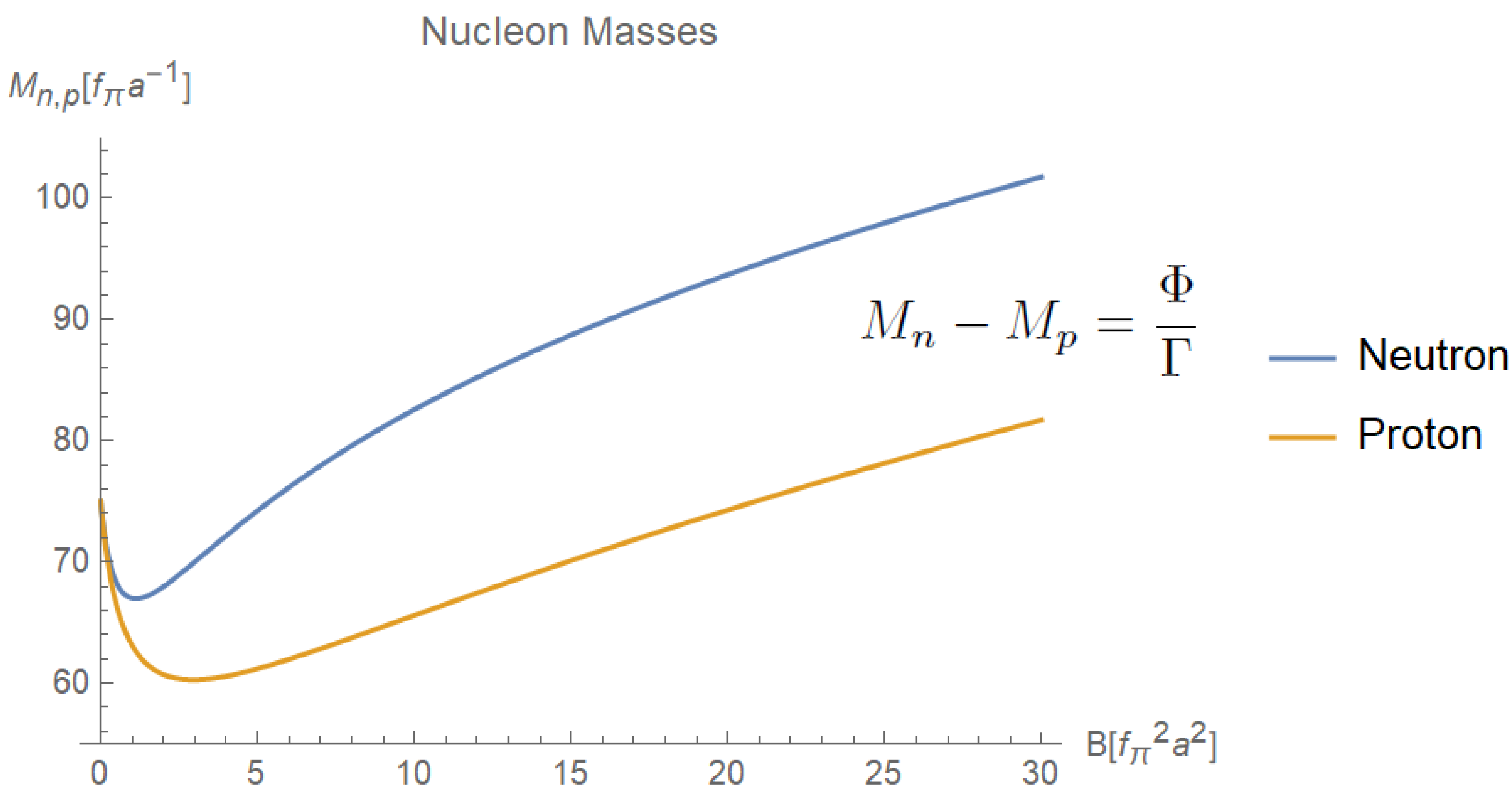
Quantum Numbers

$$J_3 = -I_3 = \beta; \quad \beta = \frac{2n-1}{2}, \quad n \in \mathbb{Z}$$

Nucleon eigenstates

$$\text{Ground state: } |p\downarrow\rangle\ ; \ \text{First excitation: } |n\uparrow\rangle$$

