New results on convergence of hydrodynamic gradient expansion

Michal P. Heller

aei.mpg.de/GQFI

“old” results reviewed in **1707.02282** with Florkowski & Spaliński

new results: **2007.05524** + work in progress with Serantes, Spaliński, Svensson & Withers

New results on convergence of hydrodynamic gradient expansion (and on hydrodynamic attractors)

Michal P. Heller

aei.mpg.de/GQFI

"old" results reviewed in **1707.02282** with Florkowski & Spaliński

new results: **2007.05524** + work in progress with Serantes, Spaliński, Svensson & Withers

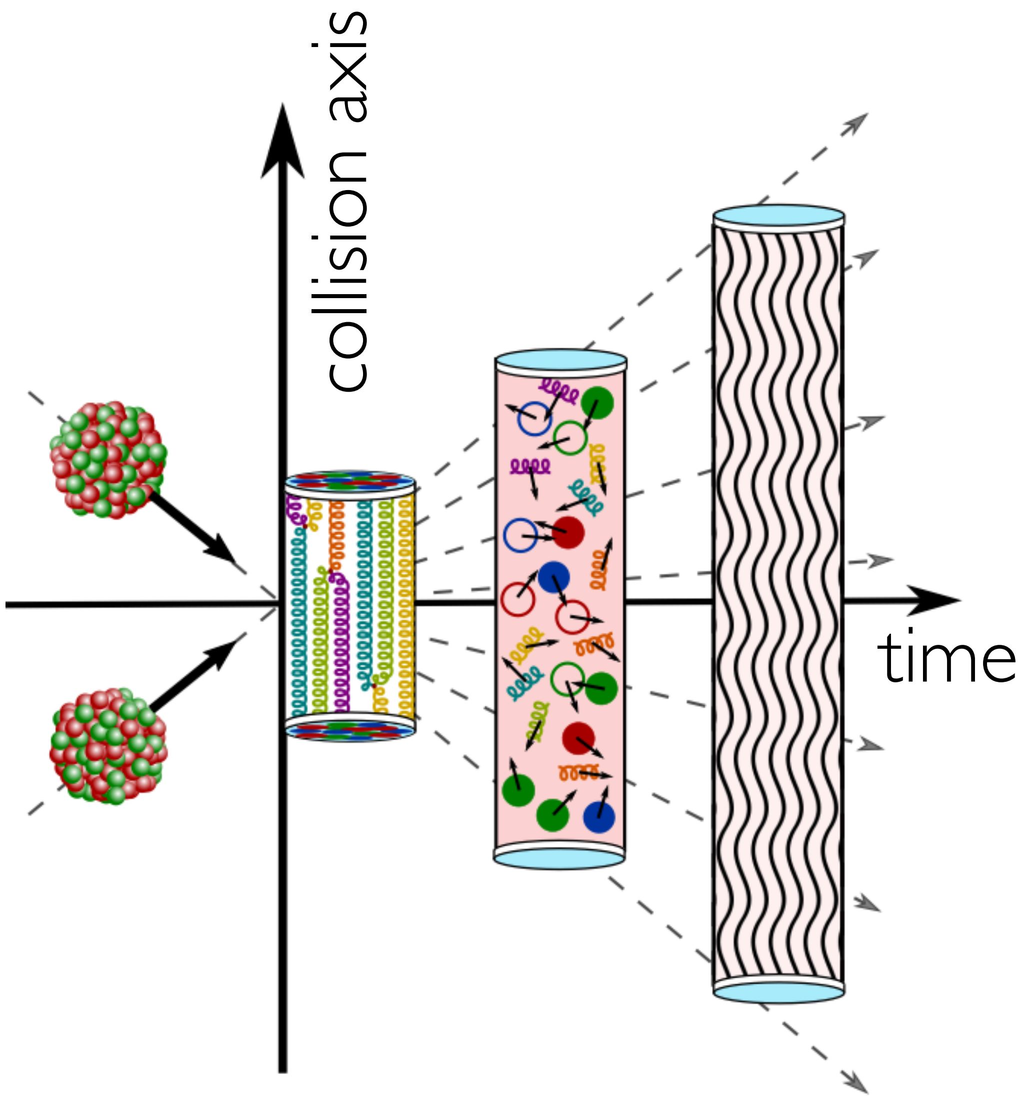
attractors: old **1503.07514** with Spaliński; new **2003.07368** also with Jefferson & Svensson

“Old” results (2011-2018)

reviewed in **1707.02282** with Florkowski & Spaliński

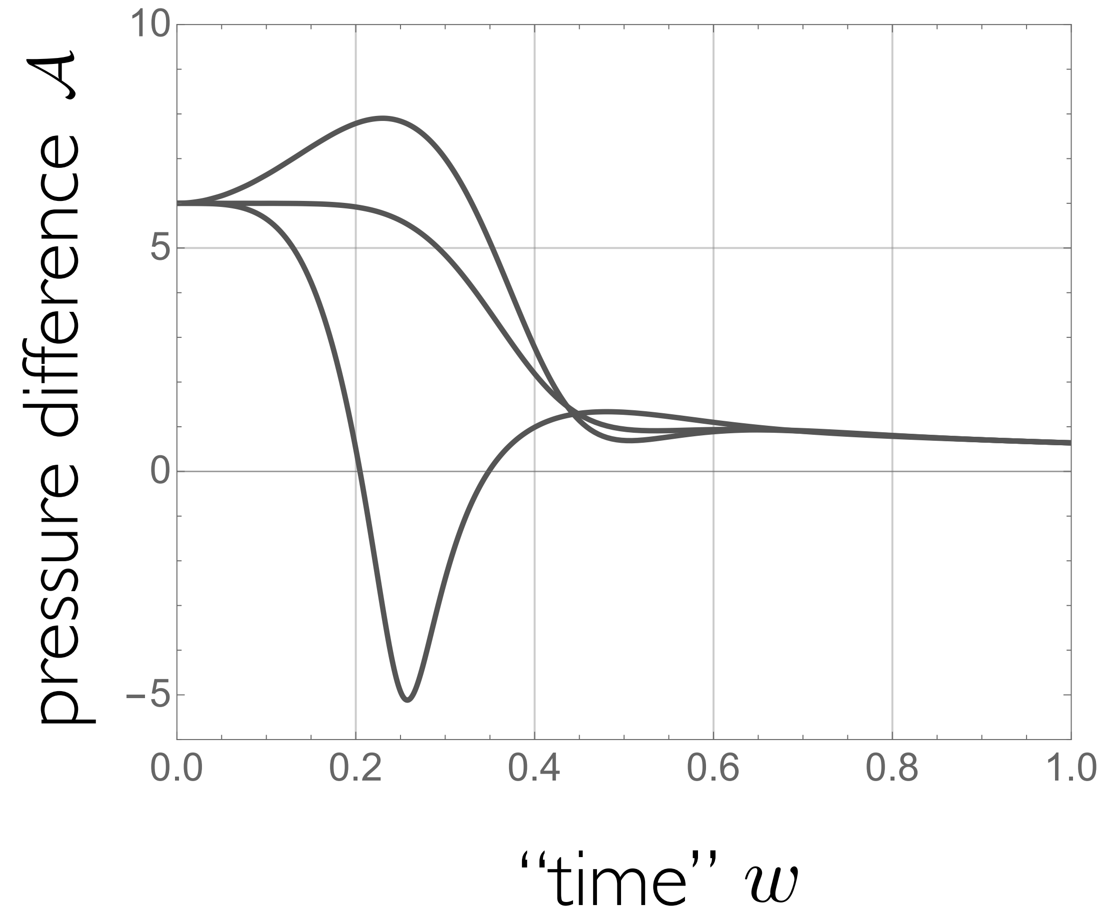
Motivation

heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

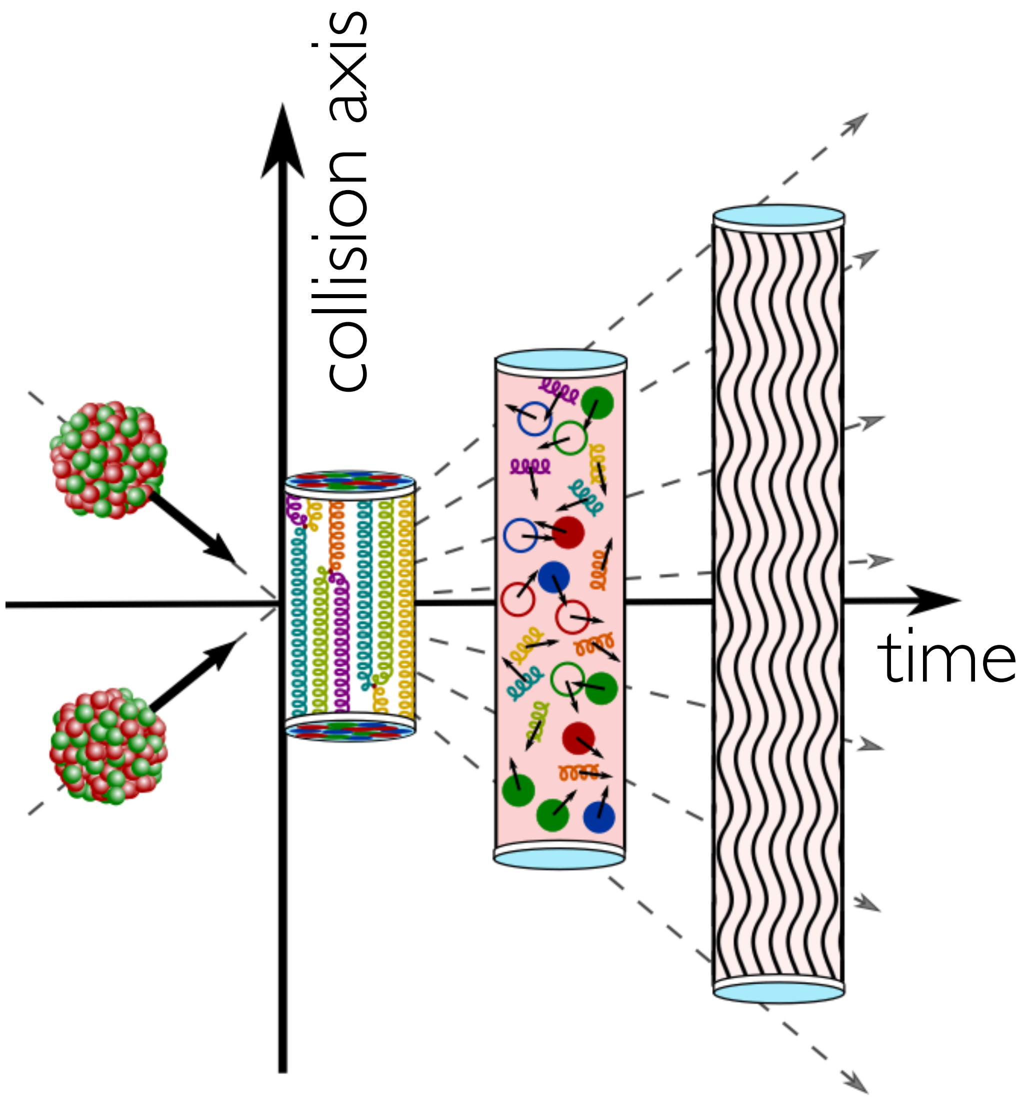
behaviour in 3 classes
of theoretical models
(here: holography)



