Spin Hydrodynamics and Symmetric Energy-Momentum Tensors

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Ref:

1. K. Fukushima, SP, 2010.01608
2. Lecture Note: K. Fukushima, SP, 2001.00359
3. Review: J.H. Gao, G.L. Ma, SP, Q. Wang, Nucl. Sci. Tech 31 (2020) 9, 90

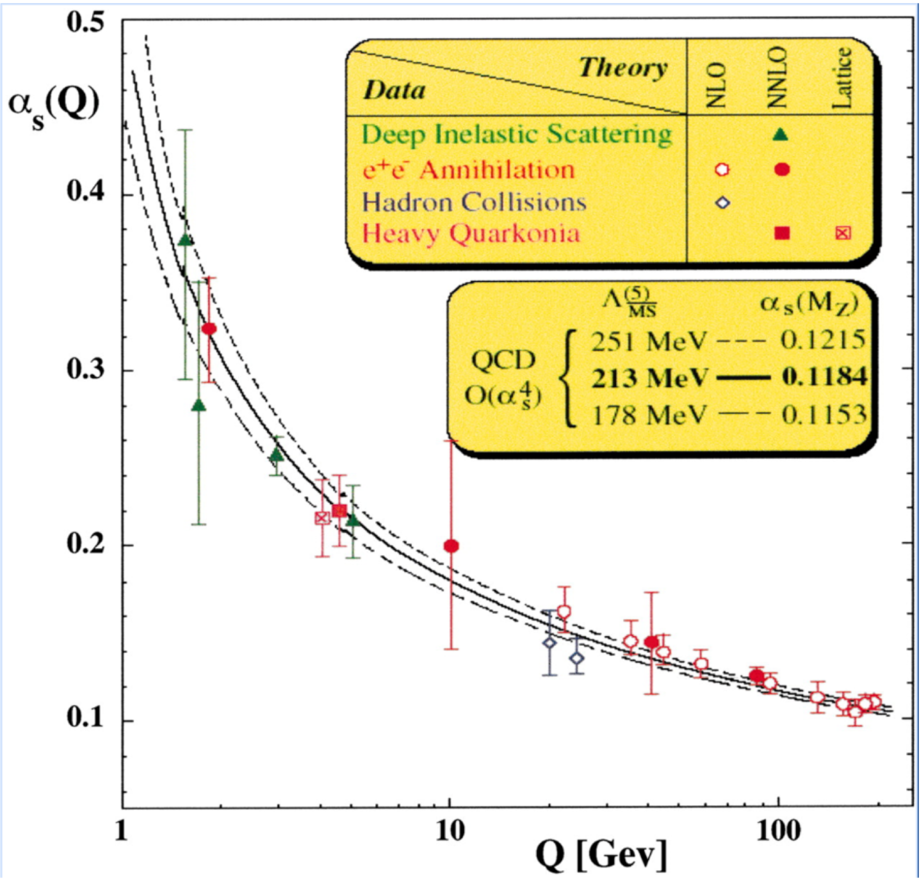
Outline

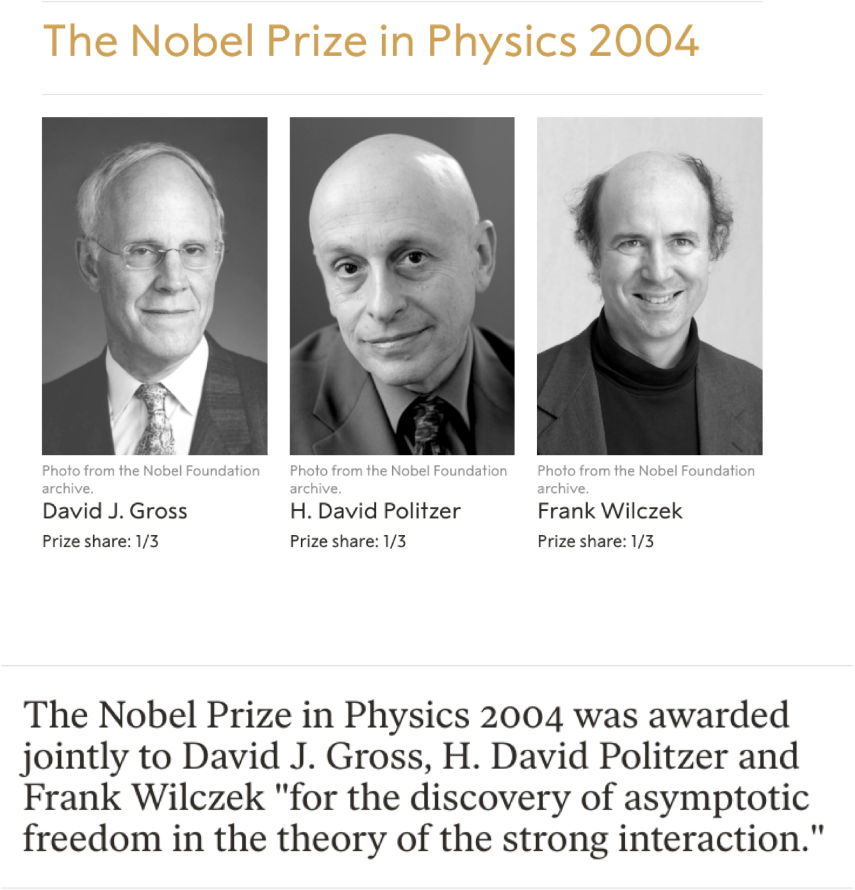
- **Introduction**
  - Why we need the relativistic spin hydro for heavy ion collisions?
  - Connection to other fields -- proton spin puzzle
  - Canonical, Belinfante and GLW decomposition of angular momentum
- **Canonical form of spin hydrodynamics**
- **Belinfante form of spin hydrodynamics**
- **Puzzles, outlook and summary**

Introduction

- Why we need the relativistic spin hydro for heavy ion collisions?
- Connection to other fields -- proton spin puzzle
- Canonical, Belinfante and GLW decomposition of angular momentum

Asymptotic freedom of QCD





The Nobel Prize in Physics 2004

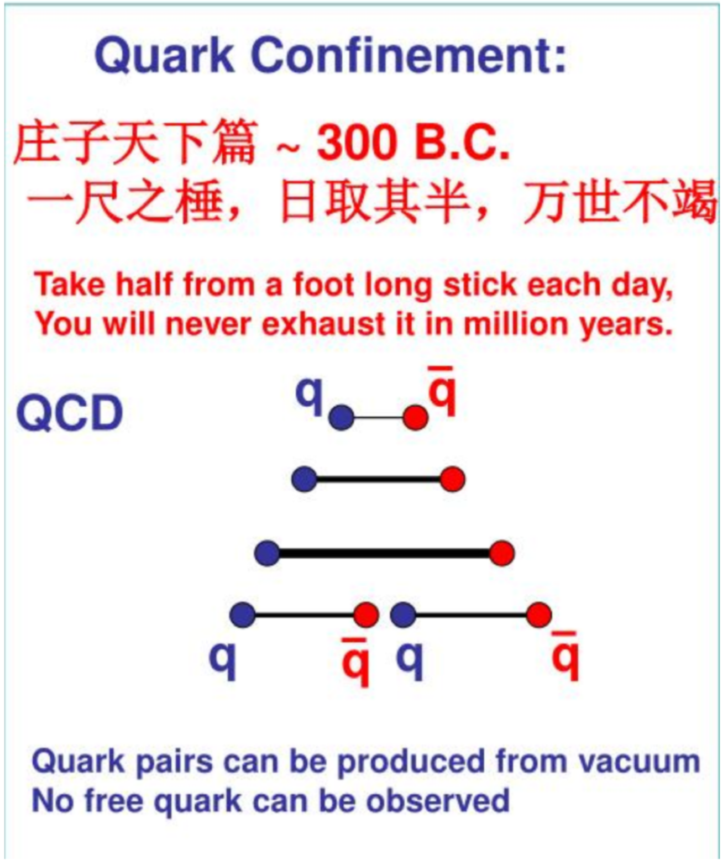
Photo from the Nobel Foundation archive.
David J. Gross
Prize share: 1/3

Photo from the Nobel Foundation archive.
H. David Politzer
Prize share: 1/3

Photo from the Nobel Foundation archive.
Frank Wilczek
Prize share: 1/3

The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek "for the discovery of asymptotic freedom in the theory of the strong interaction."

