From Kadanoff-Baym to Boltzmann equations for massive spin-1/2 fermions

Xin-Li Sheng (盛欣力)

Department of physics,
Central China Normal University, Wuhan

2021/05/11

QCD theory seminars

Contents

華中師範大學
HUAZHONG NORMAL UNIVERSITY

Spin

⇓

$$J^\mu = \xi_B B^\mu + \xi \omega^\mu$$
$$J^\mu_5 = \xi_{B5} B^\mu + \xi_5 \omega^\mu$$

Chiral Magnetic Effect

D.E.Kharzeev, et al. Nucl.Phys.A803,227 (2008)

Chiral Separation Effect

D.E.Kharzeev, et al. Prog.Part.Nucl.Phys.88,1 (2016)

Chiral Vortical Effect

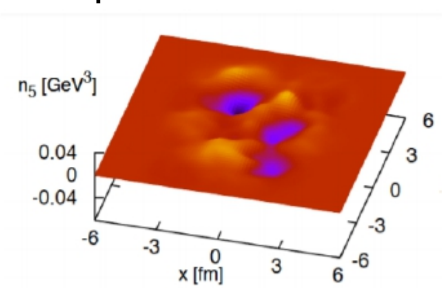
D.T.Son and P.Surowka, Phys.Rev.Lett.103,191601 (2009)

Axial Chiral Vortical Effect

D.E.Kharzeev, et al. Prog.Part.Nucl.Phys.88,1 (2016)

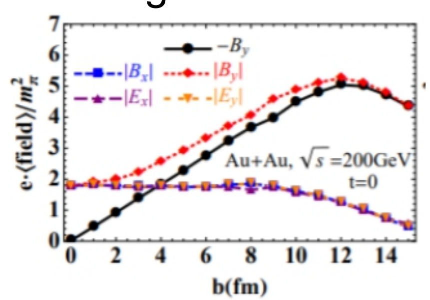
- Heavy-ion collisions provide a good platform for spin studies

spin imbalance



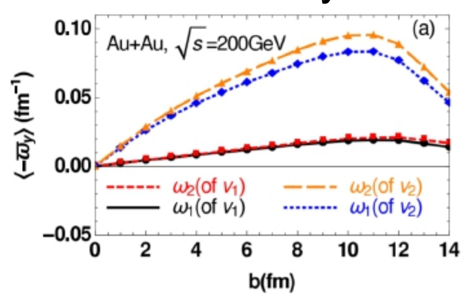
Y.Hirono, et al. arXiv:1412.0311.

magnetic field



W.-T.Deng, X.-G.Huang, Phys.Rev.C85,044907 (2012).

vorticity

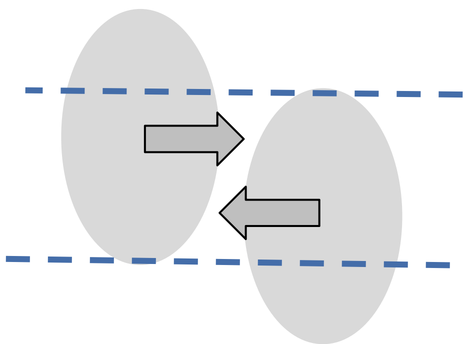


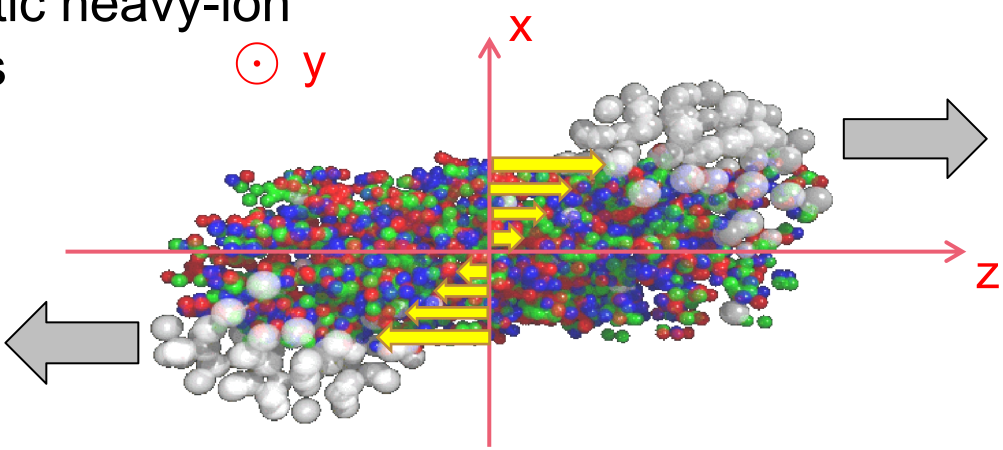
W.-T.Deng, et al. J.Phys.Conf.Ser.779,012070 (2017)

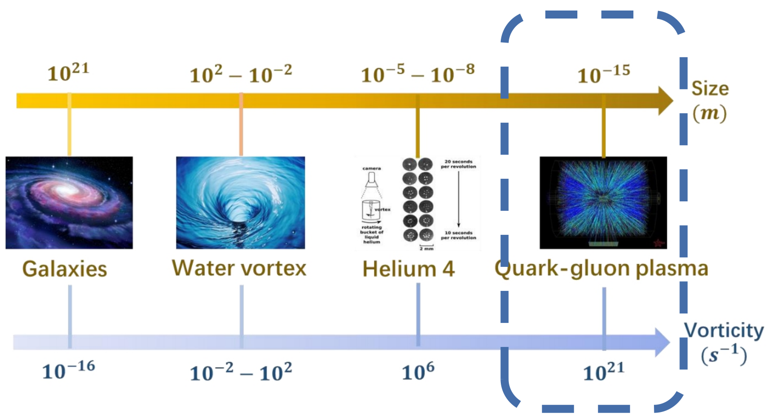
Vorticity in QGP

華中師範大學 HUAZHONG NORMAL UNIVERSITY

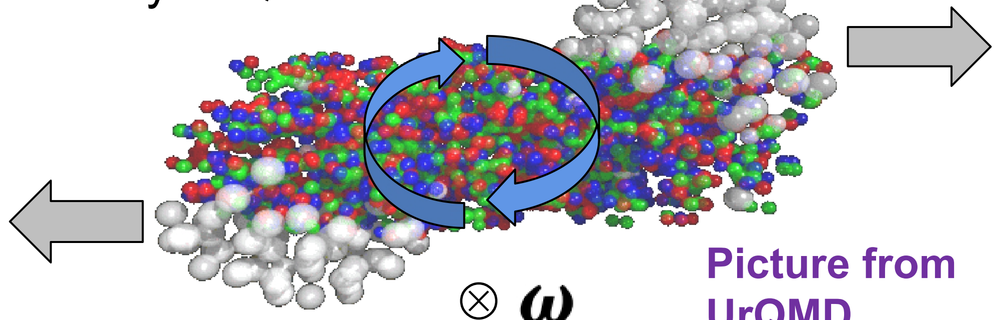
Relativistic heavy-ion collisions







Vorticity in QGP

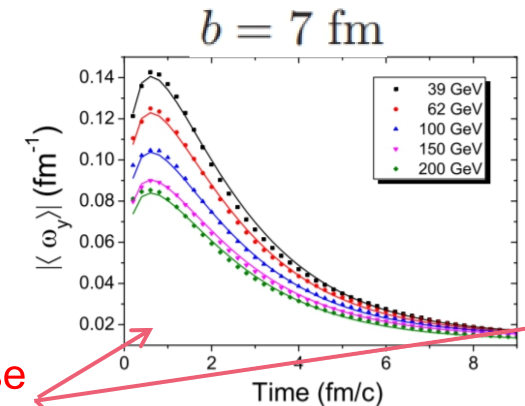


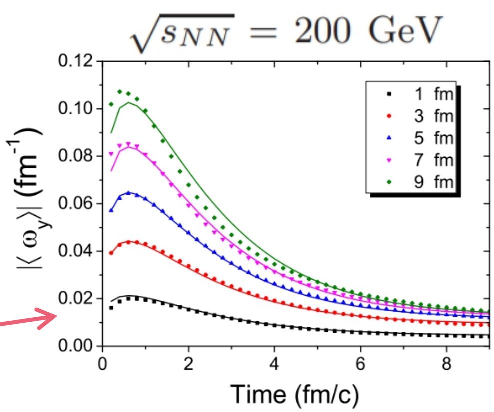
Picture from UrQMD

The most vortical field human ever made

Vorticity in QGP

- Space-time dependence of vorticity

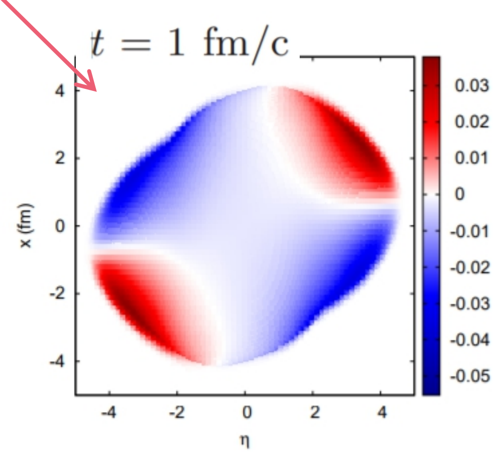


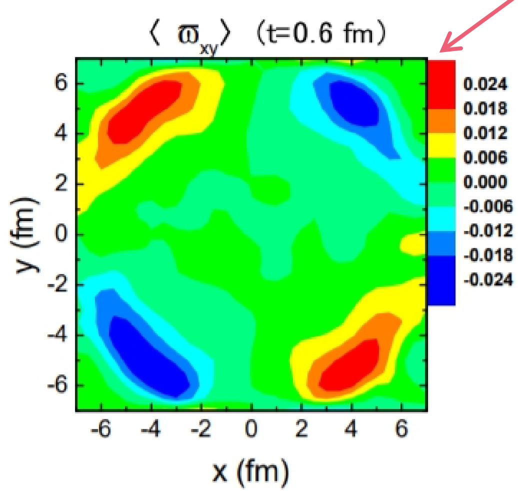


AMPT simulation by Y. Jiang, Z.-W. Lin, and J. Liao, PRC 94, 044910 (2016)