1103.3452 with Janik & Witaszczyk

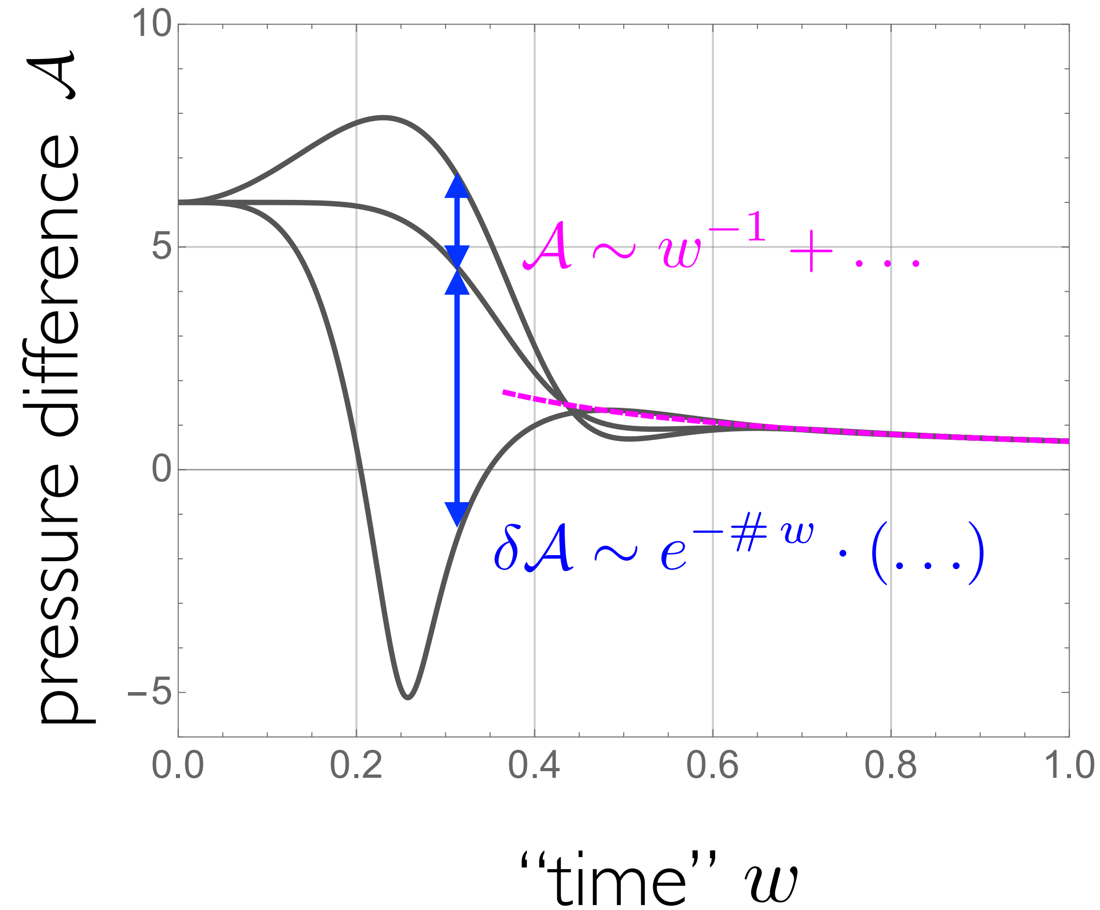
Motivation

heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

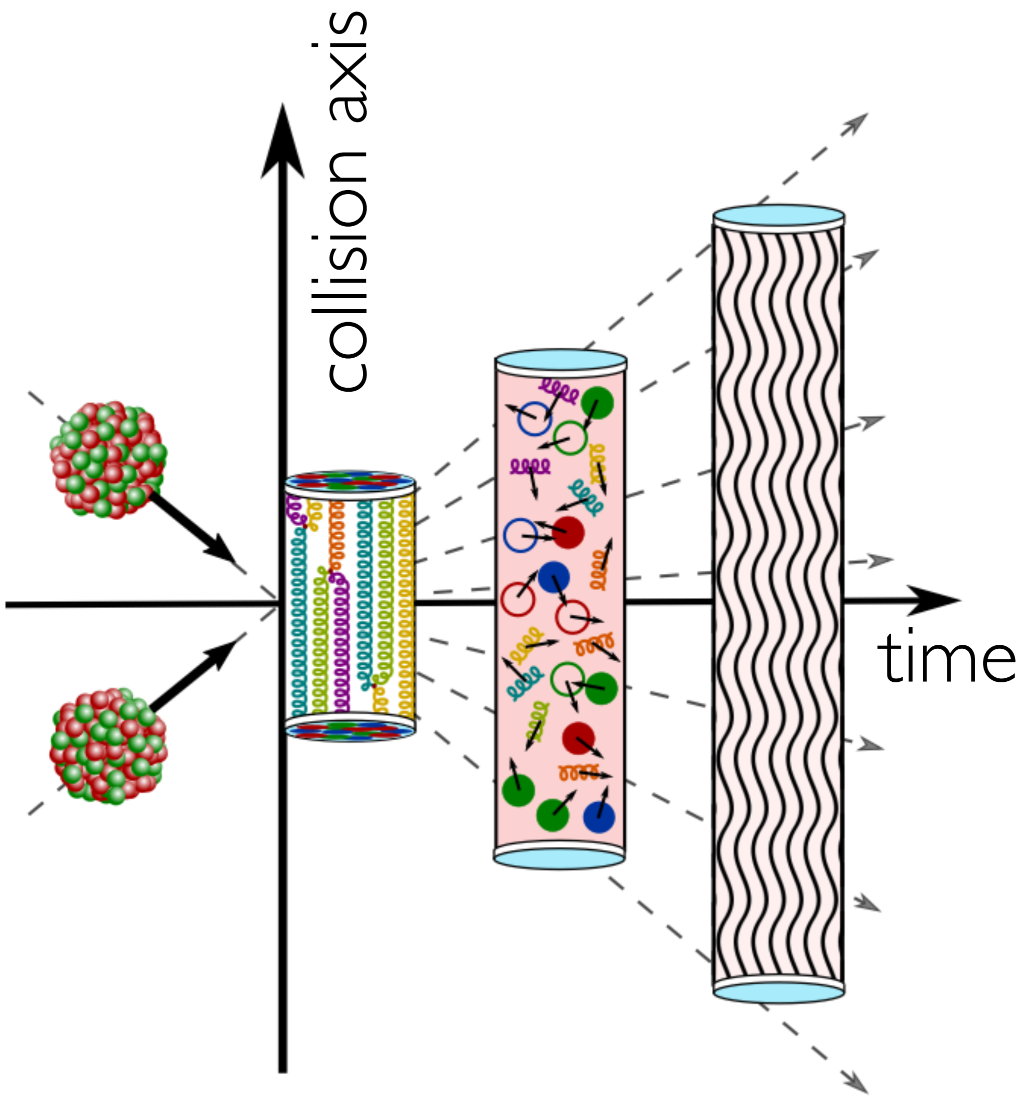
behaviour in 3 classes
of theoretical models
(here: holography)