Quantization: Spectrum



$$M_n - M_p = \frac{\Phi}{\Gamma}$$

Skyrme Crystal: Background

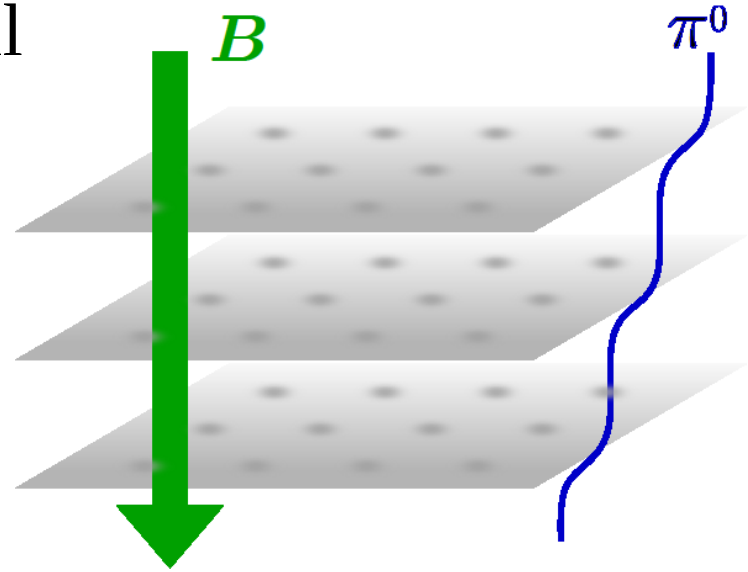
Skyrme Crystal & π^0 domain wall

I. R. Klebanov, Nucl. Phys. B 262, 133 (1985).

$$\Sigma(x_i) = \tau_i \Sigma^\dagger(-x_i)\tau_i; \quad i = x, y, z$$

Klebanov's:

$$\begin{aligned}\Sigma &= \tau_y \Sigma(x+c)\tau_y \\ &= \tau_z \Sigma(y+c)\tau_z \\ &= \tau_x \Sigma(z+c)\tau_x\end{aligned}$$



T. Brauner and N. Yamamoto, JHEP 04, 132 (2017).

Ours: "twisted" periodicity & reflection for 2D crystal.

$$\tau^3\Sigma\tau^3 = e^{i\lambda ByQ}\Sigma(x+2\lambda)e^{-i\lambda ByQ} = e^{-i\lambda BxQ}\Sigma(y+2\lambda)e^{i\lambda BxQ}$$

Skyrme Crystal: EOM

Energy functional in Cartesian coordinates:

$$\frac{M}{8} = \int d^3x \left(\frac{f_\pi^2}{2} \left| \boldsymbol{D}\Pi_i \right|^2 + \frac{1}{4a^2} \left| \boldsymbol{D}\Pi_i \times \boldsymbol{D}\Pi_j \right|^2 \right)$$

Covariant derivative notations:

$$\boldsymbol{D}\Pi_1 \equiv \boldsymbol{\nabla}\Pi_1 - \boldsymbol{A}\Pi_2, \;\; \boldsymbol{D}\Pi_1 \equiv \boldsymbol{\nabla}\Pi_2 + \boldsymbol{A}\Pi_1; \;\; \boldsymbol{A} = \frac{1}{2}\left(yB, -xB, 0\right)$$

Equation of Motion (with Langrange multiplier γ):

$$\frac{\delta}{\delta \Pi_i}\left[M + \gamma \left(\sum_{i=1}^{4} \Pi_i^2 - 1 \right) \right] = 0, \quad \left(\text{and } \frac{\delta}{\delta \gamma}\left[\ldots\right] = 0 \right)$$