Transverse vorticity

Longitudinal vorticity





Au + Au

$\sqrt{s} = 19.6$ GeV

centrality region 20-50%

$\eta = 0$

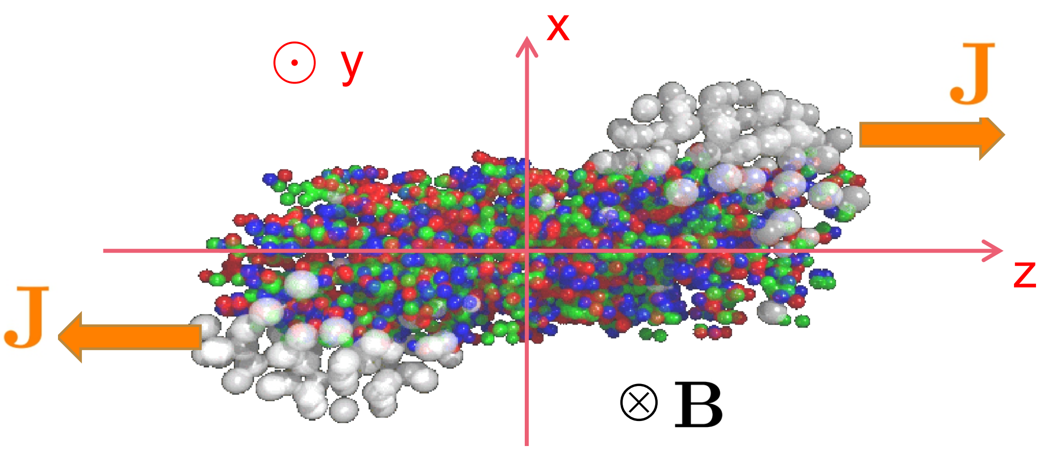
$t = 0.6$ fm

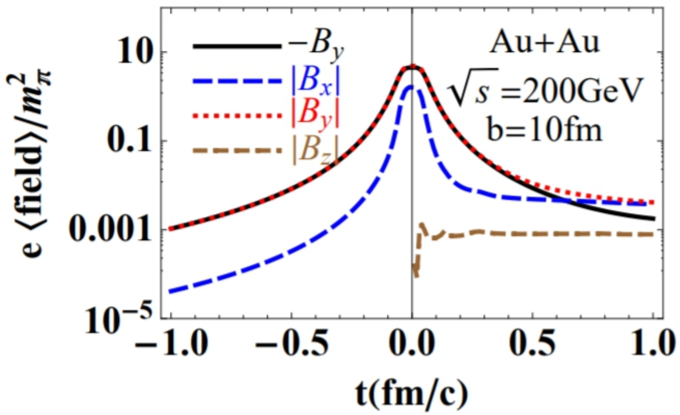
F. Becattini, et. al., EPJC 75, 406 (2015)

W.-T. Deng. X.-G. Huang, PRC 99, 014905 (2019)

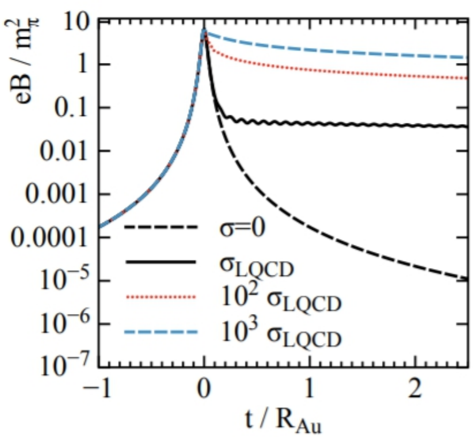
Magnetic field

Relativistic heavy-ion collisions





W.-T. Deng, X.-G. Huang, PRC 85, 044907 (2012).



Magnetic field decreases slower at later stages if we include medium feedback.

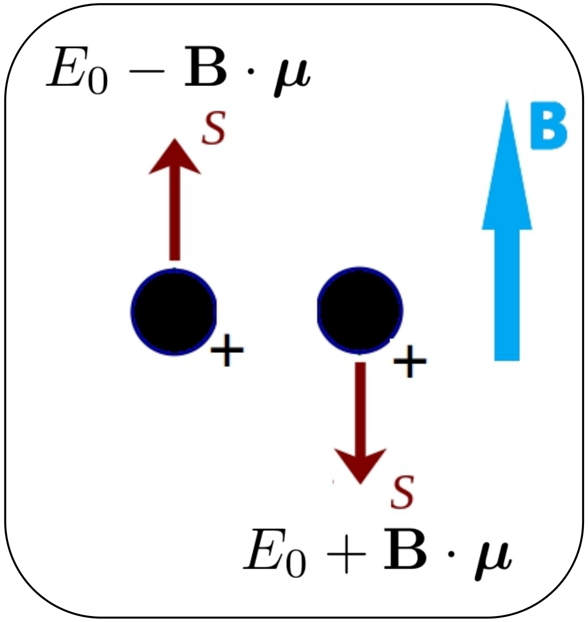
L. Mclerran and V. Skokov, NPA 929 (2014) 184.

Also see:

Y. Chen, XLS, G.-L. Ma, NPA 1011 (2021) 122199.

L. Yan and X.-G. Huang, e-Print: 2104.00831.

Spin polarization



$$f_{\pm} \sim \exp\left[-(E_0 \mp \mathbf{B} \cdot \boldsymbol{\mu}) / T\right]$$

at thermal equilibrium

Magnetic moment: $\boldsymbol{\mu} = \frac{Q}{m}\mathbf{S}$

Polarization through spin-magnetic coupling:

$$P = \frac{f_+ - f_-}{f_+ + f_-} \sim \frac{\mathbf{B} \cdot \boldsymbol{\mu}}{T}$$

$$f_s \sim \exp\left[-\left(E_0 - \boldsymbol{\omega} \cdot \mathbf{S}\right) / T\right]$$

- E_0: free particle's energy
- $\boldsymbol{\omega}$: kinetic vorticity $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{v}$
- $\mathbf{S}$: spin vector

Vorticity → Spin

Spin-orbit coupling

S. A. Voloshin, arXiv:nucl-th/0410089.
Z.-T. Liang, X.-N. Wang, PRL 94 (2004) 039901.
J.-J. Zhang, R.-H. Fang, Q. Wang, and X.-N. Wang, PRC 100 (2019) 064904.

Spin polarization

华中师范大学 HUAZHONG NORMAL UNIVERSITY

- Average spin polarization in global thermal equilibrium

$$P^{\mu}_{\pm}(x,p) = \frac{1}{2m}\left(\tilde{\omega}^{\mu\nu}_{\text{th}} \pm \frac{1}{E_p T} Q\tilde{F}^{\mu\nu}\right) p_{\nu}\left[1 - f_{FD}(E_p \mp \mu)\right]$$

Dual thermal vorticity

$$\tilde{\omega}^{\mu\nu}_{\text{th}} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}\omega^{\text{th}}_{\sigma\rho}$$

$$\omega^{\text{th}}_{\sigma\rho} = \frac{1}{2}[\partial_{\sigma}(\beta u_{\rho}) - \partial_{\rho}(\beta u_{\sigma})]$$

Dual electromagnetic fields

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\sigma\rho}F_{\sigma\rho}$$

F. Becattini, V. Chandra, et. al., Annals Phys. 338 (2013) 32.

D.-L. Yang, PRD 98 (2018) 7, 076019.

Y.-G. Yang, R.-H. Fang, Q. Wang, and X.-N. Wang, PRC 97 (2018) 034917.

Additional shear-induced polarization in local thermal equilibrium

S. Liu and Y. Yin, e-Print: 2103.09200.
B. Fu, et. al., e-Print: 2103.10403.
F. Becattini, et. al., e-Print: 2103.10917; e-Print: 2103.14621.