1103.3452 with Janik & Witaszczyk

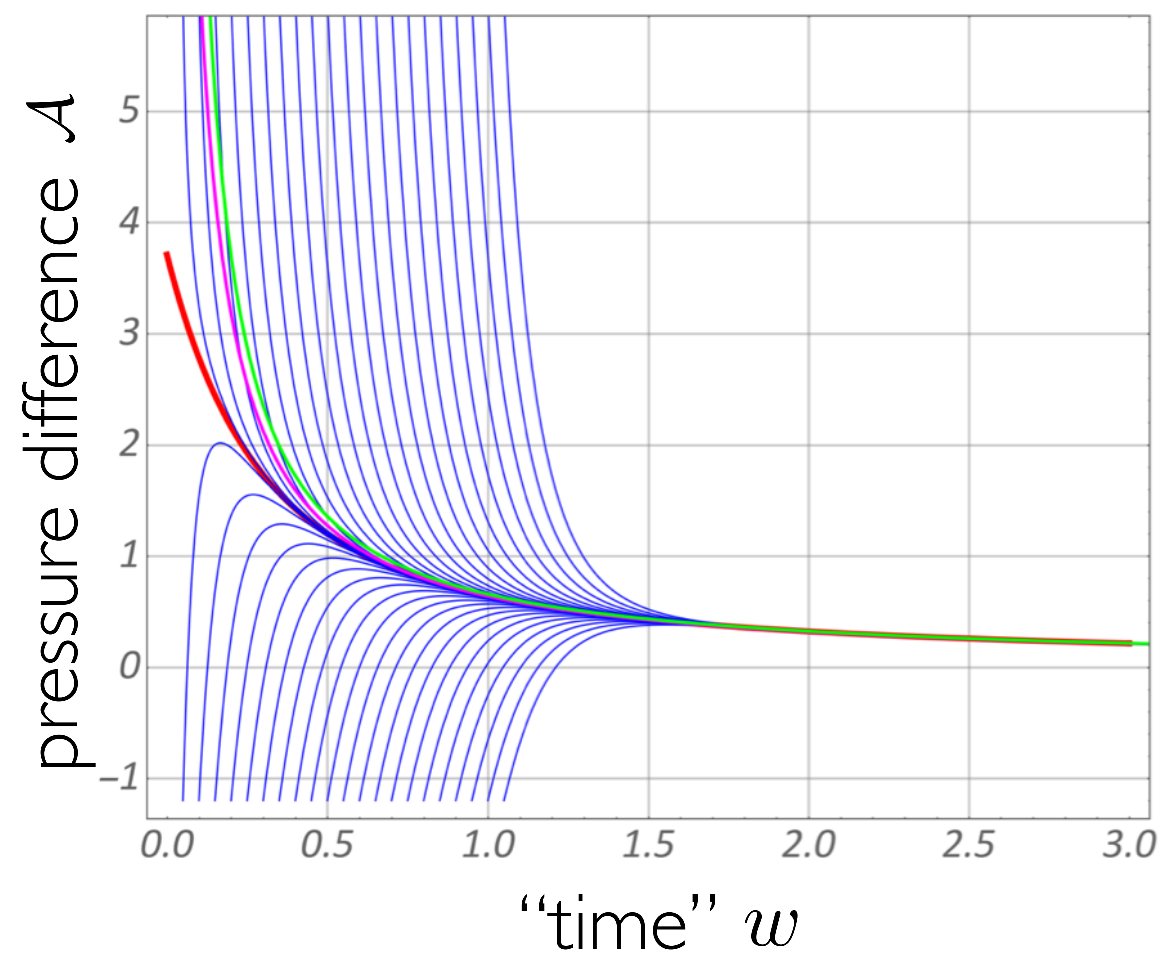
Motivation

heavy-ion collisions
at RHIC and LHC



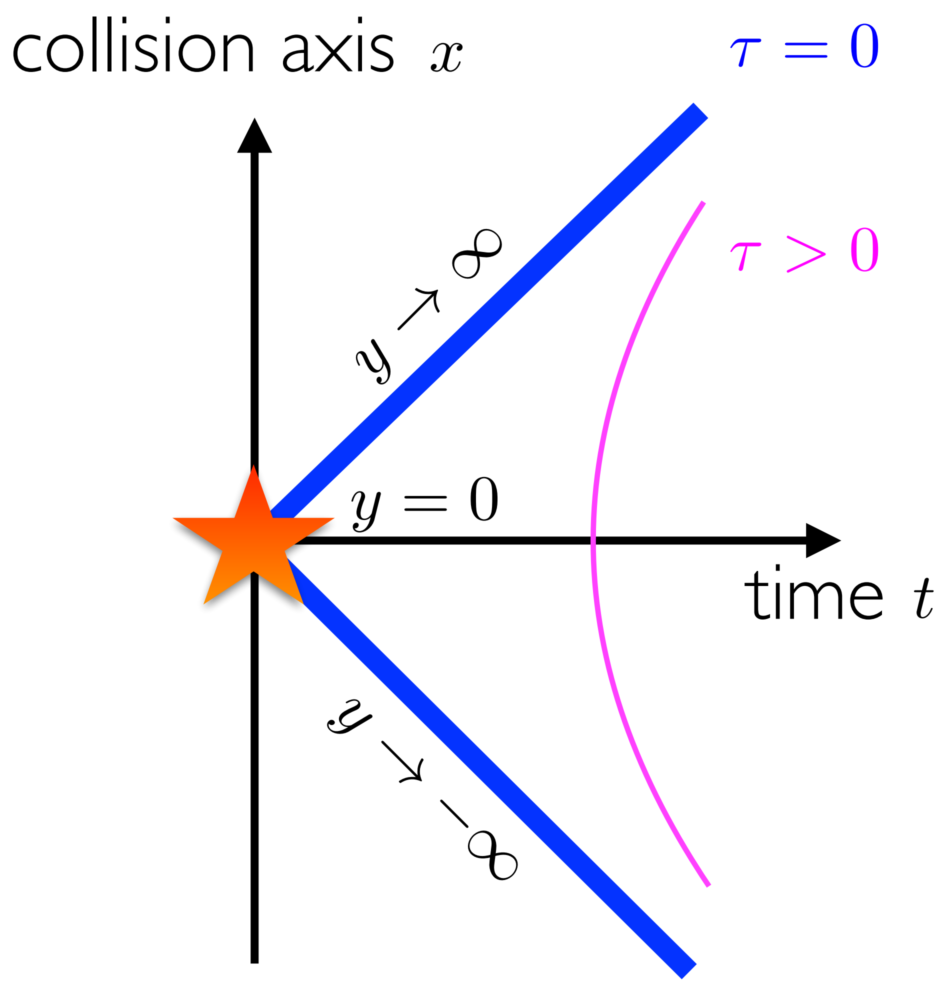
2005.12299
with Berges, Mazeliauskas & Venugopalan

behaviour in 3 classes
of theoretical models
(here: MIS)



1503.07514 with Spaliński

Bjorken flow: basics Bjorken 1982



+ two transverse coordinates

Bjorken's simplification: physics is the same in all longitudinally boosted reference frame; this is Lorentzian analogue of rotational invariance.

analogue of the radius: $\tau = \sqrt{t^2 - x^2}$ analogue of the angle: $y = \mathrm{arccosh}(t/x)$

Relativistic hydrodynamics: basics

e.g. **1205.5040** by Kovtun; **1707.02282** with Spaliński & Florkowski; **1712.05815** by Romatschke[2]

In relativistic systems like nuclear matter at RHIC and LHC energy and momentum are encoded in the energy-momentum tensor $T^{\mu\nu}$

General energy-momentum tensor has 10 components subject to 4 conservation equations $\nabla_\mu T^{\mu\nu} = 0$

Relativistic hydrodynamics is built on an assumption that if we wait long enough the number of degrees of freedom (independent components of $T^{\mu\nu}$) reduces to these specifying a local equilibrium state

$$T^{\mu\nu} = \mathcal{E}(T) u^\mu u^\nu + \mathcal{P}(T) \left(g^{\mu\nu} + u^\mu u^\nu \right) + \ldots \quad \text{with} \quad u_\alpha u^\alpha = -1$$

This stress tensor defines perfect fluid hydrodynamics: $\nabla_\mu (s(T) u^\mu) = 0$

Relativistic hydrodynamics: dissipation

e.g. **1205.5040** by Kovtun; **1707.02282** with Spaliński & Florkowski; **1712.05815** by Romatschke[2]

Realistic fluids dissipate and in hydrodynamics this is encapsulated by some of the corrections to the perfect fluid description

$$T^{\mu\nu} = \mathcal{E}(T) u^\mu u^\nu + \mathcal{P}(T) \left(g^{\mu\nu} + u^\mu u^\nu \right) + \pi^{\mu\nu}$$

To the leading order in derivatives, the dissipative terms are

this talks considers conformal fluids

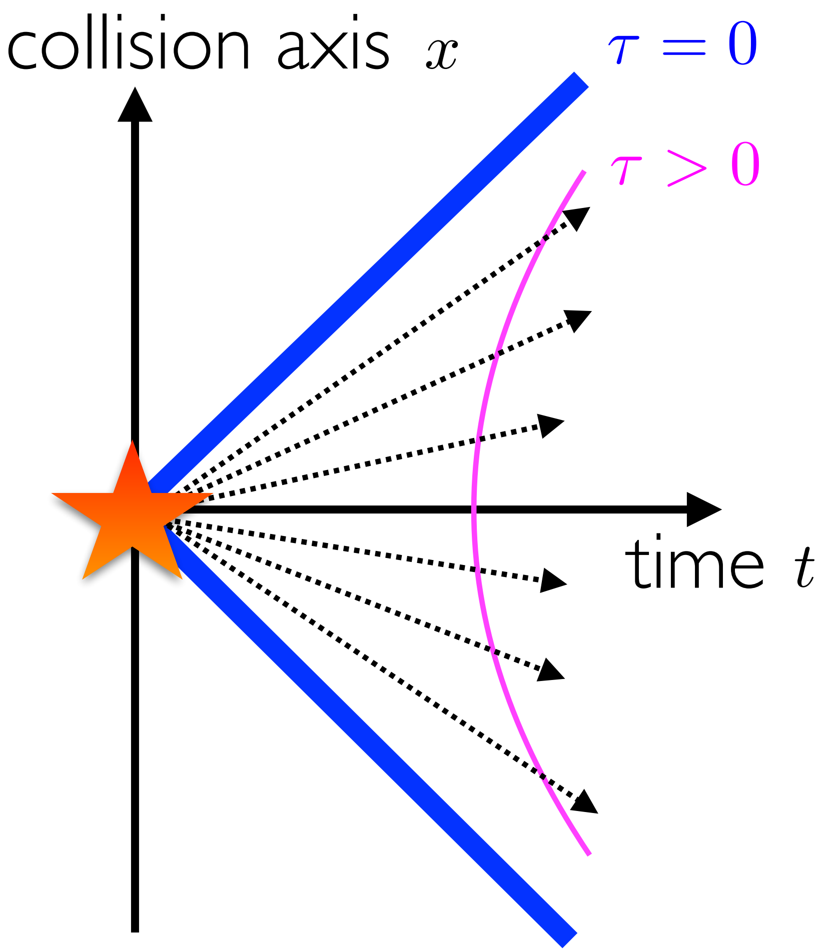
$$\pi^{\mu\nu} = \underbrace{-\eta(T)\, \nabla^{\langle\mu} u^{\nu\rangle}}_{\text{shear term}} \underbrace{\cancel{- \zeta(T)(g^{\mu\nu} + u^\mu u^\nu)\nabla_\alpha u^\alpha}}_{\text{bulk term}} + \mathcal{O}(\nabla^2)$$

@ conformality:
2 order: 5 terms
0712.2451 by Baier et al.
3 order: ~20 terms
1507.02461 by Grozdanov & Kaplis

Such corrections
- allow for dissipation
- influence properties of solutions of hydrodynamics
- therefore of interest to pheno at RHIC/LHC

Bjorken flow and relativistic hydrodynamics

e.g. 1707.02282 with Spaliński & Florkowski



Bjorken flow is a comoving flow in Minkowski:

$$u^\mu \partial_\mu = \partial_\tau \quad \text{and} \quad ds^2 = -d\tau^2 + \tau^2 dy^2 + d\mathbf{x}_\perp^2$$

$$\downarrow$$

$$\nabla^\mu u^\nu \sim \frac{1}{\tau} \quad \text{etc}$$

It is an intrinsically nonlinear phenomenon

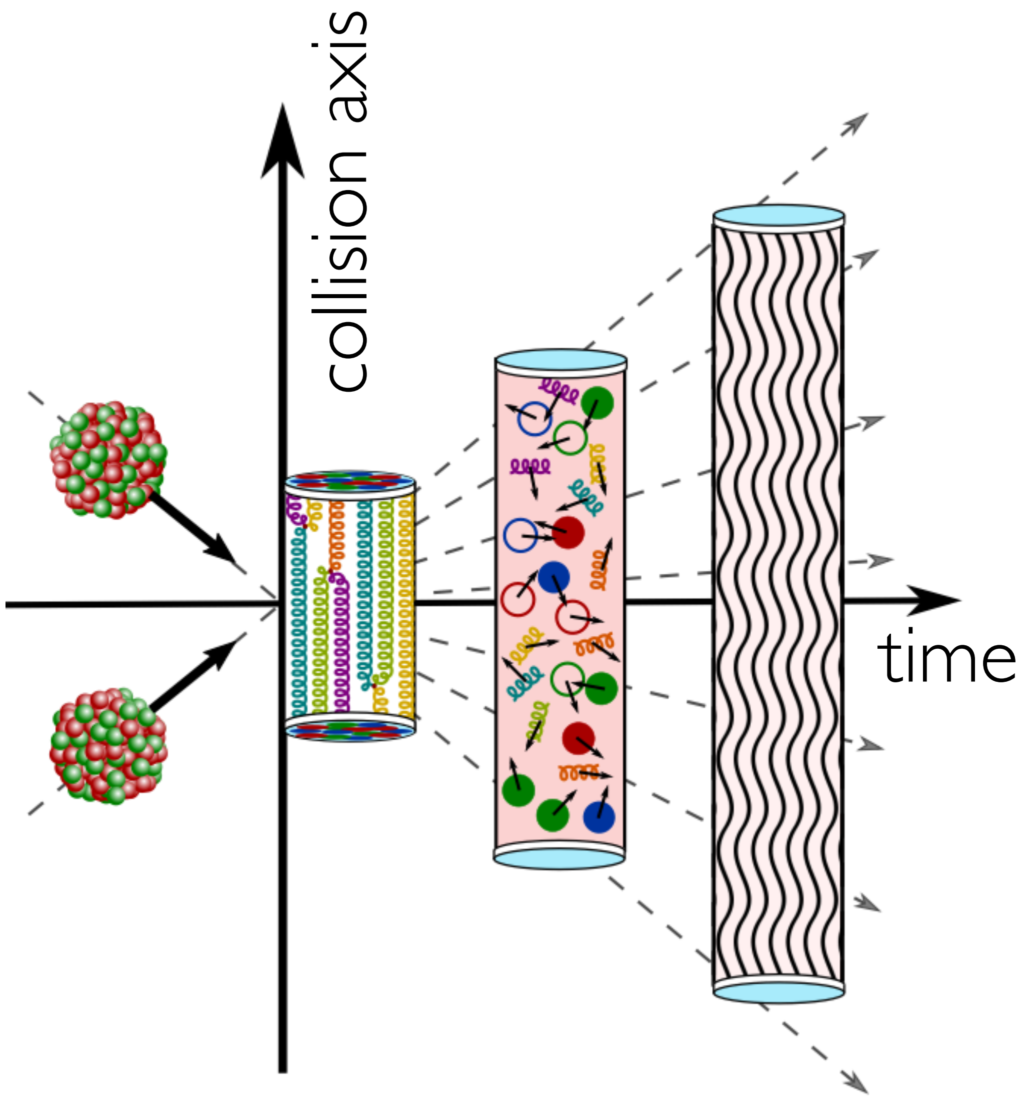
For conformal fluids $\mathcal{E} = 3\,\mathcal{P} \sim T^4$ and $\eta \sim T^3$