Quark Confinement



Quark Confinement:

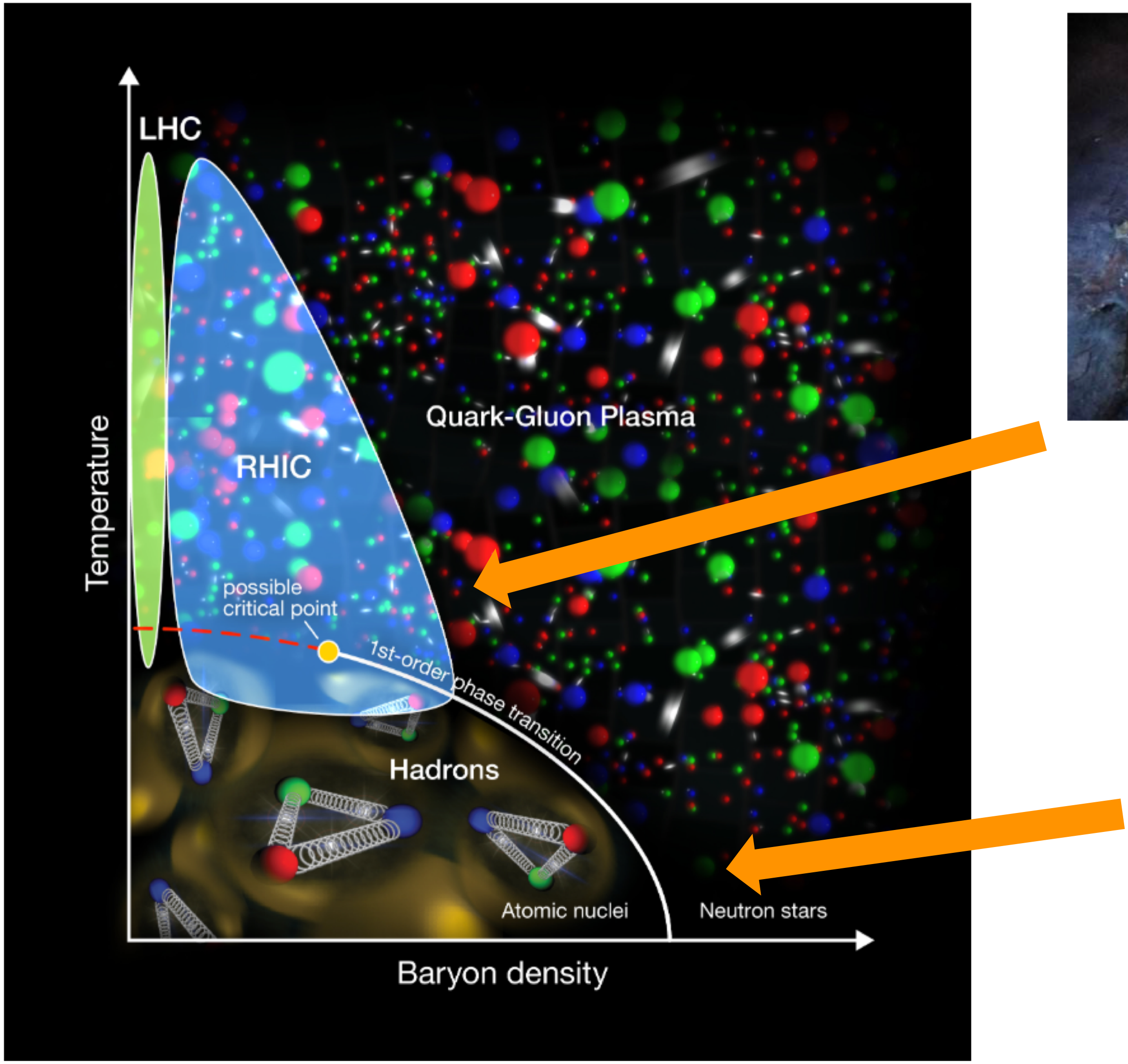
庄子天下篇 ~ 300 B.C.
一尺之棰，日取其半，万世不竭

Take half from a foot long stick each day,
You will never exhaust it in million years.