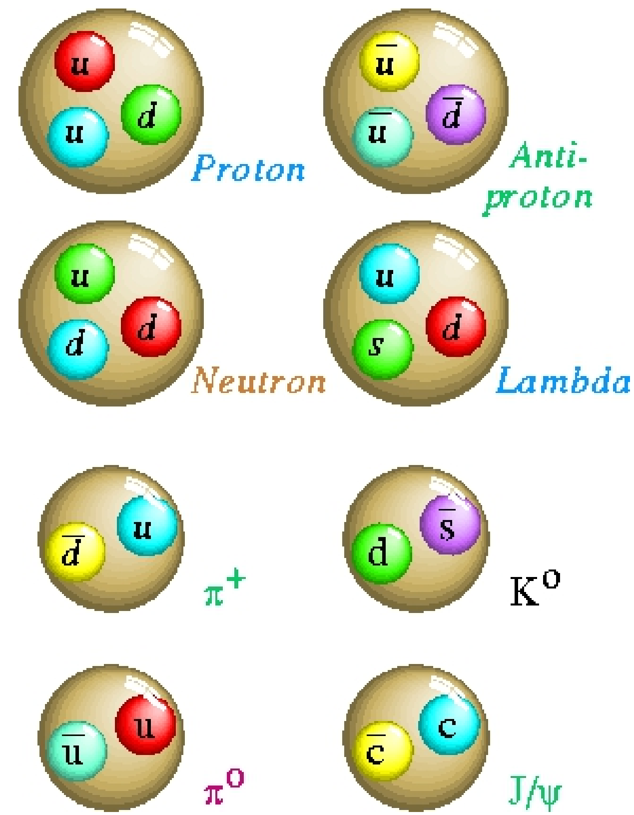
Hadron’s polarization

HUAZHONG NORMAL UNIVERSITY

Polarized quarks → Coalescence models with spin → Polarized hadrons

$$P^{y}_{\Lambda/\bar{\Lambda}}(t,\mathbf{x}) \;=\; \frac{1}{2}\omega_y \pm \frac{Q_s}{2m_s T}B_y$$

$$\rho^{\phi}_{00}(t,\mathbf{x}) \;\approx\; \frac{1}{3} - \frac{4}{9}\int \frac{d^3\mathbf{p}}{(2\pi)^3} P^{y}_{s}(\mathbf{p})P^{y}_{\bar{s}}(-\mathbf{p})\,|\psi_{\phi}(\mathbf{p})|^2$$



Z.-T. Liang, X.-N. Wang, PLB 629 (2005) 20;
PRL 94 (2005) 102301; PRL 96 (2006) 039901.

Y.-G. Yang, R.-H. Fang, Q Wang, and X.-N. Wang, PRC 97 (2018) 3, 034917.

XLS, L. Oliva, Q. Wang, PRD 101 (2020) 9, 096005.

XLS, Q. Wang, X.-N. Wang, PRD 102 (2020) 5, 056013.

Λ's decay

- Spin polarization of Λ hyperons

Weak decay   CP violation

$$\Lambda \rightarrow \mathrm{p} + \pi^-$$

	(uds)	(uud)	$(\bar{\mathrm{u}}\mathrm{d})$
strangeness	1	0	0

- Angle distribution of daughter protons in Λ's rest frame

$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}\left(1 + \alpha_H \mathscr{P}_\Lambda \frac{\mathbf{n}^* \cdot \mathbf{p}^*}{|\mathbf{p}^*|}\right)$$

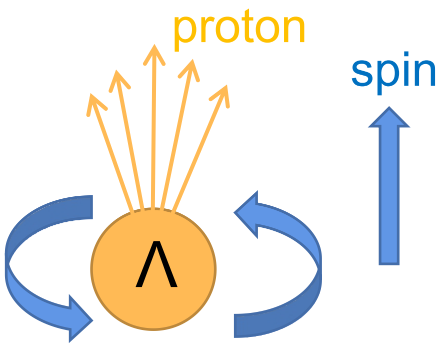
Λ's polarization direction in rest frame

proton's momentum in Λ's rest frame

decay parameter (constant)

Λ's spin polarization

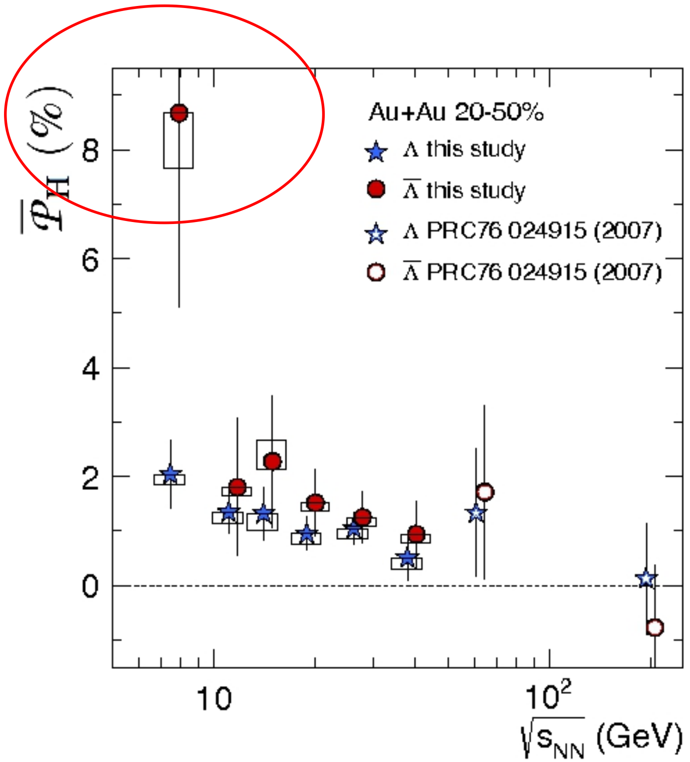
$\cos\theta^*$



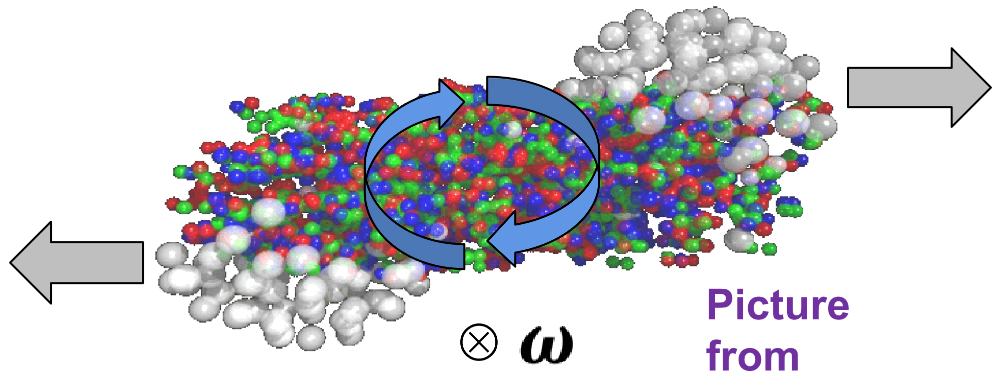
Λ's polarization

Global polarization

???



L. Adamczyk, et al. (STAR), Nature 548 (2017) 62.



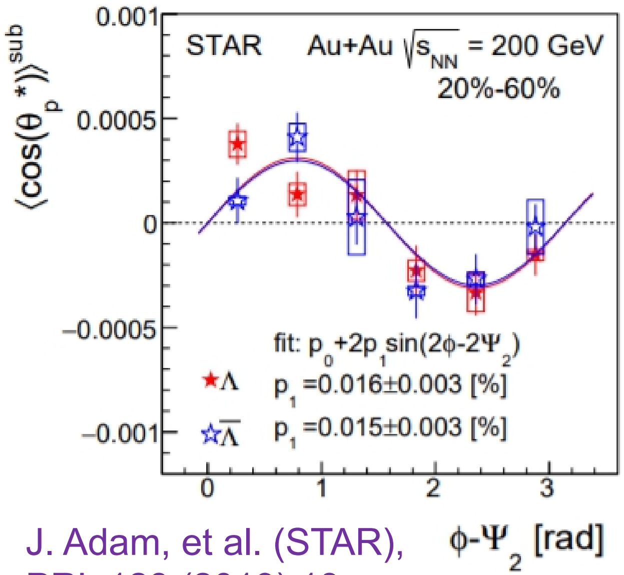
Picture from UrQMD

- Polarization along direction of total angular momentum
- Agree with theoretical prodiction
- Difference between Λ and anti-Λ may be contribution from magnetic field

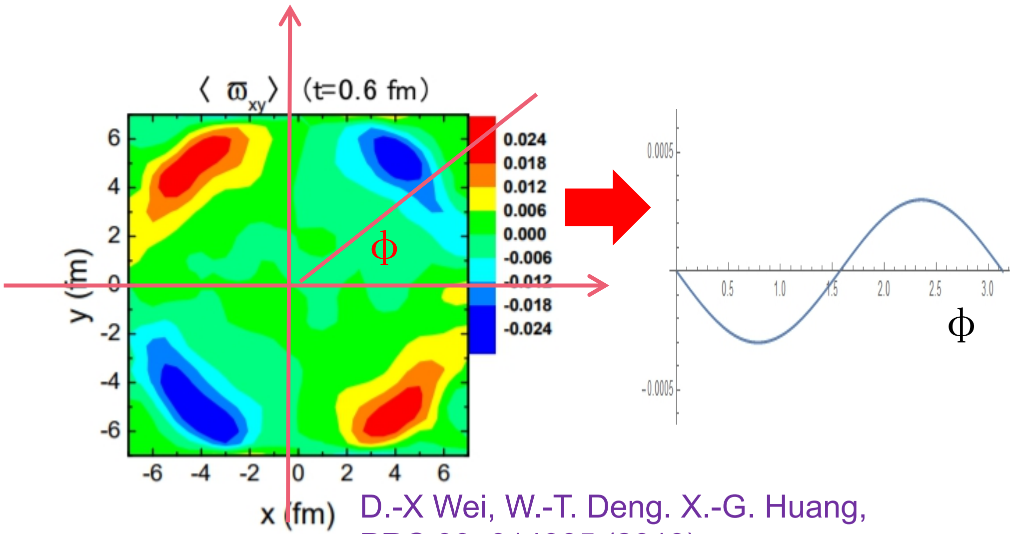
- Strange enhancement for anti-Λ's polarization at very low energy