$$\downarrow$$

$$\mathcal{A} \equiv \frac{\pi^{\perp}_{\perp} - \pi^{y}_{y}}{\mathcal{E}/3} = 8\,\frac{\eta}{s}\overbrace{\boxed{\frac{1}{\tau\, T(\tau)}}}^{\equiv w^{-1}} + \mathcal{O}(\nabla^2)$$

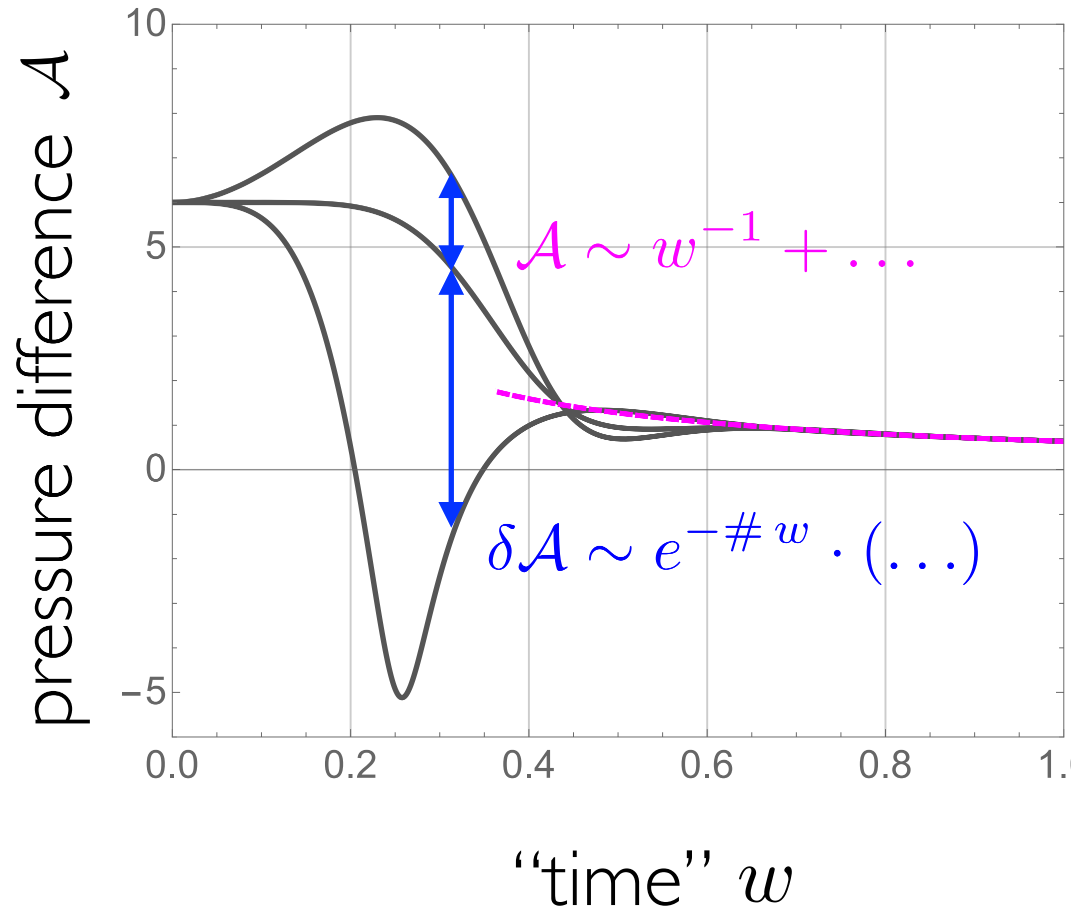
Motivation

heavy-ion collisions
at RHIC and LHC



2005.12299
with Berges, Mazeliauskas & Venugopalan

behaviour in three classes
of theoretical models



1103.3452 with Janik & Witaszczyk

Bjorken flow as a tool

Asymptotics* of hydrodynamic gradient expansion

$$\pi^{\mu\nu} = -\eta(T)\,\nabla^{\langle\mu}u^{\nu\rangle} + \boxed{\mathcal{O}(\nabla^2)}$$

is a question with a foundational, as well as a phenomenological component

The key simplifying feature that triggered progress on this problem is the fact that velocity in Bjorken flow is entirely fixed by the symmetry: $u^\mu\partial_\mu = \partial_\tau$

This allows to define a <u>version</u> of on-shell gradient expansion of the form

$$\mathcal{A} \equiv \frac{\pi^{\perp}{}_{\perp} - \pi^{y}{}_{y}}{\mathcal{E}/3} = 8\frac{\eta}{s}\underbrace{\frac{1}{\tau\,T(\tau)}}_{\equiv w^{-1}} + \mathcal{O}(\nabla^2) = \sum_{n=1}^{\infty} a_n w^{-n} + \ldots$$

~ Knudsen number: $\left(\frac{\ell_{\text{micro}}}{\ell_{\text{macro}}}\right)^n$

which is soluble among a whole class of models giving rise to hydrodynamics

The main Bjorken flow result

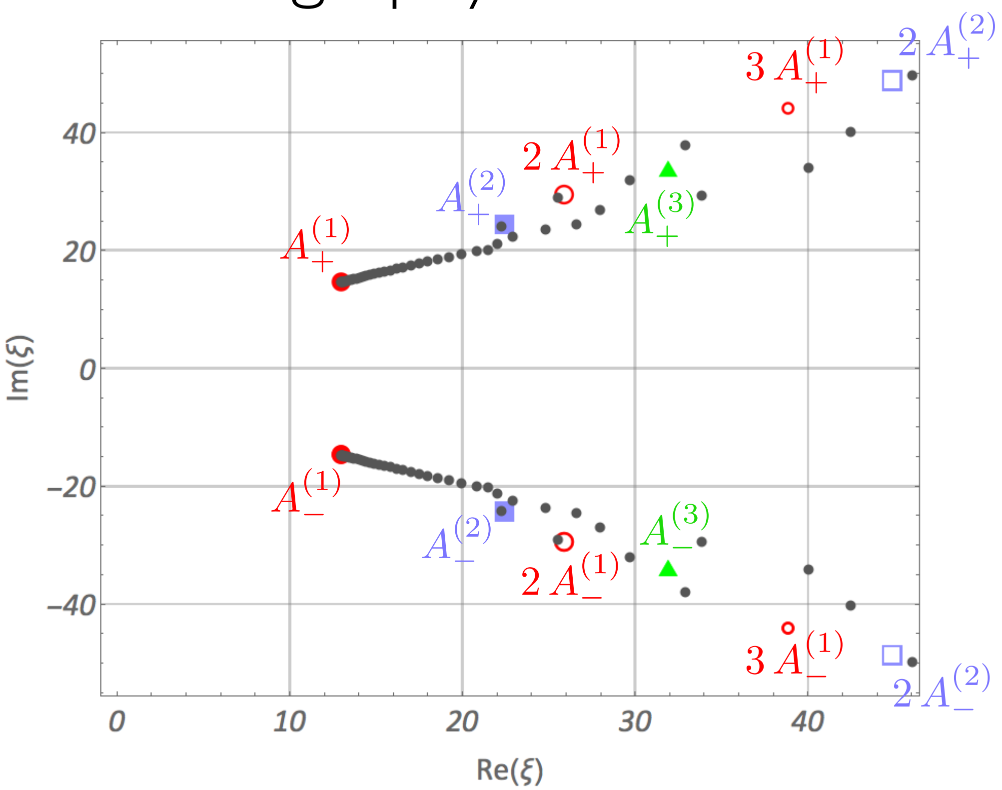
In all but one example of a microscopic model*, one gets that $a_n \sim n!$

Therefore hydrodynamic gradient expansion can diverge factorially on-shell

A standard tool in asymptotic series is a sequence

$$\mathcal{A}(w) \approx \sum_{n=1}^{\infty} \frac{a_n}{w^n} \xrightarrow[\text{Borel trafo.}]{} B\mathcal{A}(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n!}\,\xi^n \approx \frac{b_0 + \ldots + b_{100}\,\xi^{100}}{c_0 + \ldots + c_{100}\,\xi^{100}}$$