QCD

Quark pairs can be produced from vacuum
No free quark can be observed

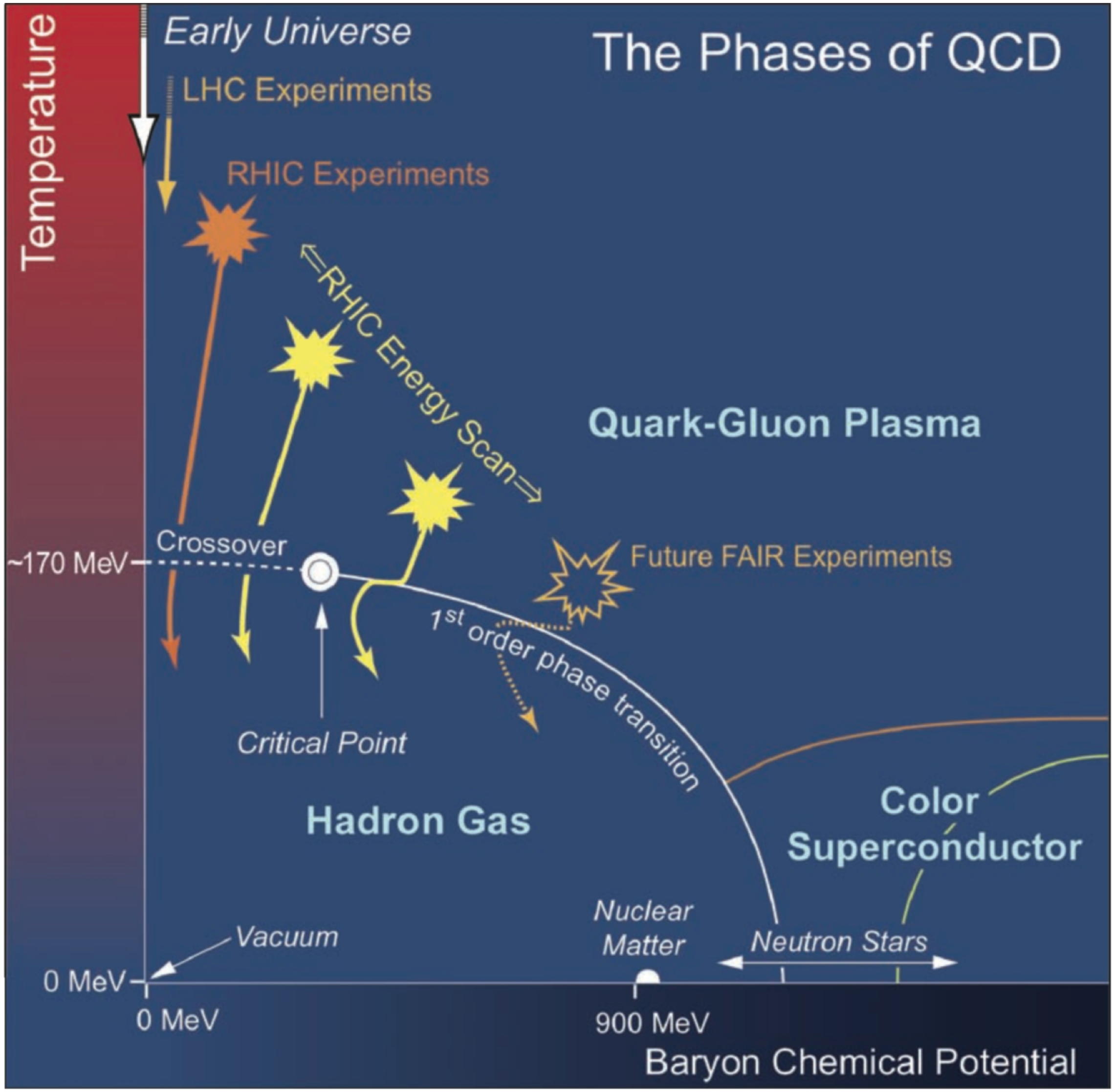
Deconfinement phase transition



High temperature

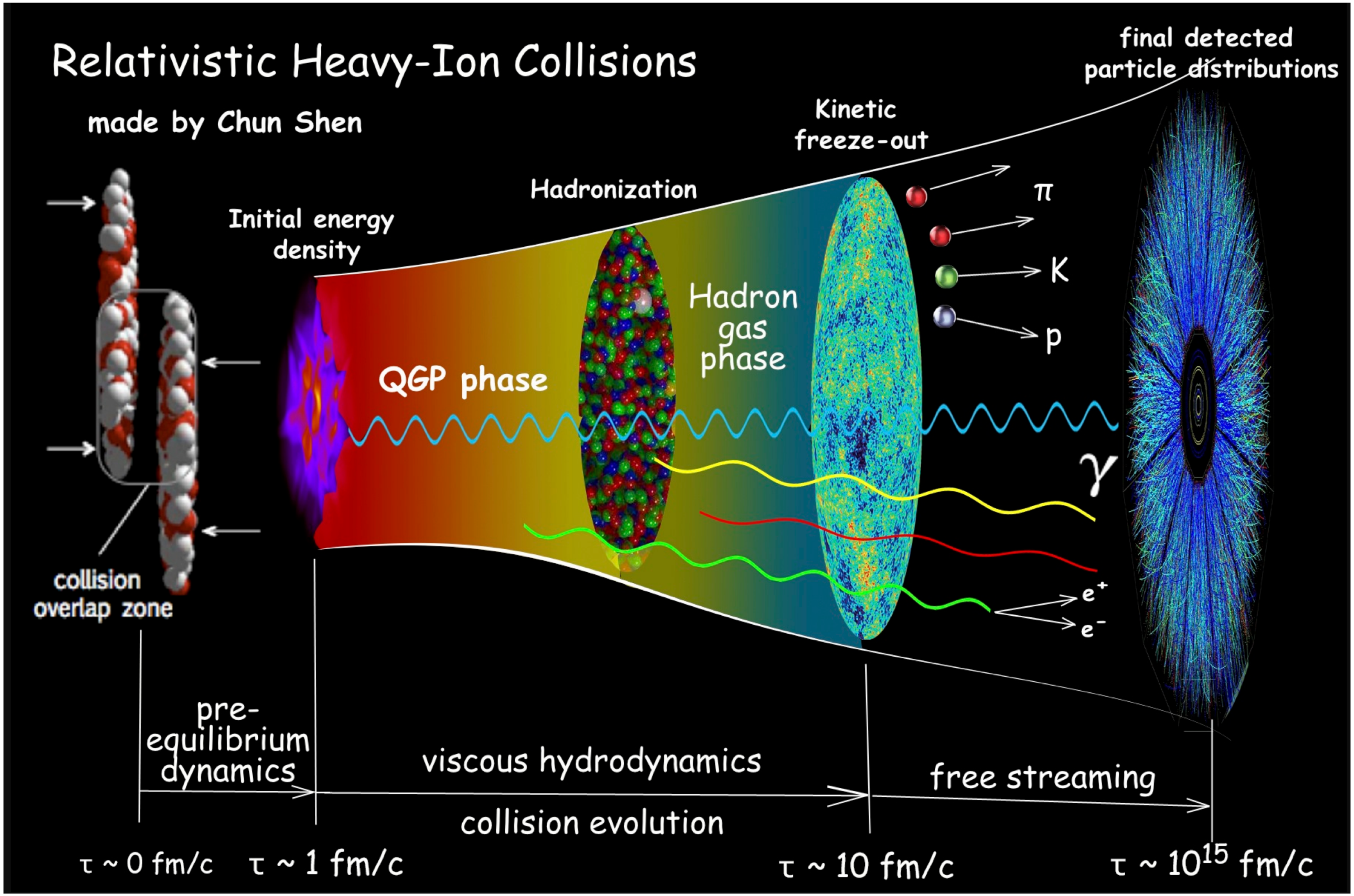
High pressure

Phases of QCD

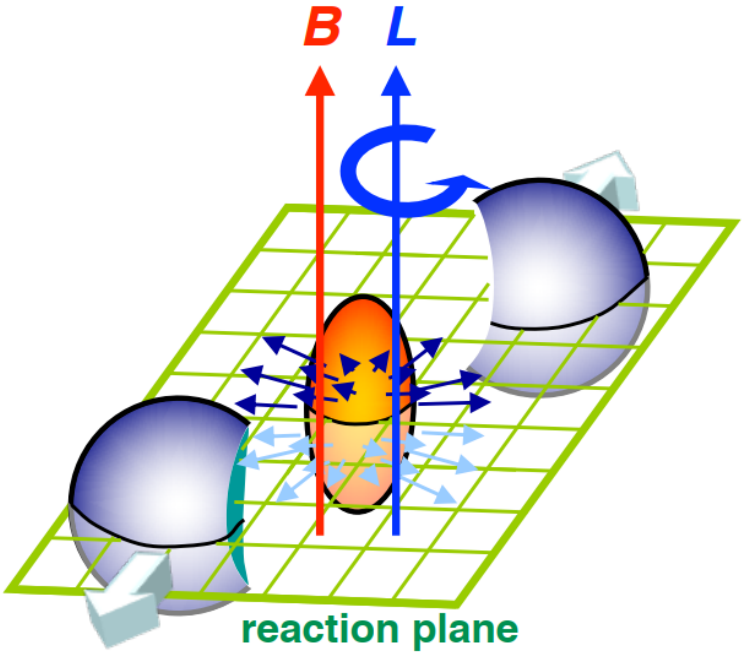


T.D. Lee (1974) and Collins (1975): Heavy ion collision to create a new form of matter!