Λ's polarization

Longitudinal polarization



J. Adam, et al. (STAR), PRL 123 (2019) 13.



D.-X Wei, W.-T. Deng. X.-G. Huang, PRC 99, 014905 (2019)

- Sign problem
- Possible reason:

  Angular momentum conservation;
  T-vorticity; shear-induced polarization

S. Liu, Y. Sun, C. M. Ko, PRL 125, 062301 (2020).
H.-Z. Wu, et. al., PRR 1, 033058 (2019).
W. Florkowski, et. al., PRC 100, 054907 (2019).
S. Liu and Y. Yin, e-Print: 2103.09200.
B. Fu, et. al., e-Print: 2103.10403.
F. Becattini, et. al., e-Print: 2103.10917; e-Print: 2103.14621.

Vector mesons

- Spin alignments of vector mesons

Decay of vector mesons

$$\phi \rightarrow K^+ + K^-$$

$$K^{*0} \rightarrow K^+ + \pi^-$$

$$J^P \quad 1^- \quad 0^- \quad 0^-$$

pesudo-scalar mesons

Angle distribution of daughter particle depends on the wave function of mother meson

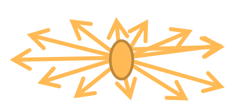
For vector meson with S = 1, m = S_z $\quad \frac{dN}{d\Omega} = |Y_{1,S_z}(\theta,\phi)|^2$

Y.-G. Yang, R.-H. Fang, Q. Wang, X.-N. Wang, PRC 97, 034917 (2017).

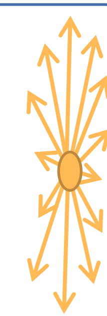
- Polar angle distribution of daughter particles

$$W(\theta) = \frac{3}{4}[(1-\rho_{00}) + (3\rho_{00}-1)\cos^2\theta].$$

$\rho_{00} < \frac{1}{3}$ More mesons have magnetic quantum number +1 or -1



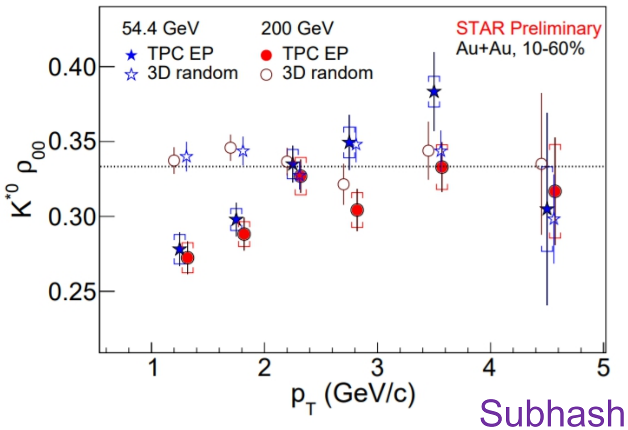
$\rho_{00} > \frac{1}{3}$

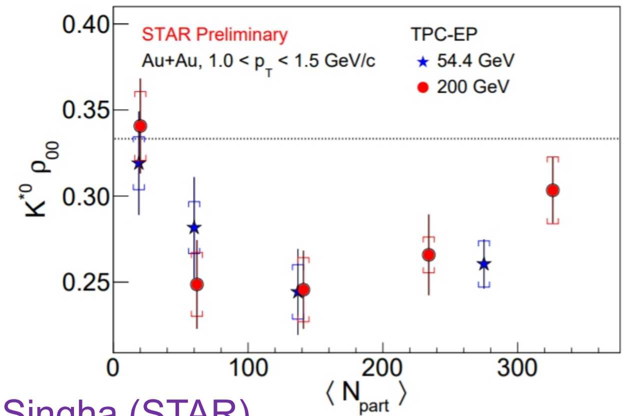


Meson's decay

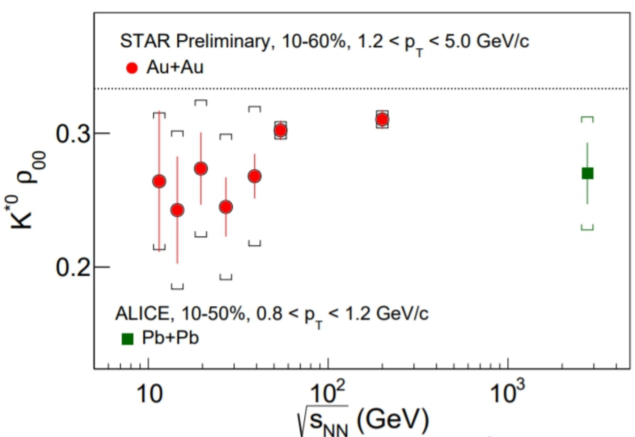
Spin alignments of vector mesons

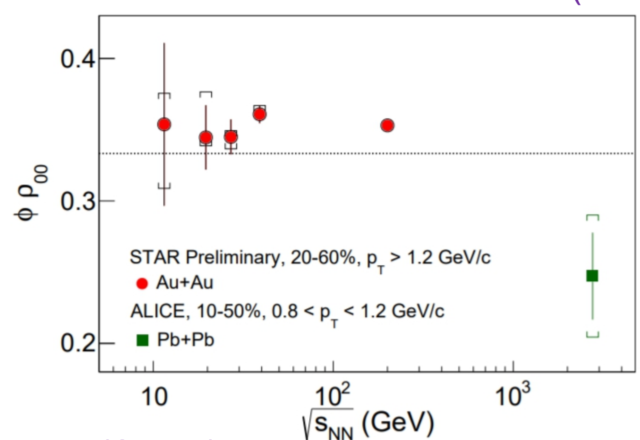
Significant derivation from 1/3

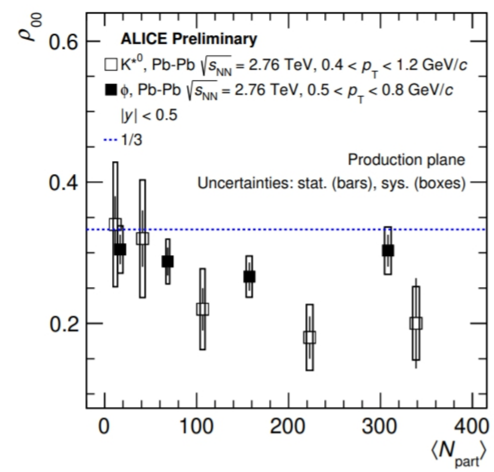




Subhash Singha (STAR), NPA 1005 (2021) 121733.



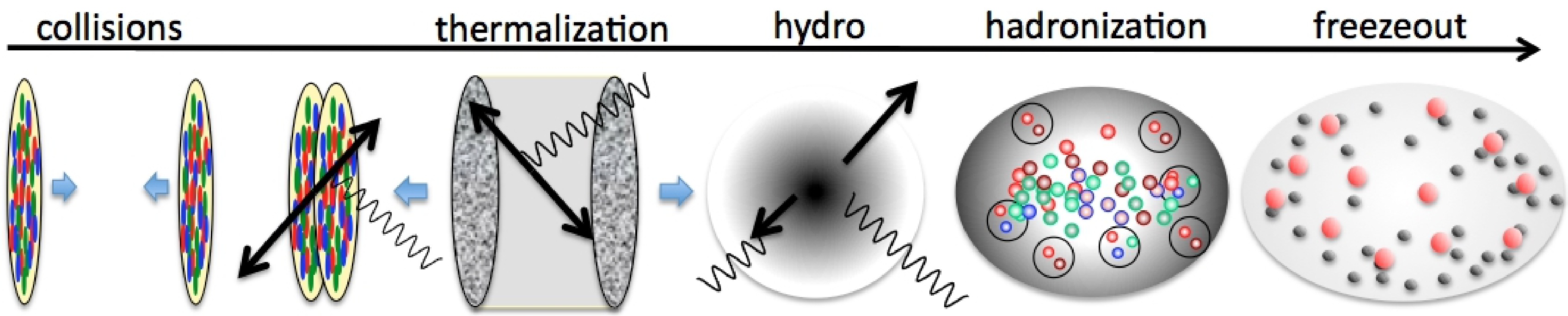




Subhash Singha (STAR), NPA 1005 (2021) 121733.

Sourav Kundu (ALICE), NPA 1005 (2021) 121912

Motivation



Initial fluctuation → Initial state: quarks are randomly polarized ☆

Evolution of spin →

hydrodynamic model → Final state: quarks are polarized along vorticity

Combination of polarized quarks →

final state interactions → Freeze-out: Polarized hadrons

→ Detected through angle distributions of decay daughters

In this talk, we will neglect electromagnetic fields.