which in holography reveals **1302.0697** with Janik and Witaszczyk

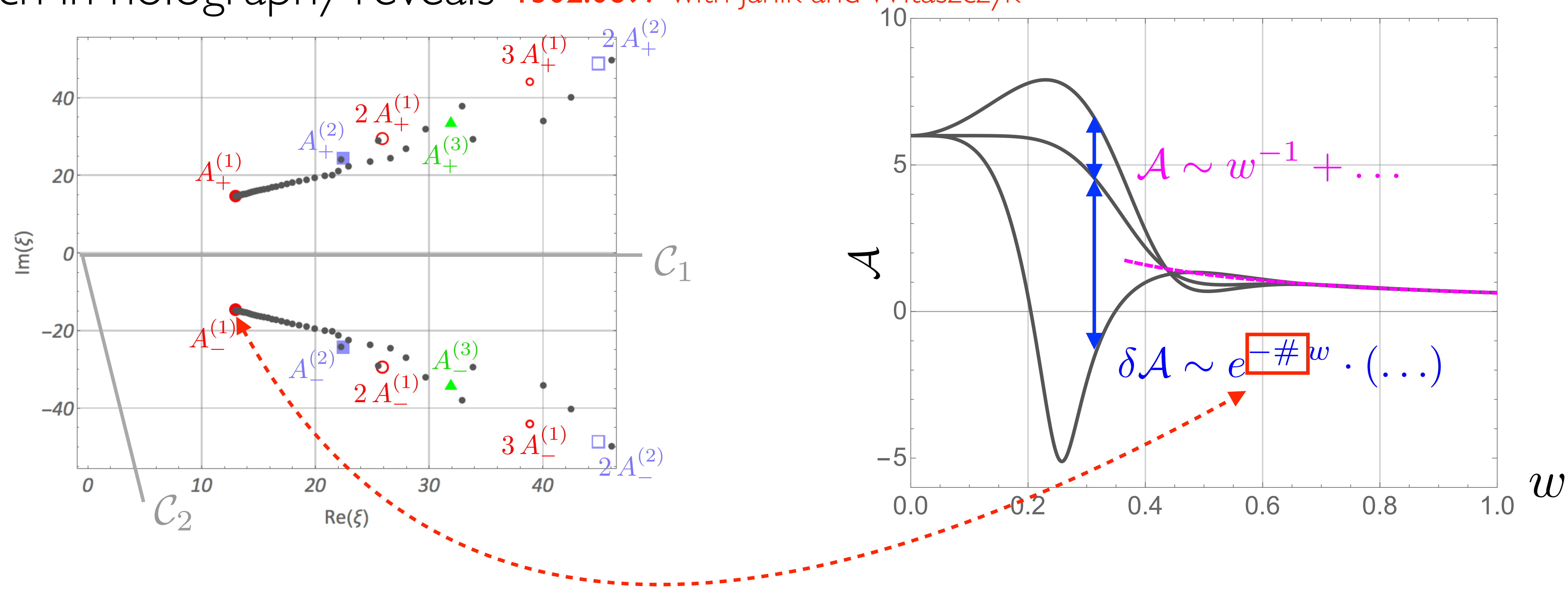


The main Bjorken flow result

A standard tool in asymptotic series is a sequence

$$\mathcal{A}(w) \approx \sum_{n=1}^{\infty} \frac{a_n}{w^n} \xrightarrow[\text{Borel trafo.}]{} B\mathcal{A}(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n!}\,\xi^n \approx \frac{b_0 + \ldots + b_{100}\,\xi^{100}}{c_0 + \ldots + c_{100}\,\xi^{100}}$$

which in holography reveals **1302.0697** with Janik and Witaszczyk



ambiguity in the inverse Borel transform $\left(\int_{\mathcal{C}_1} d\xi - \int_{\mathcal{C}_2} d\xi\right) [e^{-w\,\xi}\, w\, B\mathcal{A}(\xi)]$

reveals that gradient expansion diverges here due to transients

Making our life simple(r)

Holography is an ab initio description of a class of strongly-coupled QFTs

Its equations of motion are PDEs in one higher dimension

In order to make progress it is certainly useful to simplify the problem; in the end, what one wants is a system with hydrodynamics and transients

A particularly insightful system are Müller-Israel-Stewart theories providing hyperbolic formulations of viscous relativistic hydrodynamics (at a price):

$$\tau_\pi(T) u^\alpha \nabla_\alpha \pi^{\mu\nu} = -\pi^{\mu\nu} - \eta(T) \nabla^{\langle\mu} u^{\nu\rangle} + \ldots$$

exponential decay "over $\tau_\pi(T)$"
to a viscous prediction of $\pi^{\mu\nu} = -\eta(T)\nabla^{\langle\mu} u^{\nu\rangle} + \ldots$
supplemented by conservation equations:

$$\nabla_\mu T^{\mu\nu} = \nabla_\mu \left\{ \mathcal{E}(T)\, u^\mu u^\nu + \mathcal{P}(T)\left(g^{\mu\nu} + u^\mu u^\nu\right) + \pi^{\mu\nu} \right\} = 0$$

New look at "old" results

One can repeat the analysis done in holography in MIS:

$$\tau_\pi(T) u^\alpha \nabla_\alpha \pi^{\mu\nu} = -\pi^{\mu\nu} - \eta(T) \nabla^{\langle\mu} u^{\nu\rangle} + \ldots \longrightarrow \text{1st order ODE for } \mathcal{A}(w)$$

This led to recognizing trans-series structure in $\mathcal{A}(w)$ with hydro being the perturbative part and transients non-perturbative corrections and attractors

1503.07114 with Spaliński

However, the true value of these results depends on how general they are:

Bjorken flow is highly symmetric; is its divergence due to some finte-tuning?

Can hydrodynamics diverge non-factorially or even converge?

If it diverges, is it due to a growing number of transport coeffs: 1→5→~20…?

How general is the interplay between transients and hydro seen for Bjorken?

New results (2020-now)

2007.05524
+ work in progress

with Serantes, Spaliński, Svensson & Withers

Linear response theory

see, for example, **1707.02282** with Florkowski and Spaliński

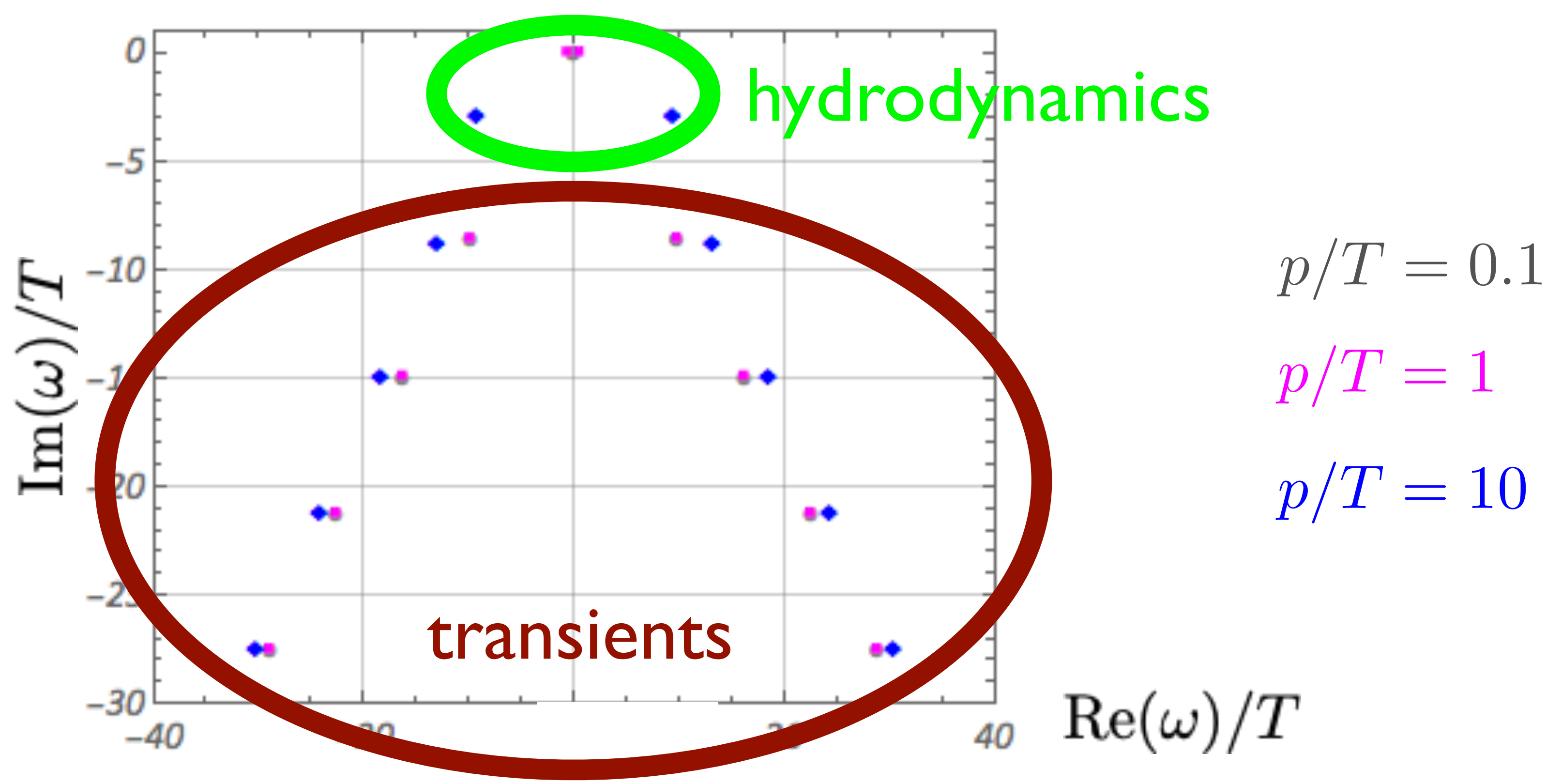
Linear response theory on top $\rho \sim e^{-\frac{1}{T}H}$ is the simplest and very powerful model of nonequilibrium physics with a hydrodynamic tail

$$\delta\langle\mathcal{O}(t,p)\rangle = \int_{-\infty}^{\infty} d\omega\, e^{-i\,\omega\,t}\, G_R^{\mathcal{O}}(\omega,p)\, \mathcal{J}(-\omega,-p)$$

with $G_R^{\mathcal{O}}(t,x) = i\,\theta(t)\,\mathrm{tr}\,(\rho\ [\mathcal{O}(t,x),\mathcal{O}(0,0)])$

The response is encoded in singularities in ω of $G_R^{\mathcal{O}}(\omega,p)$ at fixed value of p

$\mathcal{O} = T_{\mu\nu}$
in a holographic CFT:



New insight from linear response theory

1803.08058 by Withers; 1904.01018 and 1904.12862 by Grozdanov, Kovtun, Starinets, Tadić

At the level of the linear response theory, hydrodynamic gradient expansion leads to the expansion of hydrodynamic mode frequencies ω_h in p, e.g.

$$\omega_h = -i\frac{\eta}{s\,T}p^2 - i\left(\frac{\eta^2\,\tau_\pi}{s^2 T^2} - \frac{\theta_1}{2\,s\,T}\right)p^4 + \ldots$$

The quoted works showed that this expansion has a finite radius of convergence set by the smallest $|p_*|$ for which $\omega_h(p_*) = \omega_{some\,transient}(p_*)$

For example, in the simple model we considered in 1503.07114 with Spaliński based on MIS and having only 1 transient one has

$$\omega_h = i\frac{-1+\sqrt{1-4\,\frac{\eta}{s\,T}\,\tau_\pi\,p^2}}{2\,\tau_\pi} \quad \text{and} \quad \omega_{transient} = i\frac{-1-\sqrt{1-4\,\frac{\eta}{s\,T}\,\tau_\pi\,p^2}}{2\,\tau_\pi}$$

In this case p_* is real and equals $\pm\frac{1}{2}\sqrt{\frac{s\,T}{\eta\,\tau_\pi}}$

Basic idea

2007.05524 with Serantes, Spaliński, Svensson, Withers

Linearized hydrodynamics, $u^\mu \partial_\mu = \partial_t + u^j \partial_j$ and $\mathcal{E} = \mathcal{E}_0 + \epsilon$, allows us to overcome the need of high symmetry, as was the case for the Bjorken flow