Relativistic heavy ion collisions



Strong magnetic fields

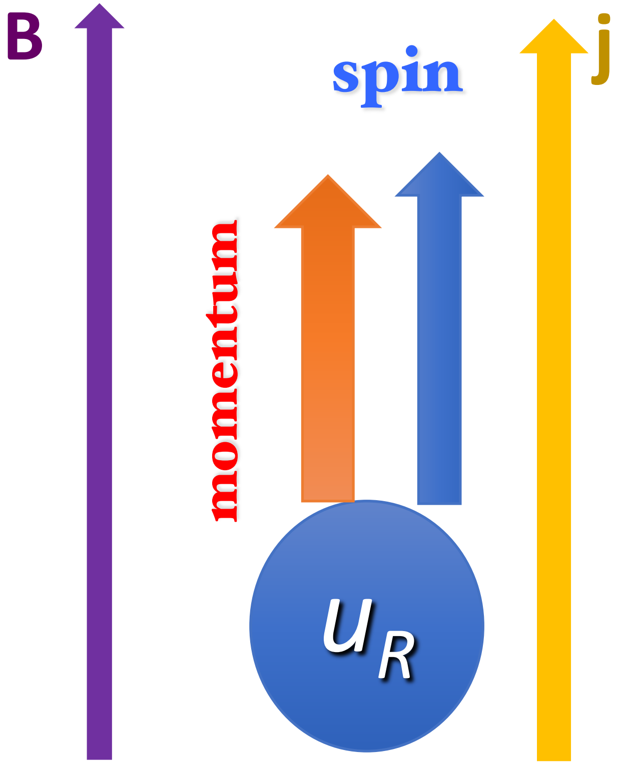


- **Very strong magnetic fields are produced**

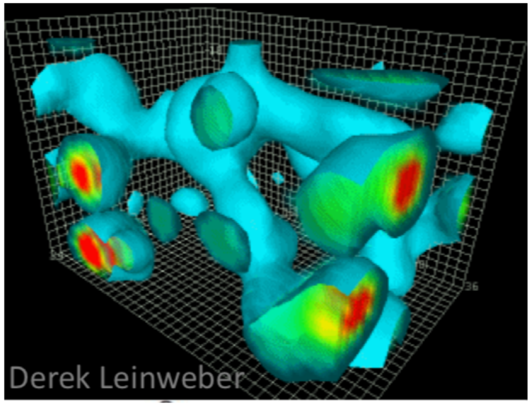
$$eB\sim10^{17}-10^{18}\text{Gauss}$$

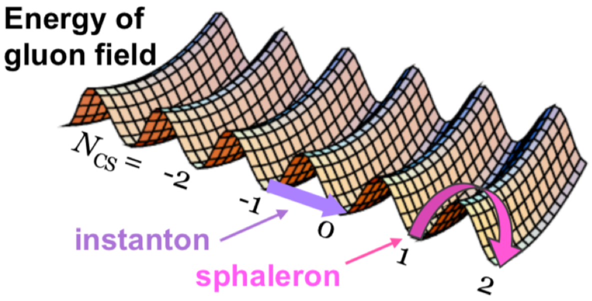
- **Novel quantum transport effects induced by magnetic fields**

Chiral Magnetic Effect



- **Magnetic fields**
- **Nonzero axial chemical potential**
- **Number of Left handed fermions ≠ Number of Right handed fermions**



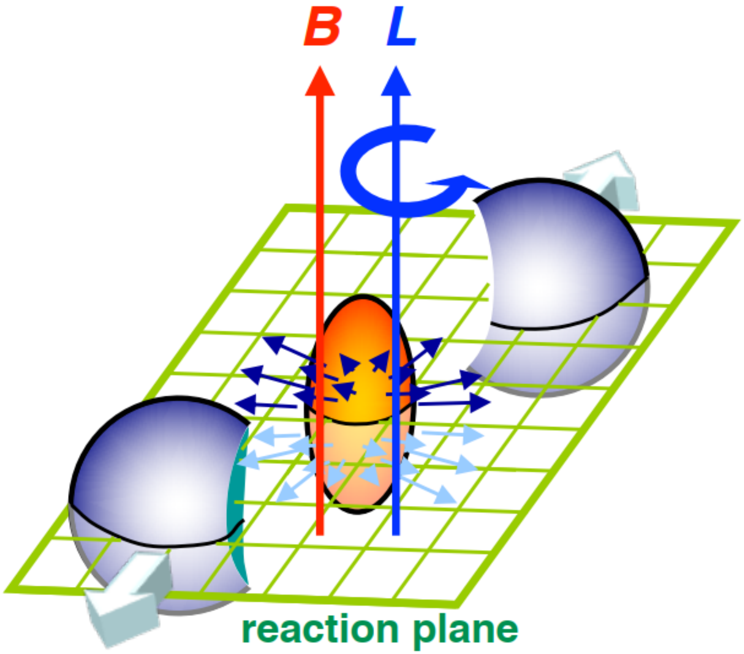


- **Charge current: charge separation**

$$\boxed{\mathbf{j} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B},}$$

Kharzeev, Fukushima, Warrigna, (08,09), etc. ...

Huge angular momentum

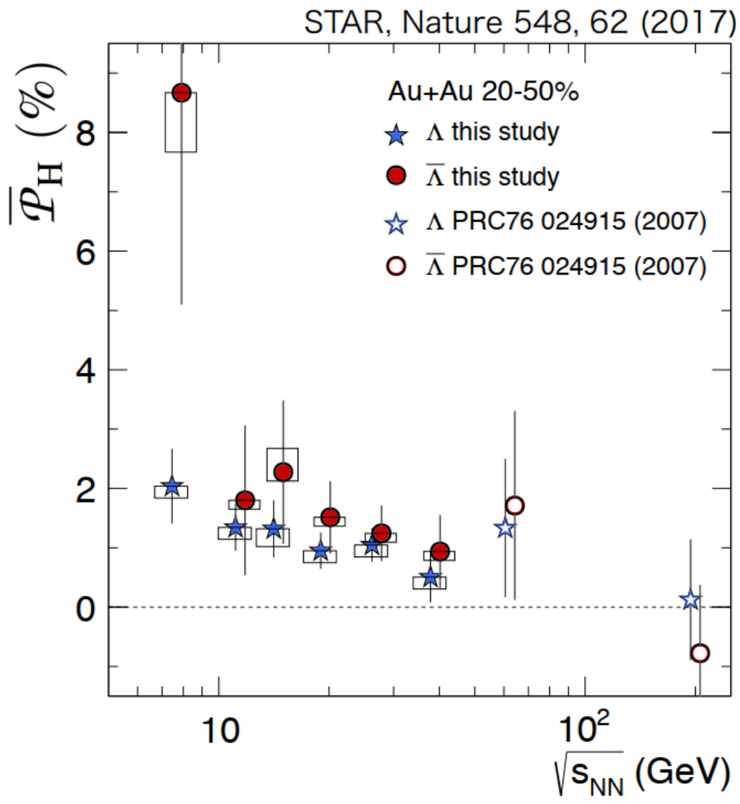


- **Huge global orbital angular momenta are produced**

$$L{\sim}10^5\hbar$$

- **How do orbital angular momenta be transferred to the matter created?**

Global Polarization of Λ and $\overline{\Lambda}$


STAR, Nature 548, 62 (2017)

- **$\sqrt{s_{NN}}$ < 62.4GeV , we observe the signal for polarization of Λ and $\overline{\Lambda}$**
- **The lower energy, the stronger polarization effects**
- **$P_{\overline{\Lambda}} > P_{\Lambda}$**

$$P_{\Lambda} \approx \frac{1}{2}\frac{\omega}{T} + \frac{\mu_{\Lambda} B}{T}$$
$$P_{\overline{\Lambda}} \approx \frac{1}{2}\frac{\omega}{T} - \frac{\mu_{\Lambda} B}{T}$$

ω = (9 ± 1)x10^{21}/s, greater than previously observed in any system.

Liang, Wang, PRL (2005)
Betz, Gyulassy, Torrieri, PRC (2007)
Becattini, Piccinini, Rizzo, PRC (2008)
Becattini, Karpenko, Lisa, Upsal, Voloshin, PRC (2017)
Fang, Pang, Q. Wang, X. Wang, PRC (2016)