Related works

Quark polarization from parton scatterings in heavy ion collisions
J.-J. Zhang, R.-H. Fang, Q. Wang, X.-N. Wang, PRC 100 (2019) 6, 064904.

Quantum Kinetic Theory of Spin Polarization of Massive Quarks in Perturbative QCD: Leading Log
S. Li, H.-U. Yee, PRD 100 (2019) 5, 056022.

Effective quantum kinetic theory for spin transport of fermions with collsional effects
D.-L. Yang, K. Hattori, Y. Hidaka, JHEP 07 (2020) 070.

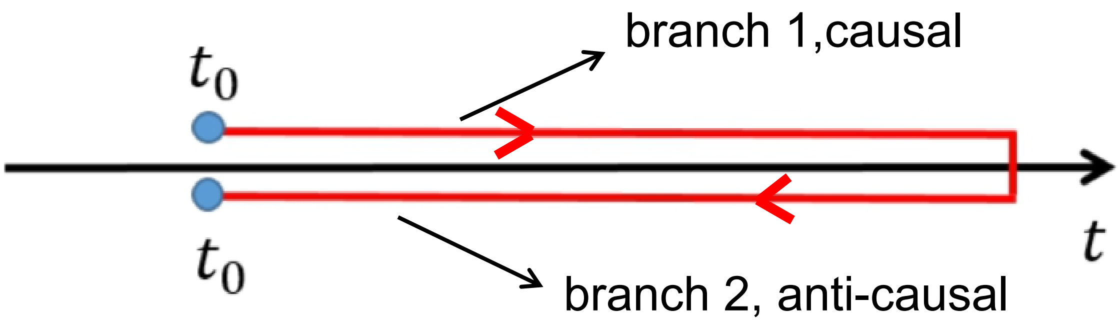
Chiral kinetic theory with small mass corrections and quantum coherent states
C. Manuel, J. M. Torres-Rincon, e-Print: 2101.05832.

Generating spin polarization from vorticity through nonlocal collisions
N. Weickgenannt, E. Speranza, XLS, Q. Wang, D. H. Rischke, e-Print: 2005.01506; e-Print: 2103.04896.

From Kadanoff--Baym to Boltzmann equations for massive spin-1/2 fermions
XLS, N. Weickgenannt, E. Speranza, D. H. Rischke, Q. Wang, e-Print: 2103.10636.

Contents

Closed time path



Closed time path contour (Schwinger-Keldysh)

P. Martin, J. S.Schwinger, PR 115 (1959) 1342.

L.V. Keldysh, Zh. Eksp. Teor. Fiz. 47 (1964) 1515.

$$G_{\alpha\beta}(x_1,x_2) = \langle T_C \psi_\alpha(x_1)\bar{\psi}_\beta(x_2)\rangle$$

$$= \left\langle T_C \psi_{I\alpha}(x_1)\bar{\psi}_{I\beta}(x_2) \exp\left[-i\int_C dt\, \mathcal{H}_I(t)\right]\right\rangle$$

In this way we establish a direct connection of observables and scattering matrix S.

Matrix form $G(x_1,x_2) = \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix}(x_1,x_2)$

	t_1	t_2	
$G^F_{\alpha\beta}(x_1,x_2) = \langle T\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle$,	branch 1	branch 1	time-ordering
$G^{\bar{F}}_{\alpha\beta}(x_1,x_2) = \langle T_A\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle$,	branch 2	branch 2	anti time-ordering
$G^<_{\alpha\beta}(x_1,x_2) = -\langle \overline{\psi}_\beta(x_2)\psi_\alpha(x_1)\rangle$,	branch 1	branch 2	
$G^>_{\alpha\beta}(x_1,x_2) = \langle \psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle$,	branch 2	branch 1	

Wigner function

- Two point Green function $G^{<}_{\alpha\beta}(x_1, x_2)$

  Fourier transform w.r.t. relative position

$$G^{<}_{\alpha\beta}(x,p) \equiv -\frac{1}{2\pi\hbar}\int d^4y\, e^{ip\cdot y/\hbar}\left\langle \overline{\psi}_\beta\left(x-\frac{y}{2}\right)\psi_\alpha\left(x+\frac{y}{2}\right)\right\rangle$$

- Wigner function (up to a constant factor)
  Heinz(1983);Vasak, Gyulassy, Elze (1987); Zhuang, Heinz (1996); etc...

  A well-defined quasi-probability distribution for quantum particles.

- Classical particle distribution

  $$f(t, \mathbf{x}, \mathbf{p})$$

  is not well defined if we consider the Heisenberg uncertainty principle.

  $$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

  p is the conjugate momentum of *x*

- Wigner function is well-defined

  $$\hat{x}^\mu \equiv \frac{1}{2}\left(\hat{x}_1^\mu + \hat{x}_2^\mu\right), \quad \hat{p}^\mu \equiv \frac{1}{2}\left(\hat{p}_1^\mu - \hat{p}_2^\mu\right)$$

  $$[\hat{x}_a^\mu, \hat{p}_b^\nu] = -i\hbar g^{\mu\nu}\delta_{ab},$$

  $$[\hat{x}^\mu, \hat{p}^\nu] = 0$$

  Average position is commutable with relative momentum.

Docomposition

- Wigner function: 4×4 complex matrix, with 16 constraints,

$$\left(G^{<}\right)^{\dagger} = \gamma^0 G^{<} \gamma^0$$

- Expansion in terms of generators of Clifford algebra,

$$\Gamma_i = \{\mathbb{I}_4,\ i\gamma^5,\ \gamma^\mu,\ \gamma^5\gamma^\mu,\ \tfrac{1}{2}\sigma^{\mu\nu}\}$$

$$\Gamma_i^{\dagger} = \gamma^0 \Gamma_i \gamma^0 \qquad 1+1+4+4+6=16$$

$$G^{<}(x,p) = \frac{1}{4}\left(\mathbb{I}_4 \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5\gamma^\mu \mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right)$$

Real functions in 8-d phase space

$$\mathrm{Tr}(\Gamma_i G^{<})$$

Physical meanings

華中師範大學
HUAZHONG NORMAL UNIVERSITY

$$G^<(x,p) = \frac{1}{4}\left(\mathbb{I}_4\mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^5\gamma^\mu\mathcal{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right)$$

	Property	Physical meaning (distribution in phase space)
$\mathcal{F}$	Scalar	Mass
$\mathcal{P}$	Pesudoscalar	Pesudoscalar condensate
$\mathcal{V}^\mu$	Vector	Net fermion current
$\mathcal{A}^\mu$	Axial-vector	Polarization (or spin current)
$\mathcal{S}^{\mu\nu}$	Tensor	Electric/magnetic dipole-moment

Physical quantities

- Net-fermion number current, spin polarization

$$\mathbb{N}^{\mu}(x)=\int d^4p\,\mathcal{V}^{\mu}(x,p),\quad \mathbb{N}_5^{\mu}(x)=\int d^4p\,\mathcal{A}^{\mu}(x,p).$$

- Energy-momentum tensor, spin tensor, and orbital angular momentum tensor

$$\mathbb{T}^{\mu\nu}_{\textbf{mat}}(x)=\int d^4p\,p^{\nu}\mathcal{V}^{\mu}(x,p)$$

canonical, in general not sysmmetric

$$\mathbb{S}^{\rho,\mu\nu}_{\textbf{mat}}(x)=-\frac{1}{2}\epsilon^{\rho\mu\nu\alpha}\int d^4p\,\mathcal{A}_{\alpha}(x,p)$$

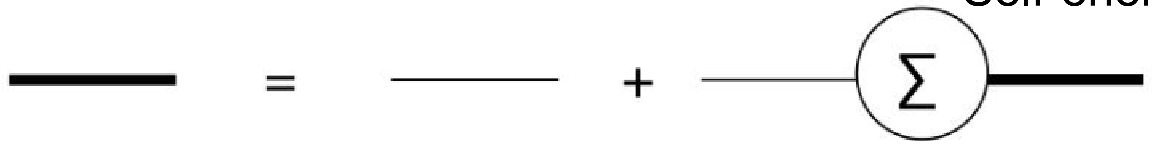
$$\mathbb{L}^{\rho,\mu\nu}_{\mathrm{mat}}(x)=x^{\mu}\mathbb{T}^{\rho\nu}_{\mathrm{mat}}(x)-x^{\nu}\mathbb{T}^{\rho\mu}_{\mathrm{mat}}(x)$$

D-S equation

HUAZHONG NORMAL UNIVERSITY

- Dyson-Schwinger equation

Self-energy



$$G = G_0 + G_0 \Sigma G$$

Integral over whole CTP

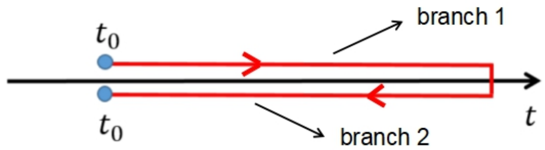
$$\pm i(i\gamma_\mu \partial^\mu_{x_1} - m)G(x_1, x_2) = \delta^{(4)}(x_1 - x_2) + \int_C dx' \Sigma(x_1, x') G(x', x_2)$$

operator G_0^{-1}

$-$ when x_1 is on branch 1

$+$ when x_1 is on branch 2

Green function (self energy) on CTP contour



$$G(x_1, x_2) = \begin{pmatrix} G^F & G^< \\ G^> & G^{\bar{F}} \end{pmatrix} (x_1, x_2)$$

Kinetic equation of Wigner function is the top-right element.