Skyrme Crystal: Boundary

Vanishing pion on upper ceiling:

$$\Pi_4(x, y, \infty) = 1$$

Requirement by Reflection symmetry:

$$\Pi_1(0, y, z) = \Pi_2(x, 0, z) = \Pi_3(x, y, 0) = 0$$

Consistency between reflection and periodicity:

$$\Pi_1(\lambda, y, z)\sin\left(\frac{1}{2}\lambda By\right) - \Pi_2(\lambda, y, z)\cos\left(\frac{1}{2}\lambda By\right) = 0$$

$$\Pi_1(x, \lambda, z)\sin\left(\frac{1}{2}\lambda Bx\right) - \Pi_2(x, \lambda, z)\cos\left(\frac{1}{2}\lambda Bx\right) = 0$$

Skyrme Crystal: Winding

Baryon density in differential forms:

$$\star j_B = \frac{1}{24\pi^2}\left\{ \mathrm{Tr}\left(L \wedge L \wedge L\right) + \frac{3}{2} F \wedge \mathrm{Tr}\left[iQ\left(L - R\right)\right]\right\}$$

$$\int d^3x j_B^0 = \frac{1}{4\pi^2}\int_S (d\varphi - A) \wedge (\Pi_4 d\Pi_3 - \Pi_3 d\Pi_4)$$

Only winding around two edges:

$$(0,0,0 \to \infty) : \frac{1}{8} n_0; \quad (\lambda, \lambda, 0 \to \infty) : \frac{1}{8} n_\lambda \left(\frac{2}{\pi}\lambda^2 B - 1\right)$$

$$n_\# = \int_0^\infty dz \left(\cos g \frac{\partial f}{\partial z} - \sin f \cos f \sin g \frac{\partial g}{\partial z}\right)\Big|_{\text{edge } \#}$$

Skyrme Crystal: Classes

Final boundary condition: unit baryon number

$$N_B = n_0 + n_\lambda \boxed{\left(\frac{2}{\pi}\lambda^2 B - 1\right)} \equiv 1$$

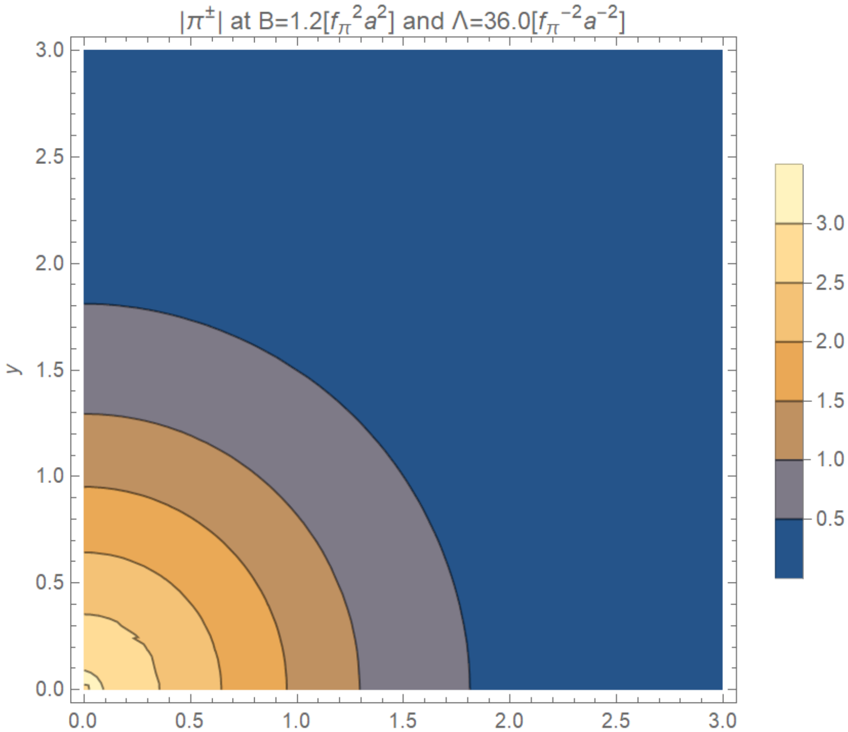
Two classes of solutions, two types of crystal:

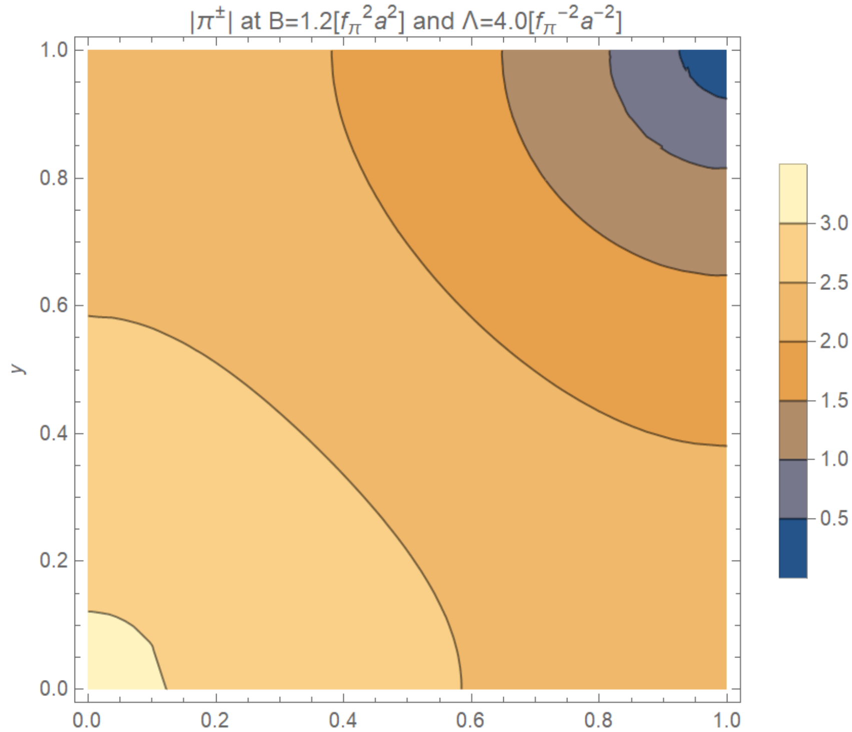
$$\text{Normal:} \quad \Pi_4\left(0,0,0\right) = -1, \quad \Pi_4\left(\lambda,\lambda,0\right) = 1$$

$$\text{Anomalous:} \quad \Pi_4\left(0,0,0\right) = -1, \quad 4\lambda^2 B = 2\pi$$

Skyrme Crystal: Solution

Anomalous Crystal = π^0 domain wall.
Then what about Normal Crystal?