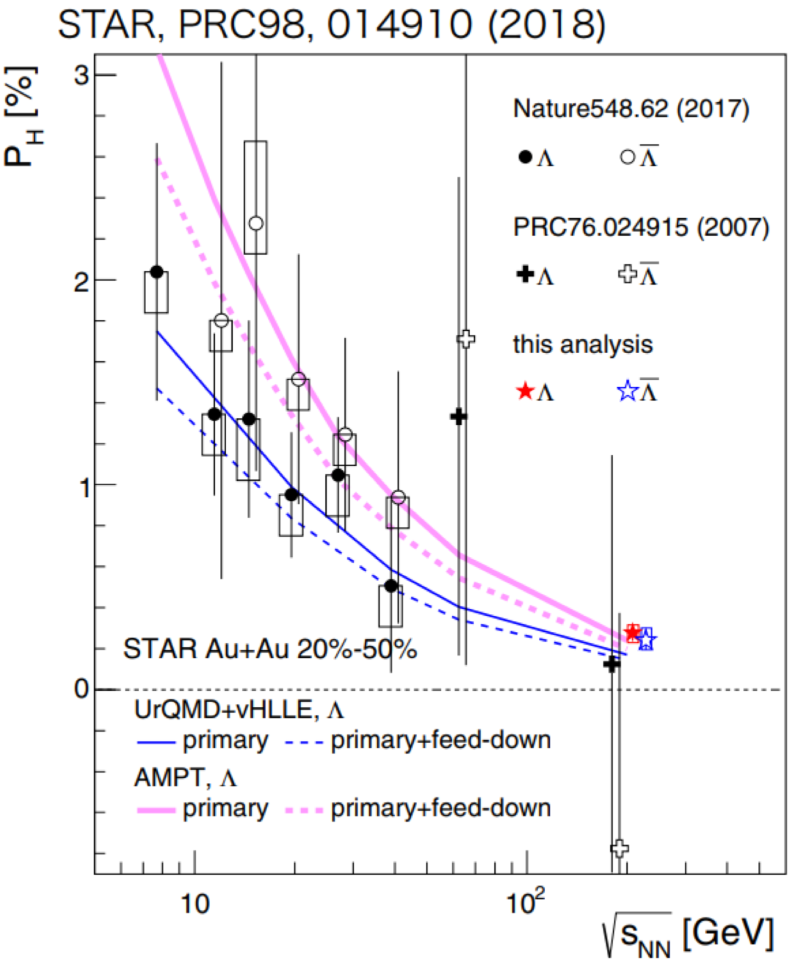
The fastest fluid

The Fastest Fluid

by Sylvia Morrow

Superhot material spins at an incredible rate.

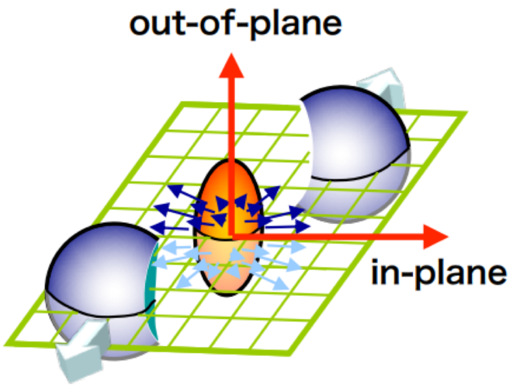
Comparison with simulation results

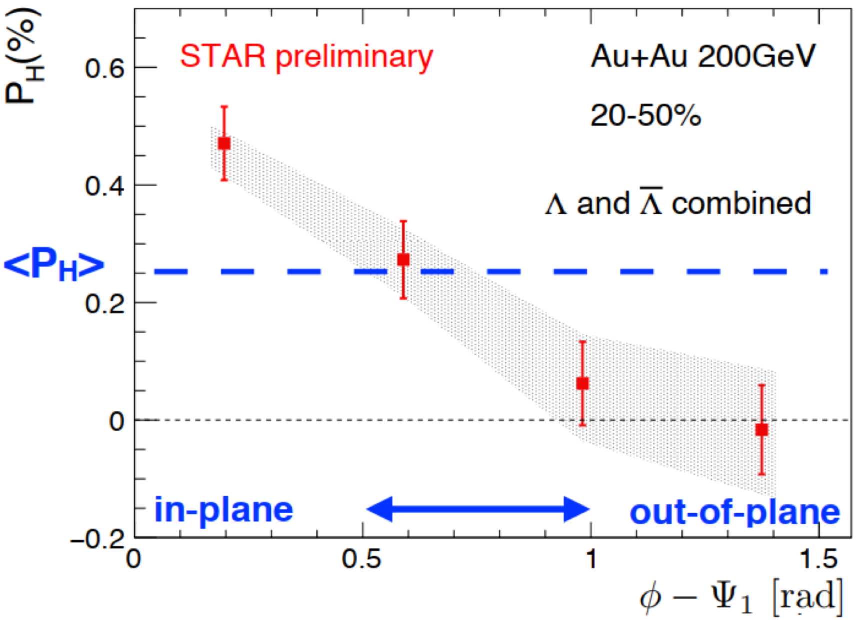


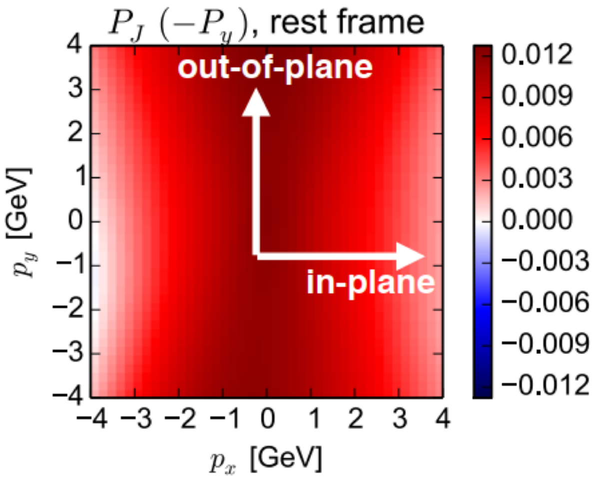
The results from both UrQMD+hydro and AMPT are consistent with the experimental data.

- ***UrQMD+vHLLE: Karpenko, Becattini, EPJC(2017)***
- ***AMPT: Li, Pang, Wang, Xia, PRC (2017)***

Local Polarization







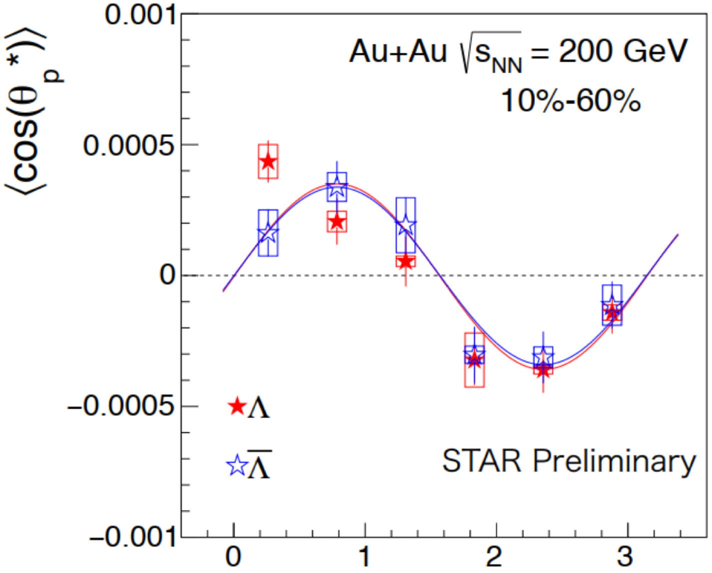
- **Exp data:**

  P_H in-plane > P_H out-of-plane

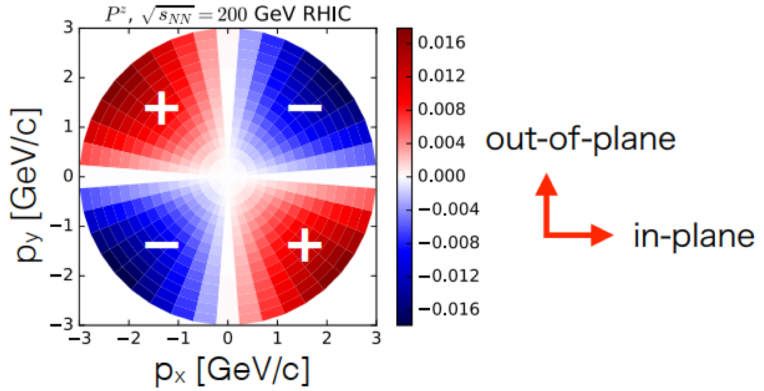
- **Simulations:**

  Sign is opposite of expected!