- Kadanoff-Baym equation

$$(\gamma \cdot K - m)\, G^{<}(x,p) = -\frac{i\hbar}{2}\left[\Sigma^{<}(x,p)G^{>}(x,p) - \Sigma^{>}(x,p)G^{<}(x,p)\right]$$
$$-\frac{\hbar^2}{4}\left[\{\Sigma^{<}(x,p), G^{>}(x,p)\}_{\mathrm{PB}} - \{\Sigma^{>}(x,p), G^{<}(x,p)\}_{\mathrm{PB}}\right]$$

$$K^{\mu} \equiv p^{\mu} + \frac{i\hbar}{2}\partial_x^{\mu}$$

- Corrections from self-energy.
- A gradient expansion is used and higher order contributions are truncated.
- Poisson bracket :

$$\{A, B\}_{\mathrm{PB}} \equiv (\partial_x A)\cdot(\partial_p B) - (\partial_p A)\cdot(\partial_x B)$$

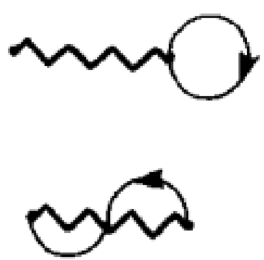
L. P. Kadanoff, G. Baym, (1962) *Quantum Statistical Mechanics.*

Collisionless: D. Vasak, M. Gyulassy, H.T. Elze, Annals Phys. 173 (1987) 462.

Self-energy

华中师范大学
HUAZHONG NORMAL UNIVERSITY

- Leading order:
  mean-field contributions



neglected

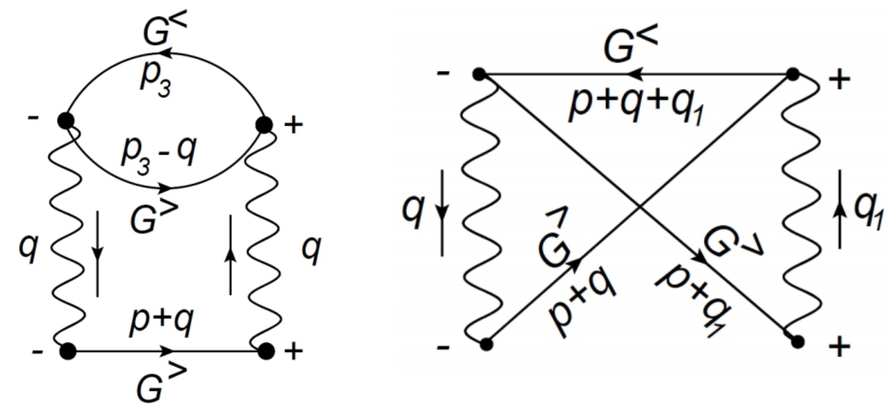
$$\Sigma(x_1,x_2) = [\Sigma_S(x_1) + \gamma_\mu \Sigma_V^\mu(x_1)]\,\delta^{(4)}(x_1-x_2)$$

effective mass

effective momentum

analogue of electromagnetic gauge potential

- Next-to-leading order: Born diagrams, scattering contributions



Solid lines: propagators of fermions
Wave lines: propagators of bosons
+/- sign: location of vertices on CTP contour

M. Schonhofen, M. Cubero, B. L. Friman, W. Norenberg, G. Wolf, NPA 572 (1994) 112.

NJL self-energy

- NJL model as an example:

$$\mathcal{L}_{\rm NJL} = \bar{\psi}(i\hbar\gamma\cdot\partial_x - m)\psi + \sum_a G_a\left(\bar{\psi}\Gamma_a\psi\right)^2$$

$$\{\mathbb{I}_4,\ i\gamma^5,\ \gamma^\mu,\ \gamma^5\gamma^\mu,\ \tfrac{1}{2}\sigma^{\mu\nu}\}$$

$$\begin{aligned}\Sigma^>(x,p) \;=\; & 4G_aG_b\int\frac{d^4p_1}{(2\pi\hbar)^4}\frac{d^4p_2}{(2\pi\hbar)^4}\frac{d^4p_3}{(2\pi\hbar)^4}(2\pi\hbar)^4\delta^{(4)}(p+p_3-p_1-p_2)\\ & \times\left\{\mathrm{Tr}\left[\Gamma_aG^<(x,p_3)\Gamma_bG^>(x,p_1)\right]\Gamma_bG^>(x,p_2)\Gamma_a\right.\\ & \left.-\Gamma_bG^>(x,p_1)\Gamma_aG^<(x,p_3)\Gamma_bG^>(x,p_2)\Gamma_a\right\}\end{aligned}$$

coupling constants

$$\Sigma^<(x,p) = \Sigma^>[G^>\leftrightarrow G^<]$$

bare vertices instead of full vertices

XLS, N. Weickgenannt, E. Speranza, D. H. Rischke, Q. Wang, e-Print: 2103.10636.

Component equations

- KB equation $$\left(\gamma\cdot p+\frac{i\hbar}{2}\gamma\cdot\partial_x-m\right)G^<(x,p)=I_{\rm coll}(x,p)$$

Decompose in terms of Γ_{i} $$G^<(x,p)=\frac{1}{4}\left(\mathbb{I}_4\mathcal{F}+i\gamma^5\mathcal{P}+\gamma^\mu\mathcal{V}_\mu+\gamma^5\gamma^\mu\mathcal{A}_\mu+\frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right)$$

Real parts

$$\begin{aligned}
p^\mu\mathcal{V}_\mu-m\mathcal{F}&={\rm Re\,Tr}\left(I_{\rm coll}\right)\ ,\\
m\mathcal{P}+\frac{\hbar}{2}\partial_x^\mu\mathcal{A}_\mu&={\rm Re\,Tr}\left(i\gamma^5I_{\rm coll}\right)\ ,\\
p_\mu\mathcal{F}-m\mathcal{V}_\mu+\frac{\hbar}{2}\partial_x^\nu\mathcal{S}_{\mu\nu}&={\rm Re\,Tr}\left(\gamma_\mu I_{\rm coll}\right)\ ,\\
\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}p^\nu\mathcal{S}^{\alpha\beta}+m\mathcal{A}_\mu-\frac{\hbar}{2}\partial_{x,\mu}\mathcal{P}&={\rm Re\,Tr}\left(\gamma^5\gamma_\mu I_{\rm coll}\right)\ ,\\
\epsilon_{\mu\nu\alpha\beta}p^\alpha\mathcal{A}^\beta+m\mathcal{S}_{\mu\nu}-\frac{\hbar}{2}\partial_{x[\mu}\mathcal{V}_{\nu]}&=-{\rm Re\,Tr}\left(\sigma_{\mu\nu}I_{\rm coll}\right)\ ,
\end{aligned}$$

Imaginary parts

$$\begin{aligned}
\frac{\hbar}{2}\partial_x^\mu\mathcal{V}_\mu&={\rm Im\,Tr}\left(I_{\rm coll}\right)\ ,\\
p^\mu\mathcal{A}_\mu&={\rm Im\,Tr}\left(-i\gamma^5I_{\rm coll}\right)\ ,\\
p^\nu\mathcal{S}_{\nu\mu}+\frac{\hbar}{2}\partial_{x,\mu}\mathcal{F}&={\rm Im\,Tr}\left(\gamma_\mu I_{\rm coll}\right)\ ,\\
p_\mu\mathcal{P}+\frac{\hbar}{4}\epsilon_{\mu\nu\alpha\beta}\partial_x^\nu\mathcal{S}^{\alpha\beta}&={\rm Im\,Tr}\left(\gamma^5\gamma_\mu I_{\rm coll}\right)\ ,\\
p_{[\mu}\mathcal{V}_{\nu]}+\frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial_x^\alpha\mathcal{A}^\beta&=-{\rm Im\,Tr}\left(\sigma_{\mu\nu}I_{\rm coll}\right)\ ,
\end{aligned}$$

Source terms from interactions

Free case has been solved analytically in

J.-H. Gao, Z.-T. Liang, PRD 100 (2019) 5; N. Weickgenannt, XLS, E. Speranza, Q. Wang, D. H. Rischke, PRD 100 (2019) 5; K. Hattori, Y. Hidaka, D.-L. Yang, PRD 100 (2019) 9.

Semi-classical expansion

- Semi-classical expansion:

  Expand Wigner function and its kinetic equations in terms of $\hbar$

  Zeroth order: classical particles

  Linear order: spin corrections

- Valid if $\left|\hbar\gamma^{\mu}\partial_{x\mu}G^{<}\right| \ll m\left|G^{<}\right|$

  Compton wave length << wave length of macroscopic fluctuations;
  Wigner function is slowly-varing in space-time.

☆ In absence of electromagnetic field, semi-classical expansion is equivalent to gradient expansion.