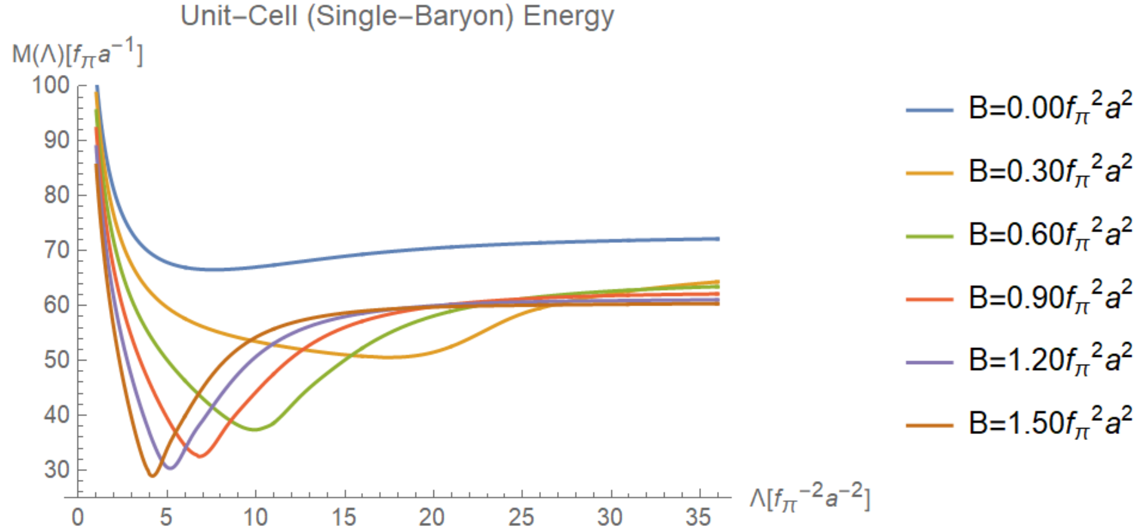


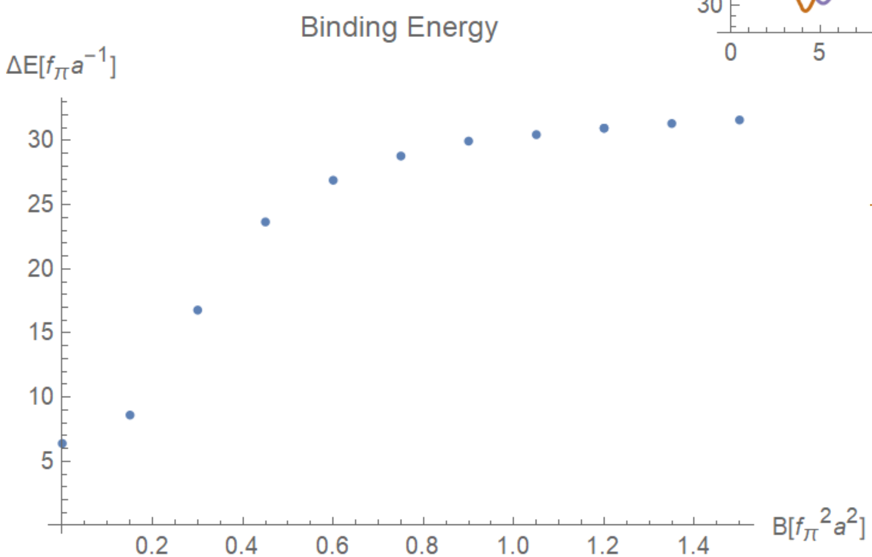


$$\Lambda = 4\lambda^2 \text{ at } 36 f_\pi^{-2} a^{-2} \text{ (Left) and } 4 f_\pi^{-2} a^{-2} \text{ (Right)}$$

Skyrme Crystal: Crystallization

$$M = 8\int_{\text{octant}} d^3x T^{00}$$





$$M(\Lambda \to \infty) \to \text{Single Soliton Mass}$$

$$\Lambda_0 : \quad M(\Lambda_0) = M_{\min}$$

$$\Delta E \equiv M(\infty) - M(\Lambda_0)$$

Thermodynamics: Basics

Helmholtz free energy of normal crystal:

$$F\left(N\Lambda, N, B\right) \equiv N\mathcal{E}\left(B, \Lambda\right)$$

Transverse pressure and chemical potential:

$$\sigma \equiv -\left(\frac{\partial F}{\partial\left(N\Lambda\right)}\right)_{B,N} = -\left(\frac{\partial \mathcal{E}}{\partial \Lambda}\right)_{B}, \quad \mu = \mathcal{E} + \sigma\Lambda$$

One less variable for anomalous crystal!