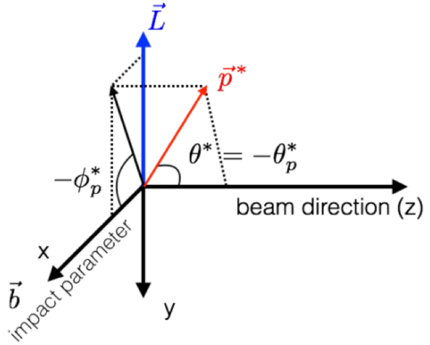
Local Polarization alone beam direction



Again, sign is opposite of expected!



UrQMD : *Becattini, Karpenko, PRL (2018)*



$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1+\alpha_{\rm H}\mathbf{P_H}\cdot\mathbf{p}_p^*)$$

$$\langle\cos\theta_p^*\rangle = \int \frac{dN}{d\Omega^*}\cos\theta_p^* d\Omega^*$$

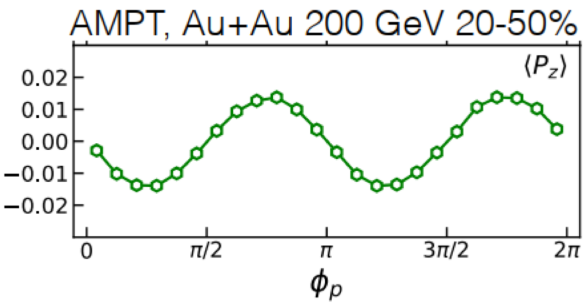
$$= \alpha_{\rm H} P_z\langle(\cos\theta_p^*)^2\rangle$$

$$\therefore\ P_z = \frac{\langle\cos\theta_p^*\rangle}{\alpha_{\rm H}\langle(\cos\theta_p^*)^2\rangle}$$

$$= \frac{3\langle\cos\theta_p^*\rangle}{\alpha_{\rm H}}$$ (if perfect detector)

α_H: hyperon decay parameter

θ_p^*: θ of daughter proton in Λ rest frame



AMPT: *Xia, Li, Tang, Wang, PRC (2018)*

Some possible ways to solve the puzzles

- **What kinds of vorticity do exp measure?**
- **The system does not reach to the global equilibrium**
- **Quantum kinetic theory with spin**
- **Spin hydrodynamics**

More discussions and references can be found in

Review:

J.H. Gao, G.L. Ma, SP, Q. Wang, Nucl.Sci.Tech 31(2020)9, 90

Spin hydrodynamics

- **Relativistic hydrodynamics + spin degree of freedom**

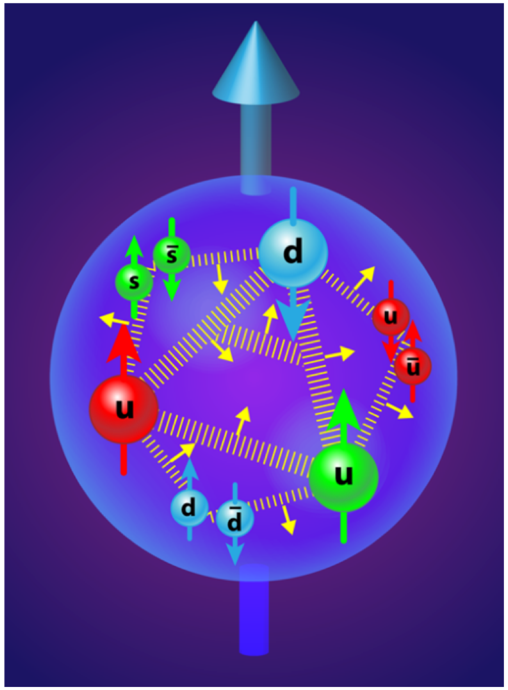
e.g. Anomalous hydrodynamics

(hydrodynamics for massless spin-1/2 fermions);

spin current is axial current j5

- **Problem: how to introduce spin of a massive fermionic fluid in a relativistic theory?**

Connection to “spin physics” (QCD)



- **Proton spin problem:**

(slides from Hatta-son’s talk)

$$\boxed{\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g}$$

Quarks’ helicity

Gluons’ helicity

Orbital angular Momentum (OAM)

Total angular momentum conservation

- **Nöther's theorem ：**

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu = (\delta^\mu_\nu + \epsilon^\mu_\nu) x^\nu \,, \quad A^\mu(x) \rightarrow A'^\mu(x) = \Lambda^\mu_\nu A^\nu(\Lambda^{-1}x),$$

$$\psi(x) \rightarrow \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x),$$

$$\Rightarrow \quad \boxed{\partial_\lambda (J_A^{\lambda\mu\nu} + J_\psi^{\lambda\mu\nu}) = 0}$$

- **Nöther current**

Gauge part

$$J_A^{\lambda\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A^\alpha)} \Delta A^{\mu\nu\alpha} = -F^\lambda_{\ \alpha}(x^\mu \partial^\nu - x^\nu \partial^\mu) A^\alpha - F^{\lambda\mu} A^\nu + F^{\lambda\nu} A^\mu \,.$$

Fermionic part

$$J_\psi^{\lambda\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi)} \Delta \psi^{\mu\nu} = \bar{\psi} i \gamma^\lambda (x^\mu \partial^\nu - x^\nu \partial^\mu - i\Sigma^{\mu\nu}) \psi$$

- **How to define the orbital and spin parts?**

Jaffe-Manohar (canonical) decomposition

- **Canonical energy momentum tensor**

Non-symmetric

Gauge part

$$T^{\mu\nu}_{A,\mathrm{can}} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\alpha)}\partial^\nu A^\alpha - g^{\mu\nu}\mathcal{L}_A = -F^\mu_{\ \alpha}\partial^\nu A^\alpha + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$

Fermionic part

$$T^{\mu\nu}_{\psi,\mathrm{can}} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial^\nu\psi - g^{\mu\nu}\mathcal{L}_\psi = \bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma^\alpha D_\alpha - m)\psi$$

- **Canonical decomposition**

$$J^{\mu\nu\lambda} = \underbrace{x^\nu T^{\mu\lambda}_{can} - x^\lambda T^{\mu\nu}_{can}}_{\text{Orbital angular momentum}} \underbrace{- \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\overline{\psi}\gamma_5\gamma_\rho\psi}_{\text{Quark helicity (spin)}} + \underbrace{F^{\mu\lambda}A^\nu - F^{\mu\nu}A^\lambda}_{\text{Gluon helicity (Spin)}}$$

- **Jaffe-Manohar decomposition (1990)**

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

Operators are not gauge invariant.