As we realized, in linearized hydrodynamics the hydrodynamic constitutive relations can be written as only three contributions at each order

$$\Pi_{jl} = -A(\partial^2)\, \sigma_{jl} - B(\partial^2)\, \pi^u_{jl} - C(\partial^2)\, \pi^\epsilon_{jl}$$

with

$$\sigma_{jl} = \left(\partial_j u_l + \partial_l u_j - \frac{2}{d-1}\delta_{jl}\partial_r u^r\right)$$

$$\pi^\epsilon_{jl} = \left(\partial_j \partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\epsilon$$

$$\pi^u_{jl} = \left(\partial_j \partial_l - \frac{1}{d-1}\delta_{jl}\partial^2\right)\partial_r u^r$$

Moreover, the properties of $A(\partial^2)$ etc determined by $\omega_h :\ A(\partial^2) = i\, s\, T\, \omega_h\big|_{p^2=-\partial^2}$

This allows to translate the properties of the expansion of $A(\partial^2)$ <u>when acting on a solution</u> to the properties of ω_h and $|p_*|$, as well as $u^j(p)$

Real space statement

2007.05524 with Serantes, Spaliński, Svensson, Withers

Let us take MIS:

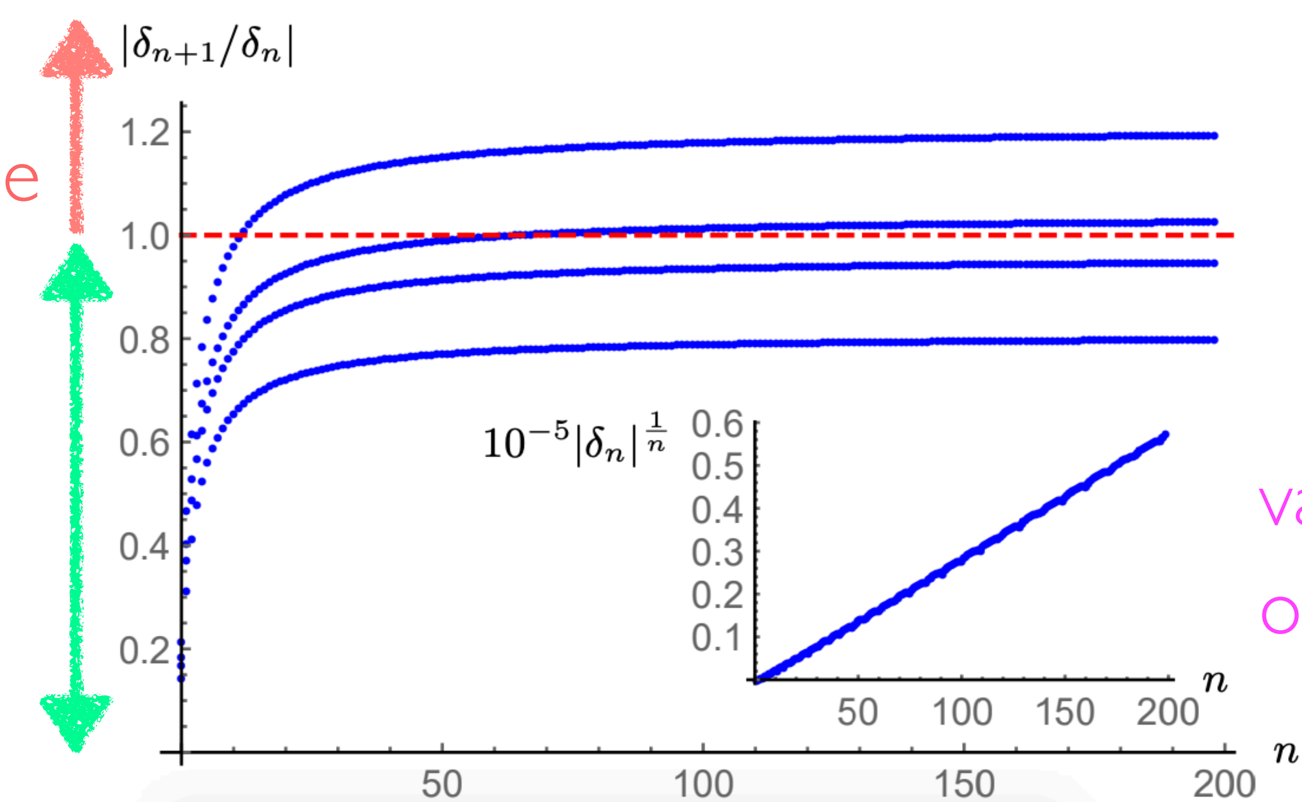
$$\omega_h = i\frac{-1+\sqrt{1-4\,\frac{\eta}{s\,T}\,\tau_\pi\,p^2}}{2\,\tau_\pi} \quad \text{and} \quad \omega_{transient} = i\frac{-1-\sqrt{1-4\,\frac{\eta}{s\,T}\,\tau_\pi\,p^2}}{2\,\tau_\pi}$$

We look at the initial conditions $u_1(0,p) = 0\,,\quad \partial_t u_1(0,p) = \frac{1}{2\pi}e^{-\frac{1}{2}\gamma^2 p^2}\Theta(p_{max}^2 - p^2)$

consider $\Pi_{1,3}(t,x) = A(\partial_x^2)\,\partial_x u_1(t,x) = \sum_{n=0}^{\infty}\delta_n$ with δ_n having $2n+1$ derivatives:

$|p_*| \leq p_{max} < \infty$
geometric divergence

$p_{max} < |p_*|$
convergence!



$p_{max} \to \infty$
vanishing radius
of convergence

This turns out to be the universal statement that we proved in our paper

First nonlinear results for general flows

210x.xxxxx with Serantes, Spaliński, Svensson, Withers

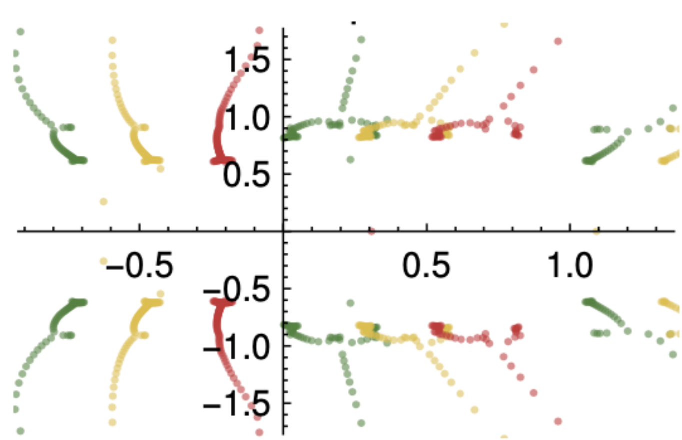
Since MIS has worked so well for the linearized hydro and the Bjorken flow, maybe one can use it to elucidate gradient expansion for more general flows?

Indeed, at least for flows depending on t and x one can solve for constitutive relations in MIS as

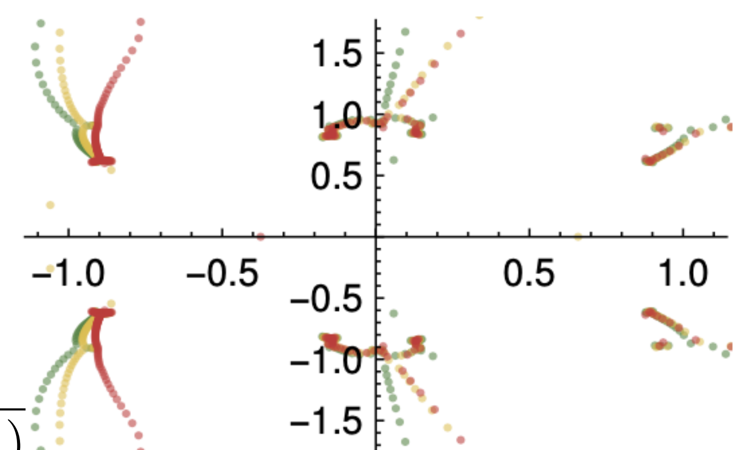
$$\tau_\pi(T)u^\alpha\nabla_\alpha\pi^{\mu\nu} = -\pi^{\mu\nu} - \eta(T)\nabla^{\langle\mu}u^{\nu\rangle} + \ldots \rightarrow \pi^{\mu\nu}\Big|_h = -\underbrace{\frac{1}{1+\tau_\pi(T)u^\alpha\partial_\alpha+\ldots}}_{\text{understood as a series}}\eta(T)\nabla^{\langle\mu}u^{\nu\rangle}$$

With numerical simulations on a spacetime lattice one can evaluate the series. At large orders geometric (lattice), intermediately (continuum) factorial

Borel singularities
at three different times



shift by $-\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau_\pi(T(\tau'))}$



New look at “old” results

One can repeat the analysis done in holography in MIS:

$$\tau_\pi(T)u^\alpha\nabla_\alpha\pi^{\mu\nu} = -\pi^{\mu\nu} - \eta(T)\nabla^{\langle\mu}u^{\nu\rangle} + \ldots \longrightarrow \text{1st order ODE for } \mathcal{A}(w)$$

This led to recognizing trans-series structure in $\mathcal{A}(w)$ with hydro being the perturbative part and transients non-perturbative corrections, attractors…

However, the true value of these results depends on how general they are:

New result provided new insights:

Bjorken flow is highly symmetric; is its divergence due to some finte-tuning?

no

Can hydrodynamics diverge non-factorially or even converge?

yes to both, however fine-tuning or lattice seem to be essential

If it diverges, is it due to a growing number of transport coeffs: 1→5→~20…?

this is certainly not necessarily, but we have not ruled it out

How general is the interplay between transients and hydro seen for Bjorken?