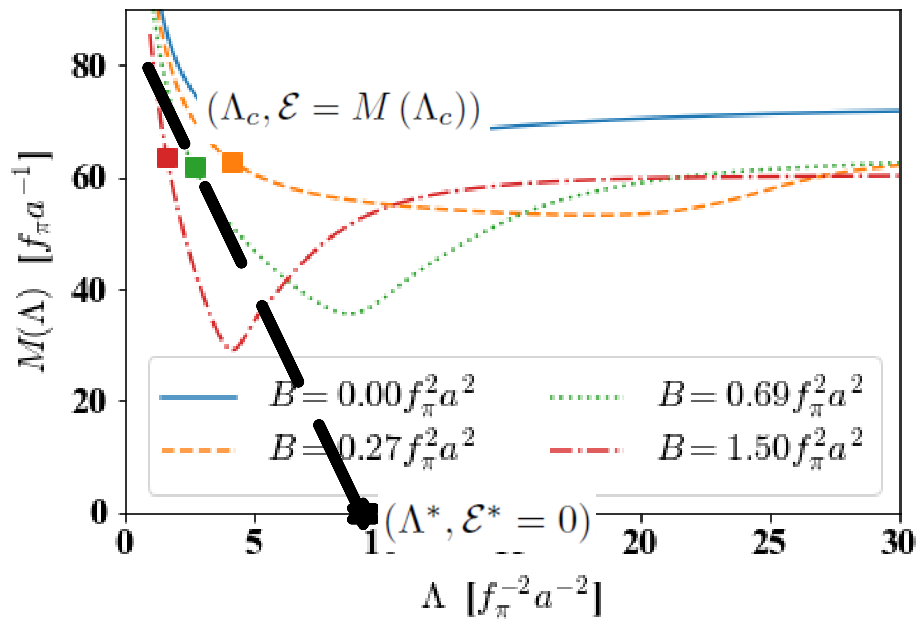
$$F^{*}\left(B, N^{*}\right) \equiv N^{*}\mathcal{E}^{*}\left(B\right)$$

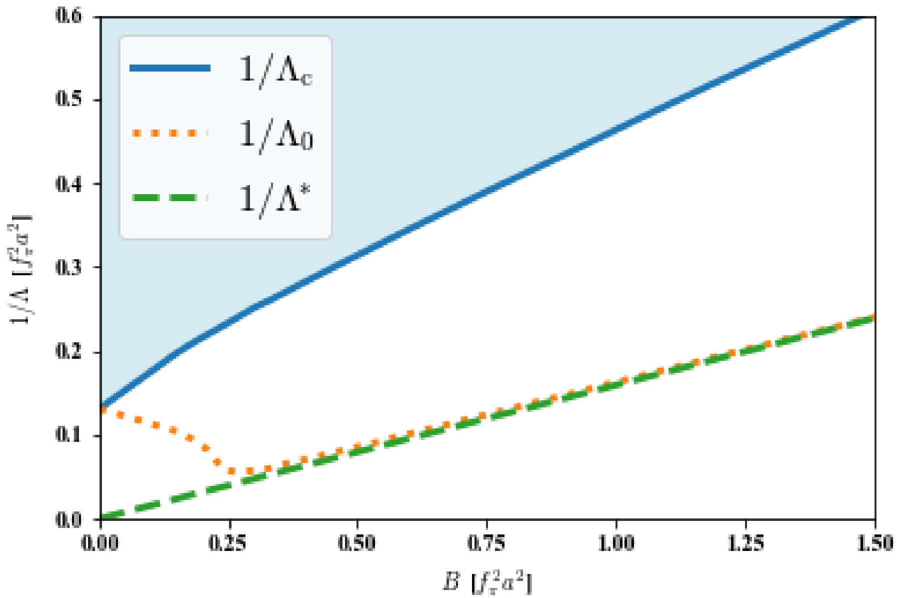
Equilibrium between normal and anomalous phases:

$$0 = \delta\left(F + F^{*}\right)\Big|_{B, N+N^{*}, N\Lambda+N^{*}\Lambda^{*}} \Rightarrow \frac{\mathcal{E} - \mathcal{E}^{*}}{\Lambda - \Lambda^{*}} = -\sigma$$

Thermodynamics: Phases

$$\Lambda_c : \quad \frac{\mathcal{E} - \mathcal{E}^*}{\Lambda - \Lambda^*} = \left(\frac{\partial \mathcal{E}}{\partial \Lambda} \right)_B$$





$$\Delta Q = M(\Lambda_c) - 0$$

$$\Delta E = M(\infty) - M(\Lambda_0)$$

(Figs from our ***arXiv: 2104.11482***)

ArXiv: 2104.11482

1. A Skyrmion is deformed into an ellipsoid under an external magnetic field.
2. The quantized Skyrmions as nucleons feature mass split by magnetic field.
3. Two classes of Skyrme Crystals exhibit $\pi^3(S^3)$ and $\pi^1(S^1)$ respectively.
4. First order phase transition takes place between the Normal Crystal and the π^0 domain wall.