Pseudo-gauge transformation

- **The variation of (canonical) spin is the anti-symmetric part of energy momentum tensor**

$$0=\partial_\lambda J^{\lambda\mu\nu}=\partial_\lambda\left(x^\mu T^{\lambda\nu}_{\mathrm{can}}-x^\nu T^{\lambda\mu}_{\mathrm{can}}+S^{\lambda\mu\nu}_{\mathrm{can}}\right)\ \Rightarrow\ T^{\mu\nu}_{\mathrm{can}}-T^{\nu\mu}_{\mathrm{can}}=-\partial_\lambda S^{\lambda\mu\nu}_{\mathrm{can}}\,,$$

- **Belinfante energy momentum tensor**

$$\boxed{T^{\mu\nu}_{\mathrm{Bel}}\equiv T^{\mu\nu}_{\mathrm{can}}+\partial_\lambda K^{\lambda\mu\nu}_{\mathrm{Bel}}}$$

$$K^{\lambda\mu\nu}=-K^{\mu\lambda\nu}$$

$$K^{\lambda\mu\nu}_{\mathrm{Bel}}=\frac{1}{2}\left(S^{\lambda\mu\nu}_{\mathrm{can}}-S^{\mu\lambda\nu}_{\mathrm{can}}+S^{\nu\mu\lambda}_{\mathrm{can}}\right)$$

Conserved $\boxed{\partial_\mu T^{\mu\nu}_{\mathrm{Bel}}=0}$

$\boxed{T^{\mu\nu}_{\mathrm{Bel}}-T^{\nu\mu}_{\mathrm{Bel}}=0}$ **Symmetric**

$$T^{\mu\nu}_{\mathrm{A,Bel}}\equiv -F^{\mu}_{\ \alpha}F^{\nu\alpha}+\frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\,,$$

$$T^{\mu\nu}_{\psi,\mathrm{Bel}}\equiv\bar{\psi}i\gamma^\mu\overleftrightarrow{D}^\nu\psi+\frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda(\bar{\psi}\gamma_5\gamma_\rho\psi)\,.$$

Gauge invariant

Ji (Belinfante) decomposition

- **Belinfante improved total angular momentum**

$$J^{\lambda\mu\nu}_{\mathrm{Bel}} \equiv J^{\lambda\mu\nu} + \partial_\rho\left(x^\mu K^{\rho\lambda\nu}_{\mathrm{Bel}} - x^\nu K^{\rho\lambda\mu}_{\mathrm{Bel}}\right)$$

$$\partial_\lambda J^{\lambda\mu\nu} = 0 \quad \Longleftrightarrow \quad \partial_\lambda J^{\lambda\mu\nu}_{\mathrm{Bel}} = 0$$

$$\boxed{J^{\lambda\mu\nu}_{A/\psi,\mathrm{Bel}} = x^\mu \tilde{T}^{\lambda\nu}_{A/\psi,\mathrm{Bel}} - x^\nu \tilde{T}^{\lambda\mu}_{A/\psi,\mathrm{Bel}}}$$

- **Ji decomposition (1997)**

$$\frac{1}{2} = J_q + J_g$$

Table of two different forms

- **Canonical (Jaffe-Manohar) decomposition**

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E}\times\boldsymbol{A}}_{\Delta G} \underbrace{-i\psi^{\dagger}(\boldsymbol{x}\times\boldsymbol{\nabla})\psi}_{L^{q}_{\text{can}}} + \underbrace{\boldsymbol{E}(\boldsymbol{x}\times\boldsymbol{\nabla})\boldsymbol{A}}_{L^{g}_{\text{can}}}$$

$$\boxed{T^{\mu\nu}_{\psi,\text{can}} = \bar{\psi}i\gamma^{\mu}\partial^{\nu}\psi - g^{\mu\nu}\bar{\psi}(i\gamma^{\alpha}D_{\alpha} - m)\psi}$$

non-symmetric
Not gauge invariant

- **Belinfante (Ji) decomposition**

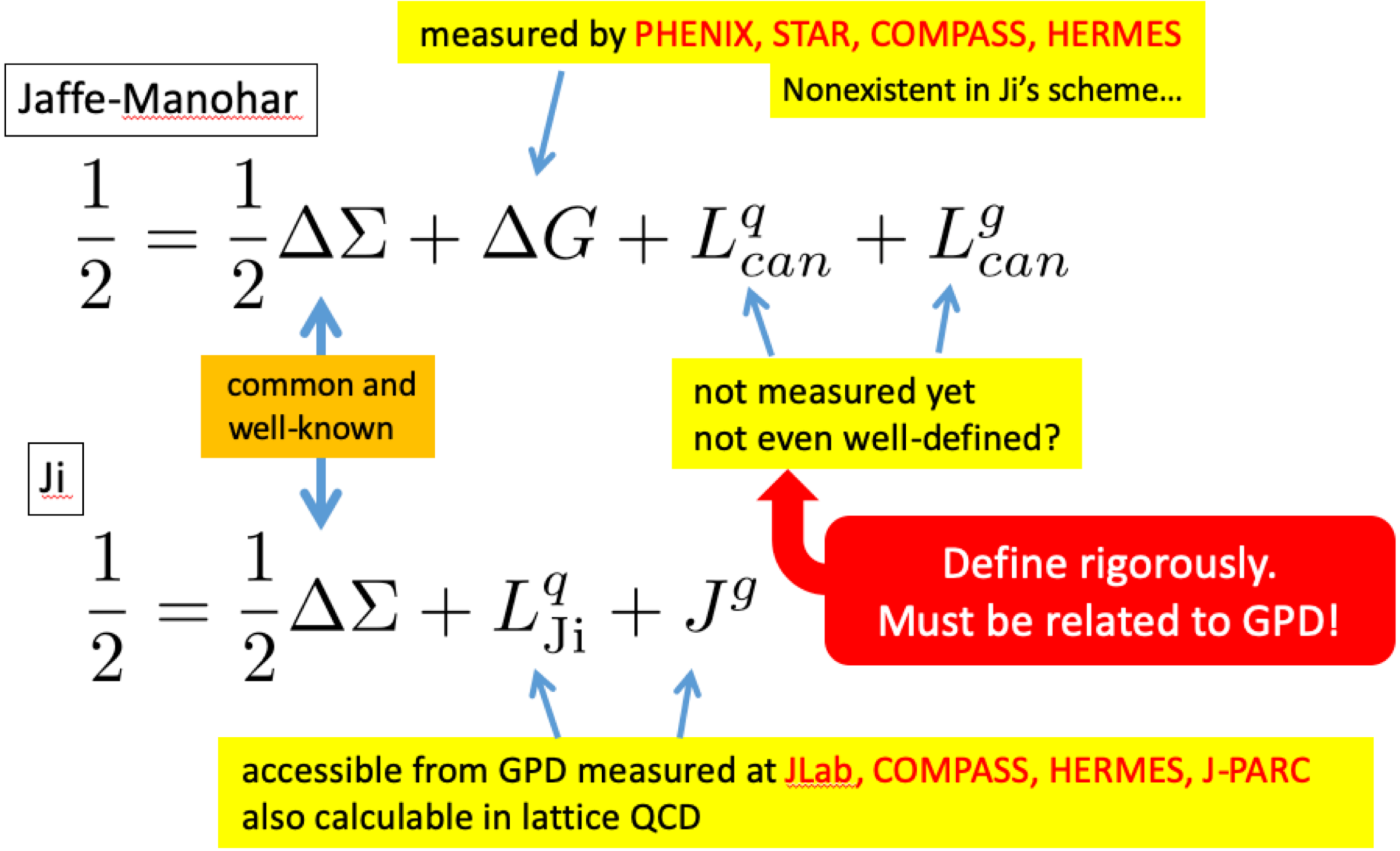
$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} \underbrace{-i\psi^{\dagger}(\boldsymbol{x}\times\boldsymbol{D})\psi}_{L^{q}_{\text{Ji}}} + \underbrace{\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})}_{J^{g}_{\text{Ji}}}$$

Connected by pseudo gauge transformation

$$\boxed{T^{\mu\nu}_{\psi,\text{Bel}} \equiv \bar{\psi}i\gamma^{\mu}\overleftrightarrow{D}^{\nu}\psi + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\partial_{\lambda}(\bar{\psi}\gamma_5\gamma_{\rho}\psi)}$$

Symmetric
Gauge invariant

Two spin communities divided



measured by PHENIX, STAR, COMPASS, HERMES

Nonexistent in Ji's scheme...

Jaffe-Manohar

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q_{can} + L^g_{can}$$

common and well-known

not measured yet
not even well-defined?