very general, visible both in momentum and real space

New results
on hydrodynamics attractors

2003.07368 with Jefferson, Spaliński & Svensson

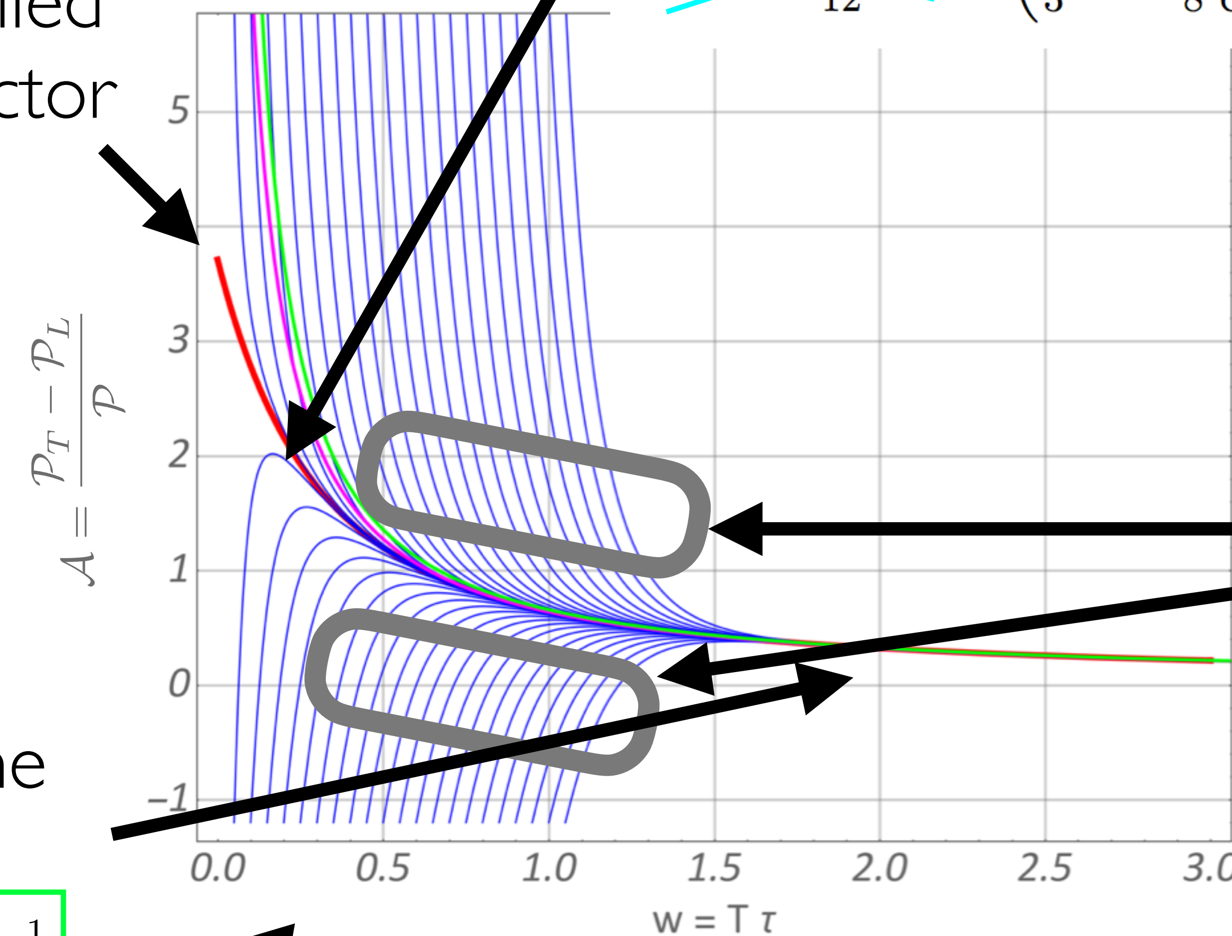
Hydrodynamic attractors

1503.07514 with Spaliński

At w = 0 there is only 1 sensible solution with finite $\mathcal{A}$ and we called it the (hydro) attractor

“slow roll” approximates it well:

$$C_{\tau_\pi} \cancel{w\,(1+\frac{1}{12}\mathcal{A})}\,\mathcal{A}' + \left(\frac{1}{3}C_{\tau_\pi} + \frac{1}{8}\frac{C_{\lambda_1}}{C_\eta}w\right)\mathcal{A}^2 + \frac{3}{2}w\,\mathcal{A} - 12\,C_\eta = 0$$



built-in exponential decay

1st or 2nd order gradient expansion work well since some threshold time:

$$\mathcal{A}_H(w) = \boxed{\boxed{8\,C_\eta \frac{1}{w}} + \frac{16}{3}C_\eta\,(C_{\tau_\pi} - C_{\lambda_1})\,\frac{1}{w^2}} + \ldots$$

divergent series, see also 1509.05046 by Basar & Dunne

attractor = a particular resummation

Since 2015: many studies of finite / slow-roll solutions (= hydro attractors)

A new look at hydrodynamic attractors

The idea of attractors arose as a way of providing a preferred resummation of divergent hydrodynamic series and, eventually, of doing hydro better

similar motivation to **0704.1647** by Lublinsky & Shuryak

It is not clear at this level if solutions having a finite $\mathcal{A}(w=0)$/ slow-roll solutions are phenomenologically interesting. In particular, for a generic solution, hydro attractor of **1503.07514** becomes relevant only from some time onwards

New idea: make use of expansion and dissipation (see also **1712.03856** by Blaizot & Yan) to argue for a reduction of effective degrees of freedom in heavy-ion collision setups (~ local "attractors")

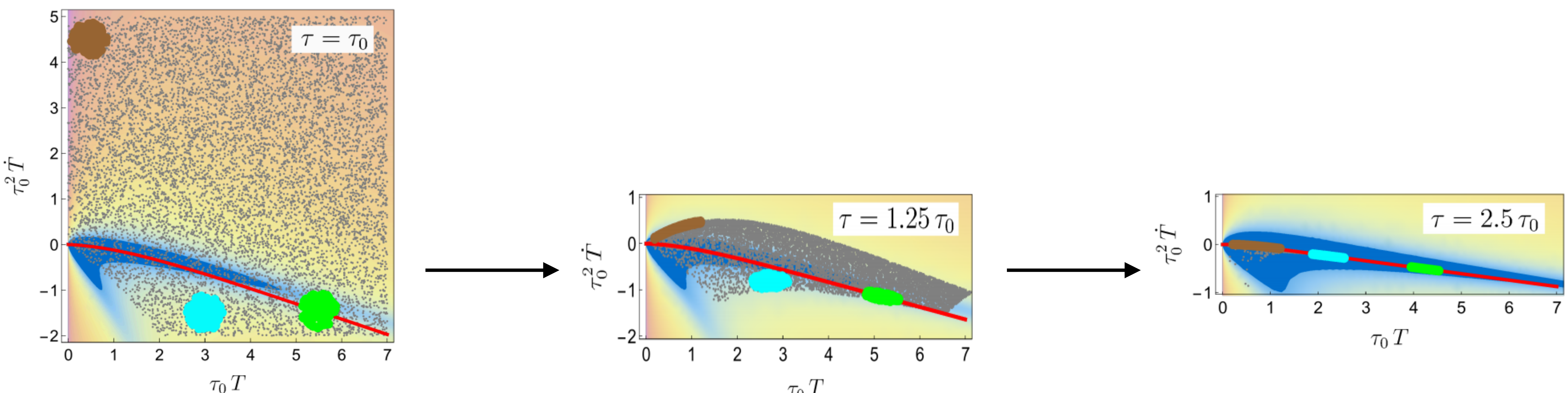
This will not require introducing any "preferred" parametrization of evolution of states akin to $\mathcal{A}(w)$

Dimensionality reduction in phase space

Let's revisit the theory of hydrodynamics: EOMs are 2nd order ODEs for $T(\tau)$

The phase space* therefore is 2D: for each τ a state can be param. by T & $\dot{T}$, however, many different choices are possible, including w & $\mathcal{A}$

Time evolution turns out to dimensionally reduce regions of phase space:

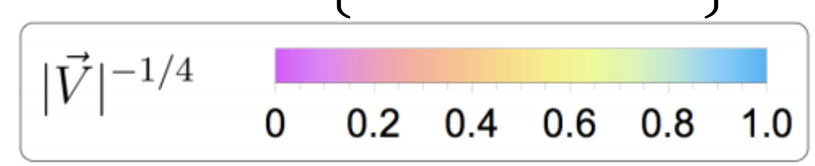


The MIS attractor is a red curve

$$\mathcal{A}(\tau T) = \frac{\pi^{\perp}_{\perp} - \pi^{y}_{y}}{\mathcal{E}/3} = 6 + 18\frac{\tau \dot{T}}{T}$$

and

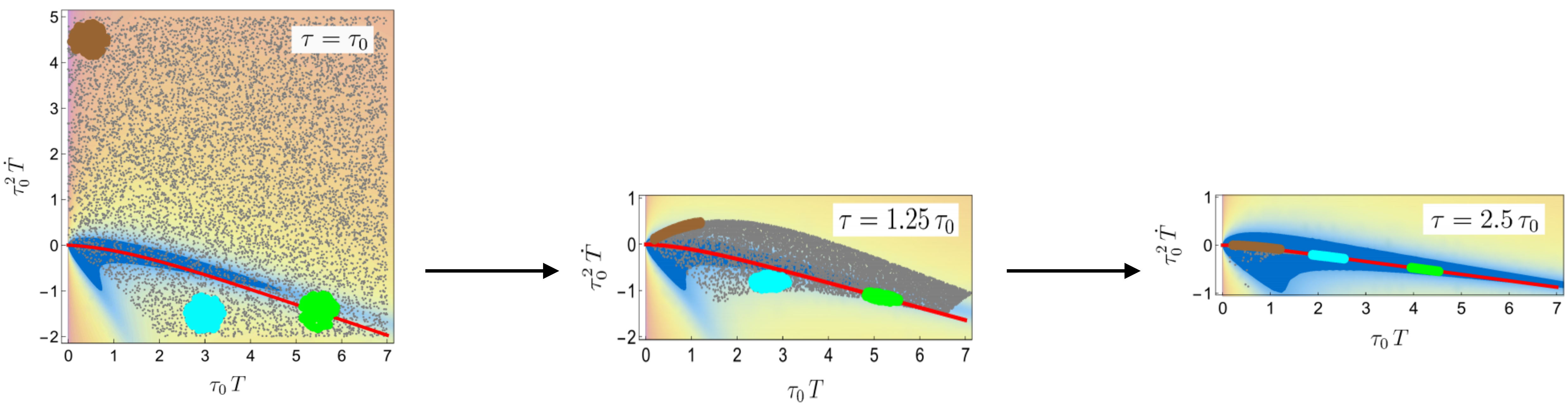
$$\vec{V} = \tau_0 \partial_\tau \left\{ \tau_0 T(\tau), \tau_0^2 \dot{T}(\tau) \right\}$$