☆ Zeroth order can contain classical spin degrees of freedom

D. Vasak, M. Gyulassy, H.T. Elze, Annals Phys. 173 (1987) 462.

- Mass-shell conditions

$$
\begin{aligned}
(p^2-m^2)\mathcal{F}^{(0)} &= 0\\
(p^2-m^2)\mathcal{F}^{(1)} &= 2m\,\mathrm{ReTr}\left(I^{(1)}_{\mathrm{coll}}\right)\\
(p^2-m^2)\mathcal{A}^{(0)}_\mu &= 0\\
(p^2-m^2)\mathcal{A}^{(1)}_\mu &= -\epsilon_{\mu\nu\alpha\beta}p^\nu \mathrm{ReTr}\left(\sigma^{\alpha\beta}I^{(1)}_{\mathrm{coll}}\right)
\end{aligned}
$$

Interactions induce mass corrections at first order

$\mathcal{P},\ \mathcal{V}_\mu,\ \mathcal{S}_{\mu\nu}$ are expressed in terms of $\mathcal{F},\ \mathcal{A}_\mu$

XLS, N. Weickgenannt, E. Speranza, D. H. Rischke, Q. Wang, e-Print: 2103.10636.

- Constraint conditions

$$p^\mu\mathcal{A}^{(0)}_\mu = 0 \quad p^\mu\mathcal{A}^{(1)}_\mu = \mathrm{ImTr}\left(-i\gamma^5 I^{(1)}_{\mathrm{coll}}\right)$$

- Boltzmann equations

$$
\begin{aligned}
p\cdot\partial_x\mathcal{F}^{(0)} &= 2m\,\mathrm{ImTr}\left(I^{(1)}_{\mathrm{coll}}\right)\\
p\cdot\partial_x\mathcal{F}^{(1)} &= 2m\,\mathrm{ImTr}\left(I^{(2)}_{\mathrm{coll}}\right)+\mathrm{ReTr}\left(\gamma\cdot\partial_x I^{(1)}_{\mathrm{coll}}\right)\\
p\cdot\partial_x\mathcal{A}^{(0)}_\mu &= -\epsilon_{\mu\nu\alpha\beta}p^\nu\mathrm{ImTr}\left(\sigma^{\alpha\beta}I^{(1)}_{\mathrm{coll}}\right)\\
p\cdot\partial_x\mathcal{A}^{(1)}_\mu &= -2p^\mu\mathrm{ImTr}\left(\gamma^5 I^{(2)}_{\mathrm{coll}}\right)-2\mathrm{ImTr}\left(\gamma\cdot p\gamma^5\gamma^\mu I^{(2)}_{\mathrm{coll}}\right)-\mathrm{ReTr}\left(\gamma^5\partial^\mu_x I^{(1)}_{\mathrm{coll}}\right)
\end{aligned}
$$

Wigner function solution

華中師範大學 HUAZHONG NORMAL UNIVERSITY

- Wigner function can be separated into three parts

$$G^{<}(x,p) \quad = \quad G^{<}_{\rm qc}(x,p) + G^{<}_{\nabla}(x,p) + G^{<}_{\rm off}(x,p)$$

On-shell

$$G^{<}_{\rm qc}(x,p) \quad = \quad \frac{1}{4m}(\gamma\cdot p + m)(\mathcal{F}_{\rm qc} + \gamma^5\gamma\cdot\mathcal{A}_{\rm qc})$$

Quasi-classical contribution

$$G^{<}_{\nabla}(x,p) \quad = \quad \frac{i\hbar}{4m}\left[\gamma_\mu, \partial^\mu_x G^{<}_{\rm qc}(x,p)\right] \; + \mathcal{O}(\text{coupling constant})$$

Gradient and collision contributions

contribution of collisions, depends on $I_{\rm coll}$

$$G^{<}_{\rm off}(x,p) \quad = \quad \frac{1}{4m}(\gamma\cdot p + m)(\mathcal{F}_{\rm off} + \gamma^5\gamma\cdot\mathcal{A}_{\rm off})$$

Off-shell contribution

In the Kadanoff-Baym equation, off-shell part of Wigner function cancels with off-shell part of collision term.

XLS, N. Weickgenannt, E. Speranza, D. H. Rischke, Q. Wang, e-Print: 2103.10636.

Matrix distributions

- Quasi-classical part takes the same form as that in absence of interactions. It can be expressed in terms of free-streaming wave functions

$$
\begin{aligned}
G^{<}_{\mathrm{qc},\alpha\beta}(x,p) &= -2\pi\hbar\delta(p^2-m^2)\theta(p_0)\sum_{rs} u_{r,\alpha}(p)\bar{u}_{s,\beta}(p) f^{(+)}_{sr}(x,p) \\
&\quad -2\pi\hbar\delta(p^2-m^2)\theta(-p_0)\sum_{rs} v_{s,\alpha}(\bar{p})\bar{v}_{r,\beta}(\bar{p})\left[\delta_{sr} - f^{(-)}_{sr}(x,\bar{p})\right]
\end{aligned}
$$

2×2 Hemitian matrix

$\bar{p}^\mu \equiv (E_{\mathbf{p}}, -\mathbf{p})$

- Scalar and axial-vector components

number density

$$
\mathcal{F}_{\mathrm{qc}}(x,p) = -2\pi\hbar\frac{m}{E_p}\left\{\delta(p_0-E_p)\,\mathrm{tr}\left[f^{(+)}(x,p)\right] + \delta(p_0+E_p)\,\mathrm{tr}\left[f^{(-)}(x,\bar{p})-1\right]\right\} + \mathcal{O}(\hbar^2)\,,
$$

$$
\mathcal{A}^\mu_{\mathrm{qc}}(x,p) = -2\pi\hbar\frac{m}{E_p}\left\{\delta(p_0-E_p)\mathrm{tr}\left[n^{(+)\mu} f^{(+)}(x,p)\right] + \delta(p_0+E_p)\mathrm{tr}\left[n^\mu(\bar{p}) f^{(-)}(x,\bar{p})\right]\right\} + \mathcal{O}(\hbar^2)\,,
$$

spin density

$$
n^{(\pm)\mu} \equiv \left(\pm\frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{m}, \boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{m(E_p+m)}\mathbf{p}\right)
$$

$(0,\boldsymbol{\sigma})$ in rest frame

$\boldsymbol{\sigma}$ are Pauli matrices

Boltzmann equations

- Boltzmann equation for particle number density

$$p \cdot \partial_x \mathrm{tr}\left[f^{(+)}(x,p)\right] = -\frac{E_p}{2\pi\hbar m}\int_0^{\infty} dp_0 \left[2m \mathrm{Im}\,\mathrm{Tr}(I_{\mathrm{coll}}) + \mathrm{Re}\,\mathrm{Tr}(\gamma \cdot \partial_x I_{\mathrm{coll}})\right]$$

$\mathcal{O}(\partial_x)$

takes out particle contributions and drops antiparticle contributions

$$I_{\mathrm{coll}} = -\frac{i\hbar}{2}\left[\Sigma^{<}(x,p)G^{>}(x,p) - \Sigma^{>}(x,p)G^{<}(x,p)\right] - \frac{\hbar^2}{4}\left[\{\Sigma^{<}(x,p), G^{>}(x,p)\}_{\mathrm{PB}} - \{\Sigma^{>}(x,p), G^{<}(x,p)\}_{\mathrm{PB}}\right] \quad \mathcal{O}(\partial_x)$$

$$G^{\lessgtr}_{\mathrm{on}}(x,p) = G^{\lessgtr}_{\mathrm{qc}}(x,p) + G^{\lessgtr}_{\nabla}(x,p)$$

$\mathcal{O}(1)$ $\quad$ $\mathcal{O}(\partial_x)$

Off-shell part of Wigner function does not contribute

$$p \cdot \partial_x \mathrm{tr}\left[f^{(+)}(x,p)\right] = \mathscr{C}_{\mathrm{scalar}}(I_{\mathrm{coll,qc}}) + \mathscr{C}_{\mathrm{scalar}}(I_{\mathrm{coll},\nabla}) + \mathscr{C}_{\mathrm{scalar}}(I_{\mathrm{coll,PB}}) + \mathscr{C}_{\mathrm{scalar}}(\partial_x I_{\mathrm{coll}})$$

$\mathcal{O}(1)$ $\quad$ $\mathcal{O}(\partial_x)$

- Boltzmann equations for spin density

$$p\cdot\partial_x \mathrm{tr}\left[n^{(+)\mu}f^{(+)}(x,p)\right] = \frac{E_p}{2\pi\hbar m}\int_0^\infty dp_0\left[\epsilon^{\mu\nu\alpha\beta}p_\nu \mathrm{Im}\,\mathrm{Tr}(\sigma_{\alpha\beta}I_{\mathrm{coll}}) + \mathrm{Re}\,\mathrm{Tr}(\gamma^5\partial_x^\mu I_{\mathrm{coll}})\right]$$

Spin polarization density in lab frame

$$p\cdot\partial_x \mathrm{tr}\left[n^{(+)\mu}f^{(+)}(x,p)\right] = \mathscr{C}^\mu_{\mathrm{pol}}(I_{\mathrm{coll,qc}}) + \mathscr{C}^\mu_{\mathrm{pol}}(I_{\mathrm{coll},\nabla}) + \mathscr{C}^\mu_{\mathrm{pol}}(I_{\mathrm{coll,PB}}) + \mathscr{C}^\mu_{\mathrm{pol}}(\partial_x I_{\mathrm{coll}})$$