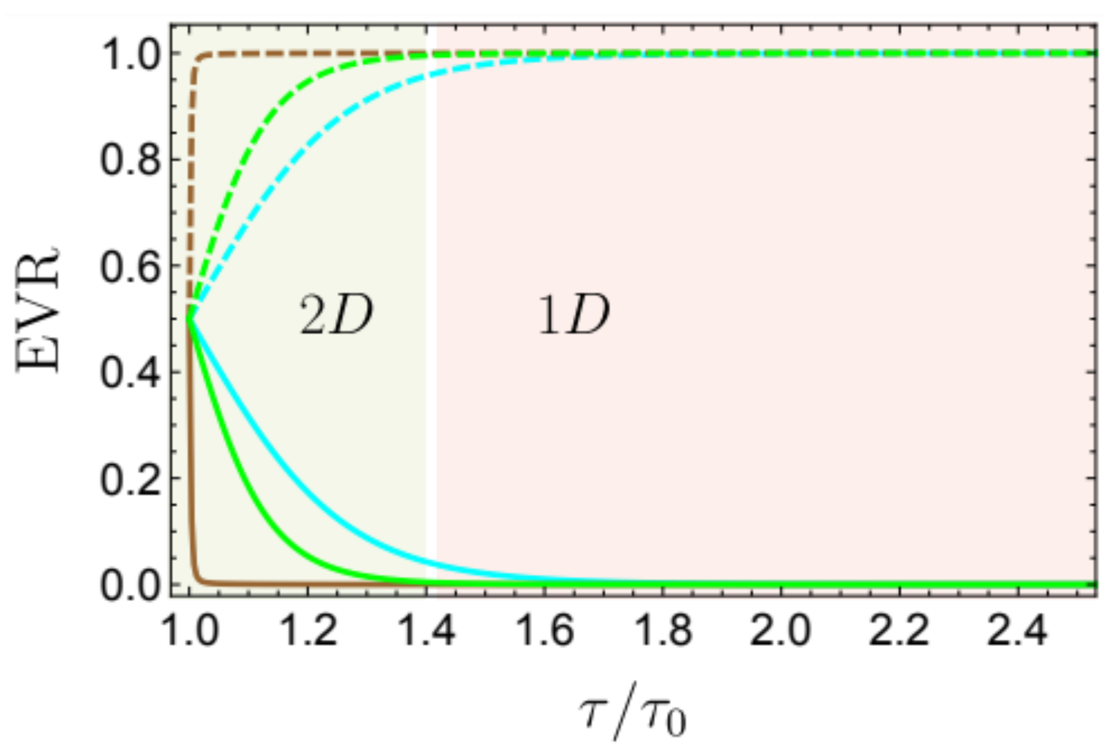


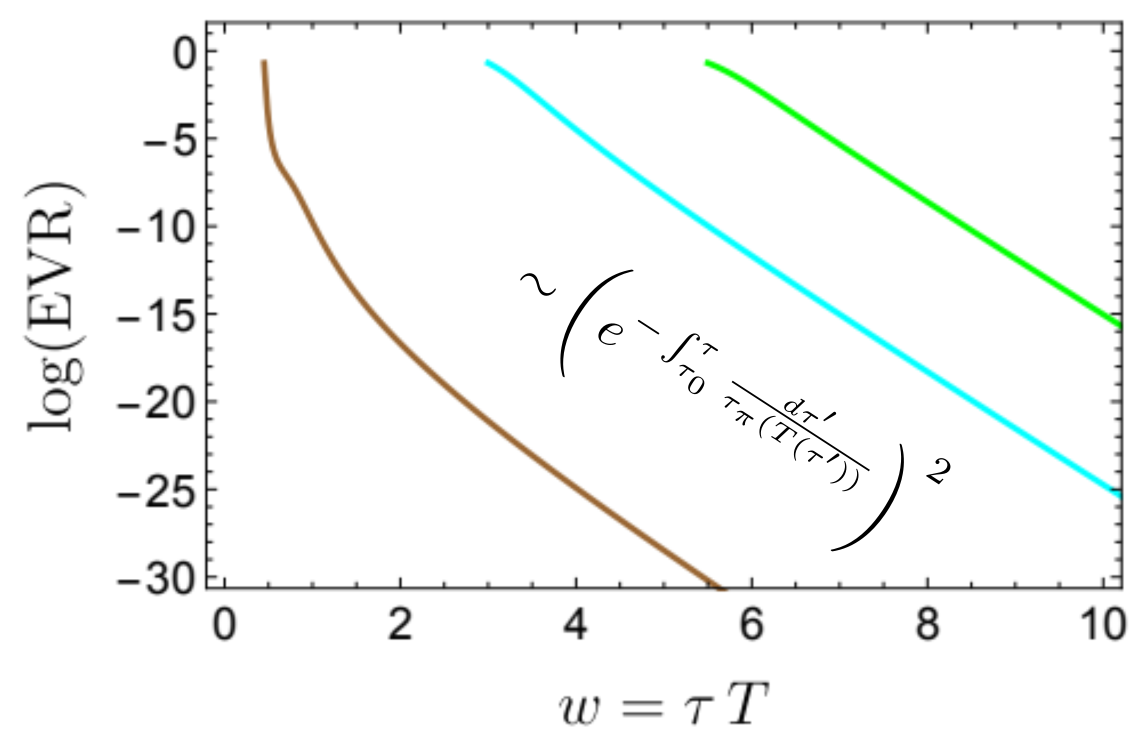
Slow regions similar to **1910.00021** by Brewer, Yan & Yin

Principal components analysis and attractors

A useful tool to quantify dimensionality reduction is PCA







Comments on the new idea

After some short time some directions in phase space become locally irrelevant

As a result, all initial states localized in a finite phase space volume end up on a lower D manifold

Details depend on parametrization and choice of phase space metric

A useful way of detecting and parametrizing dimensional reduction is the principal component analysis. We checked it works in RTA kinetic theory with effective phase space made from truncated moments

A pheno potential of this idea might lie in the fact that in heavy-ion collisions we might then not care about evolving all initial states till late times, but rather just representatives from different points of this lower D manifold

Summary and outlook

Summary

Many reasons to be interested in relativistic hydrodynamics in 2021: nuclear collisions, gravity, neutron stars mergers and, last but not least, resurgence

Behaviour of hydrodynamic expansion at large orders is a foundational question in relativistic fluid mechanics of possible practical importance

$$\pi^{\mu\nu} = -\eta(T)\,\nabla^{\langle\mu}u^{\nu\rangle} + \boxed{\mathcal{O}(\nabla^2)}$$

(Conformal) Bjorken flow is a highly symmetric non-equilibrium dynamics in which gradient expansion can be written as a late time expansion

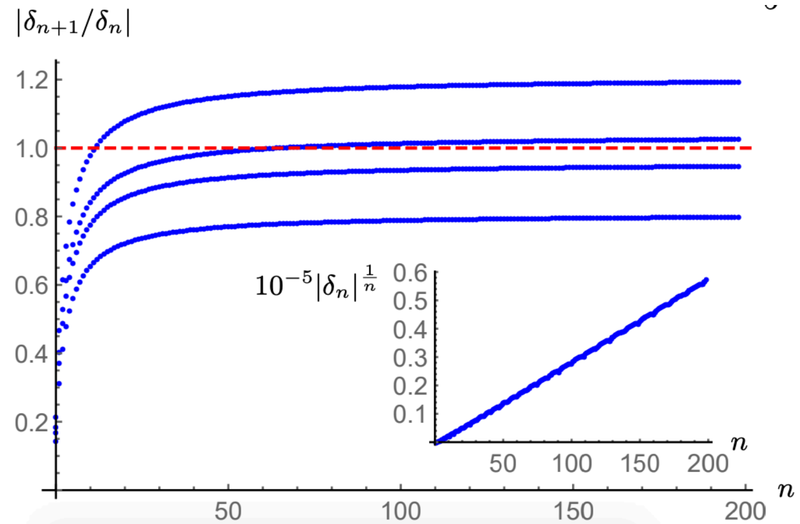
$$\mathcal{A} \equiv \frac{\pi^{\perp}_{\ \perp} - \pi^{y}_{\ y}}{\mathcal{E}/3} = 8\frac{\eta}{s}\underbrace{\frac{1}{\tau\,T(\tau)}}_{\equiv w^{-1}} + \mathcal{O}(\nabla^2) = \sum_{n=1}^{\infty} a_n w^{-n} + \ldots$$

This triggered extensive studies. In all but one* studied models $a_n \sim n!$

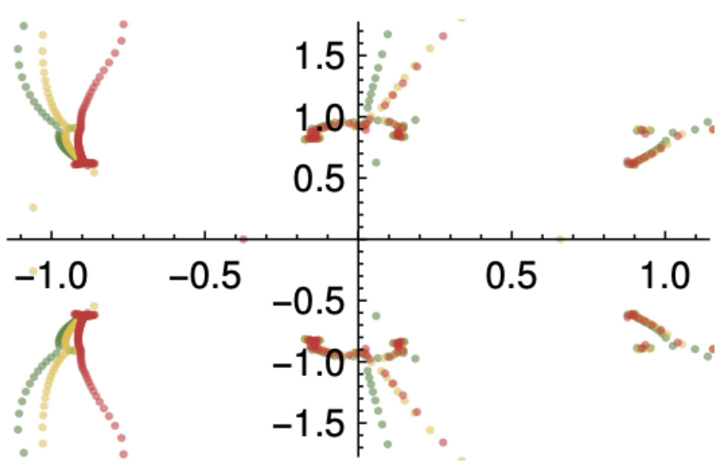
This not only brought connection with transseries to hydro, but more importantly raised many questions, effectively starting a new research area

Outlook

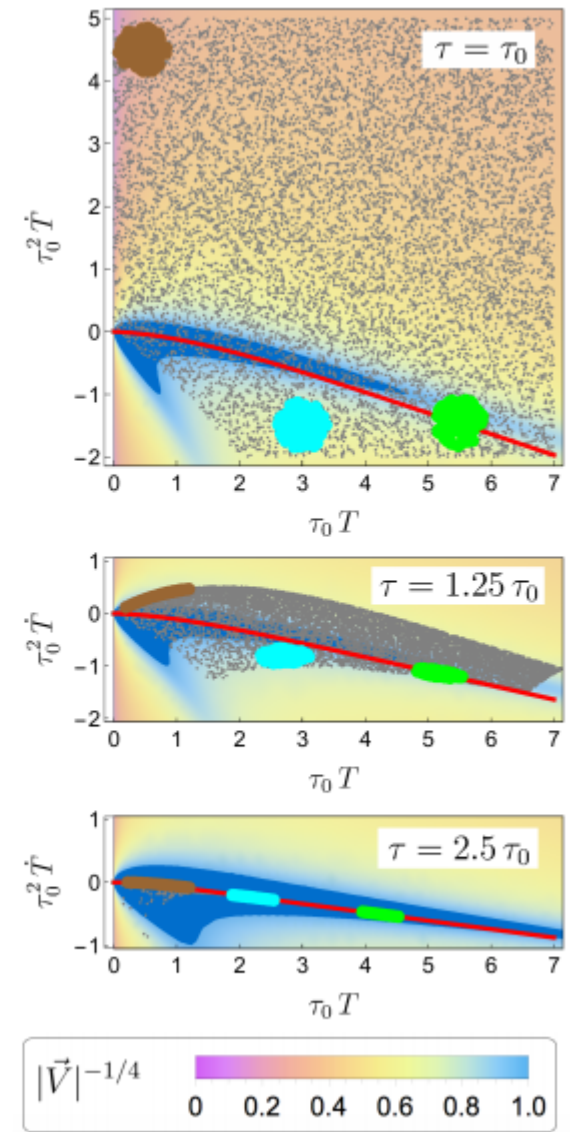
The progress I described today can be shortly summarized as going beyond the $\mathcal{A}(w)$ paradigm both for the gradient expansion and attractors



2007.05524
with Serantes, Spaliński, Svensson, Withers



210x.xxxxx
with Serantes, Spaliński, Svensson, Withers



2003.07368
with Jefferson, Spaliński & Svensson

Thank you