$\mathcal{O}(1)$ $\quad$ $\mathcal{O}(\partial_x)$

Center of energy (mass)

- Heisenberg uncertainty principle $\sigma_x \sigma_p \geq \frac{\hbar}{2}$
- Center of energy (dynamical mass, or inertia)

$$\langle x^\mu \rangle_E \equiv \frac{1}{P^0} \int d^3\mathbf{x}\, x^\mu\, T^{00}(x)$$

Total energy $P^\mu \equiv \int d^3\mathbf{x}\, T^{\mu 0}(x)$

A. D. Fokker, Relativiteitstheorie, P. Noordhoff, Groningen, 1929.

M. Born and L. Infeld, Proc. Roy. Soc. Lond. A 150, no.869, 141-166 (1935).

M.H.L. Pryce, Proc. Roy. Soc. Lond. A195, 62 (1948)

- Center of mass

$$\langle x^\mu \rangle_M \equiv \Lambda^\mu{}_\nu \langle x^\nu \rangle_{\text{rest}}$$

center of energy in rest frame

Rest frame means vanishing spatial components of total momentum

$$P^\mu_{\text{rest}} = (M, 0, 0, 0)$$

$$P^\mu = \Lambda^\mu{}_\nu P^\nu_{\text{rest}}$$

- Center of mass is a manifest Lorentz vector, but center of energy is not a Lorentz vector.

Center of energy (mass)

華中師範大學 HUAZHONG NORMAL UNIVERSITY

- Relation between center of energy (mass) and matrix-valued distribution

$$\langle \mathbf{x} \rangle_E = \frac{\int d^3\mathbf{x} \int d^3\mathbf{p} \left(\mathbf{x} + \frac{1}{2mE_p} \mathbf{p} \times \frac{\mathrm{tr}\left[\boldsymbol{\sigma} f^{(+)}(x,p)\right]}{\mathrm{tr}\left[f^{(+)}(x,p)\right]} \right) E_p \mathrm{tr}\left[f^{(+)}(x,p)\right]}{\int d^3\mathbf{x} \int d^3\mathbf{p}\, E_p \mathrm{tr}\left[f^{(+)}(x,p)\right]}$$

$$\langle \mathbf{x} \rangle_M = \frac{\int d^3\mathbf{x} \int d^3\mathbf{p}\, \mathbf{x}\, \mathrm{tr}\left[f^{(+)}(x,p)\right]}{\int d^3\mathbf{x} \int d^3\mathbf{p}\, \mathrm{tr}\left[f^{(+)}(x,p)\right]}$$

- Physical meaning

$\mathrm{tr}\left[f^{(+)}(x,p)\right]$ is number density of particles at time *t* with mass center located at $\mathbf{x}$ and energy center located at $\mathbf{x} + \frac{1}{2mE_p} \mathbf{p} \times \frac{\mathrm{tr}\left[\boldsymbol{\sigma} f^{(+)}(x,p)\right]}{\mathrm{tr}\left[f^{(+)}(x,p)\right]}$

~~~~~~~~~

Average polarization in rest frame
~~~~~~~~~

Center of energy (mass)

- Distance between energy center and mass center

$$\langle \mathbf{x} \rangle_E - \langle \mathbf{x} \rangle_M \simeq \frac{1}{2mE_p} \mathbf{p} \times \hat{\mathbf{n}}$$

- Side-jump in chiral kinetic theory

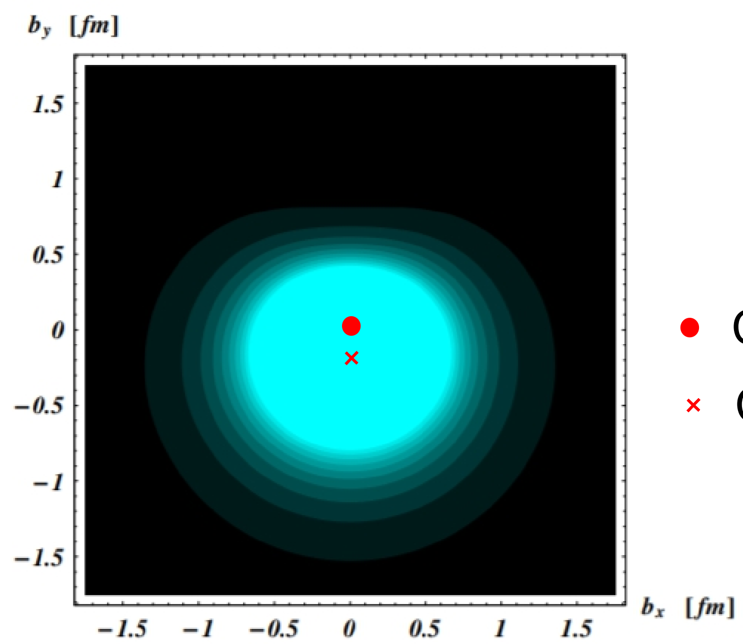
$$\langle x^\mu \rangle_1 - \Lambda^\mu{}_\nu \langle x^\nu \rangle_2$$

J.-Y. Chen, D. T. Son, M. A. Stephanov, H.-U. Yee, Y. Yin, PRL 113 (2014) 18, 182302.

J.-Y. Chen, D. T. Son, M. A. Stephanov, PRL 115 (2015) 2, 021601.

Y. Hidaka, S. Pu, D.-L. Yang, PRD 95 (2017) 9, 091901.

- Transverse charge densities for a proton polarized along x-axis. Proton is moving along z-axis.



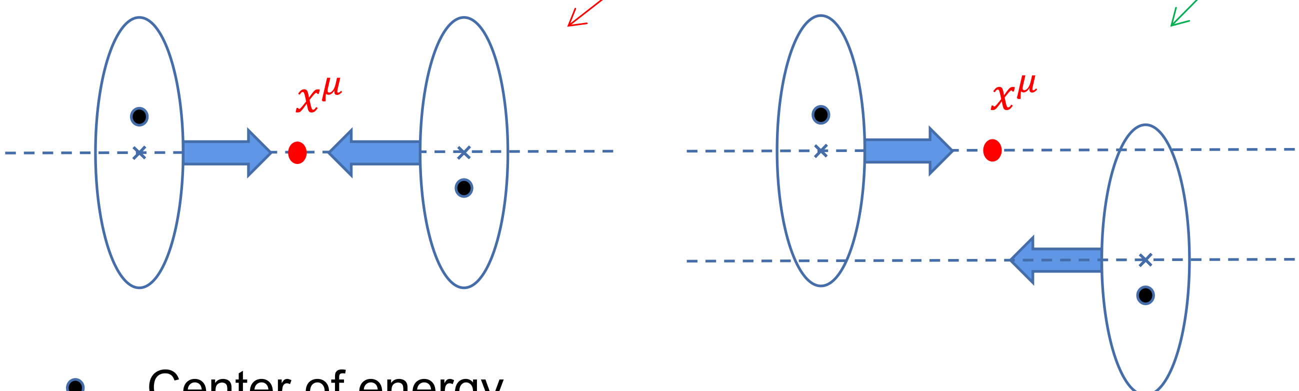
- ● Center of energy
- × Center of mass

C. E. Carlson, M. Vanderhaeghen, PRL 100 (2008) 032004.

Microscopic description

$$p\cdot\partial_x \mathrm{tr}\left[f^{(+)}(x,p)\right] = \mathscr{C}_{\mathrm{scalar}}(I_{\mathrm{coll,qc}}) + \mathscr{C}_{\mathrm{scalar}}(I_{\mathrm{coll},\nabla}) + \mathscr{C}_{\mathrm{scalar}}(I_{\mathrm{coll,PB}}) + \mathscr{C}_{\mathrm{scalar}}(\partial_x I_{\mathrm{coll}})$$

$$p\cdot\partial_x \mathrm{tr}\left[n^{(+)\mu} f^{(+)}(x,p)\right] = \mathscr{C}^{\mu}_{\mathrm{pol}}(I_{\mathrm{coll,qc}}) + \mathscr{C}^{\mu}_{\mathrm{pol}}(I_{\mathrm{coll},\nabla}) + \mathscr{C}^{\mu}_{\mathrm{pol}}(I_{\mathrm{coll,PB}}) + \mathscr{C}^{\mu}_{\mathrm{pol}}(\partial_x I_{\mathrm{coll}})$$



Interactions can modify full propagator and distribution functions

● Center of energy

× Center of mass

XLS, Y.-C. Liu, ..., in preparation

Summary

HUAZHONG NORMAL UNIVERSITY

- Derived Boltzmann equations for particle number density and spin density, respectively.
- All first order terms in space-time gradient are included.
- Microscopic descriptions are given for collision terms in Boltzmann equations.
- Transform between orbital angular momentum and spin happens at first order in spatial gradient.

Outlook

- Numerical simulating evolution of spin?
- Kinetic theory for polarized bosons and collisions between polarized fermions and bosons?
- Angular momentum conservation?

华中师范大学

HUAZHONG NORMAL UNIVERSITY

Thanks for your attention !