Ji

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + L^q_{\mathrm{Ji}} + J^g$$

Define rigorously.
Must be related to GPD!

accessible from GPD measured at JLab, COMPASS, HERMES, J-PARC
also calculable in lattice QCD

(slides from Hatta-san's talk)

E. Leader, C. Lorce, Phys. Rept. 541 (2014) 163-248

GLW decomposition

- **Another Pseudo gauge transformation**

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can} \right)$$

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can}$$

$$\Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$$

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x) + \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x).$$

$$T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm GLW} - \frac{1}{2}\partial_\lambda \left(S^{\nu,\lambda\mu}_{\rm GLW} + S^{\mu,\lambda\nu}_{\rm GLW} \right)$$

Textbook written by de Groot, van Leeuven, and van Weert

Review: ***W. Florkowski, R. Ryblewski and Avdhesh Kumar, Prog. Part. Nucl. Phys. 108 (2019) 103709***

- **Microscopic kinetic theory: GLW is the classical one.**

$$T^{\mu\nu}_{\rm GLW}(x) = \frac{1}{m}{\rm tr}_4 \int d^4k\, k^\mu\, k^\nu \mathcal{W}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu \mathcal{F}(x,k).$$

Question

- **Which kinds of energy momentum tensor are measured or preferred by the experiments?**

- **Hints:**
  - ✓ Ordinary relativistic hydrodynamics formulism are symmetric.
    (Relatively easy to be extended to spin hydro ?)
  - ✓ Anomalous (magneto-) hydrodynamics (including the spin current for massless fermions) are symmetric.
    (Relatively easy to be checked in massless limit)
  - ✓ Maybe, we eventually need to add the gluons' contributions.
    (A gauge invariant macroscopic theory may be more acceptable.)

Canonical form of spin hydrodynamics

Ref:

K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya,
"Fate of spin polarization in a relativistic fluid: An entropy-current analysis,"
Phys. Lett. B795 (2019) 100–106, arXiv:1901.06615 [hep-th].

Basic conservation equations

- **Total angular momentum conservation**

$$\partial_\alpha J^{\alpha\mu\nu}_{\mathrm{can}} = 0 \qquad J^{\alpha\mu\nu}_{\mathrm{can}} = \underbrace{x^\mu T^{\alpha\nu}_{can} - x^\nu T^{\alpha\mu}_{can}}_{\text{Orbital part}} + \underbrace{\Sigma^{\alpha\mu\nu}}_{\text{Spin tensor}}$$

$$\longrightarrow \quad \boxed{\partial_\alpha \Sigma^{\alpha\mu\nu} = T^{\nu\mu}_{can} - T^{\mu\nu}_{can}} \qquad \checkmark$$

- **Energy-momentum conservation**

$$\boxed{\partial_\mu T^{\mu\nu}_{can} = 0} \qquad \checkmark$$

- **Currents conservation**

$$\boxed{\partial_\mu j^\mu = 0} \qquad \checkmark$$

Common strategy for derivation of fluid equations

- **Tensor decomposition:**
  - Parallel or perpendicular to fluid velocity u^μ
  - Traceless part and other part

- **Gradient (∂) expansion:** $\partial X \ll X$

- **Entropy principle:**

  to derive the general expression for all components of tensors

An example: charge currents

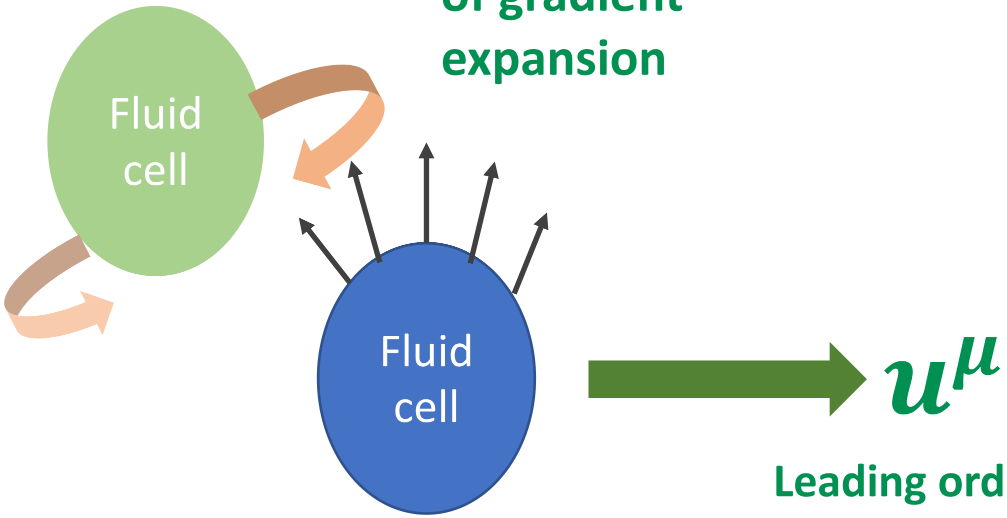
- **Charge currents:**

n: charge density

$$j^{\mu} = nu^{\mu} + j^{\mu}_{(1)}$$

Parallel to fluid velocity u^{μ}; Leading order of gradient expansion

Perpendicular to fluid velocity u^{μ}; Higher orders of gradient expansion



Higher orders: exchange the heat and particles with other cells.

Leading order: moving alone the u^{μ} in average

Spin tensor decomposition

- **Analogy to the decomposition for currents:**

$$j^\mu = \underbrace{n u^\mu}_{\text{Parallel to fluid velocity } u^\mu\text{; Leading order}} + \underbrace{j^\mu_{(1)}}_{\text{Perpendicular to fluid velocity } u^\mu\text{; Higher order}}$$

Parallel to fluid velocity u^μ;
Leading order

Perpendicular to fluid velocity u^μ;
Higher order

- **One can assume that**

Spin density

$$\Sigma^{\alpha\mu\nu} = u^\alpha S^{\mu\nu} + \Sigma^{\alpha\mu\nu}_{(1)}$$

spin tensor

Parallel to fluid velocity u^μ;
Leading order

Perpendicular to fluid velocity u^μ;
Higher order

Modified thermodynamic relations

- **Density vs Chemical potential**

Charge density: n ⟷ **Charge chemical potential: μ**

Spin density: $S^{\mu\nu}$ ⟷ **Spin chemical potential: $\omega^{\mu\nu}$**

Physical interpretation:
Variation of total energy when we add one particle with spin $S^{\mu\nu}$

- **Thermodynamic relations**

$$e + p = Ts + \mu n + \omega_{\mu\nu} S^{\mu\nu}$$

energy density · pressure · temperature X entropy density · spin chemical potential X spin density

- **Gibbs relations**

$$de = Tds + \mu dn + \omega_{\mu\nu} dS^{\mu\nu} \qquad dp = sdT + nd\mu + S^{\mu\nu} d\omega_{\mu\nu}$$

Orders of $S^{\mu\nu}$ and $\omega^{\mu\nu}$

- In Ref. *K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya, Phys. Lett. B795 (2019) 100–106.*

$$\boxed{S^{\mu\nu}, \omega^{\mu\nu} \sim O\left(\partial^1\right);\ \omega_{\mu\nu}S^{\mu\nu} \sim O\left(\partial^2\right)}$$

- In our recent work, *K. Fukushima, SP, 2010.01608*

$$\boxed{S^{\mu\nu} \sim O(1), \omega^{\mu\nu} \sim O\left(\partial^1\right);\ \omega_{\mu\nu}S^{\mu\nu} \sim O\left(\partial^1\right)}$$

Density is classic $O(1)$, but the variation of energy is quantum $O\left(\partial^1\right)$!

- We only consider the spin hydro up to the first order in gradient expansion.

Entropy production rate

- **Two ways to derive the entropy flow:**
  - Directly using

$$u_\nu \partial_\mu T^{\mu\nu}_{can} + \mu \partial_\mu j^\mu = 0$$

+ Gibbs relation

→ $\partial_\mu \mathcal{S}^\mu_{can} \geq 0$

  - Using the extended entropy flow

W. Israel, J. Stewart, Annals Phys. 118, 341 (1979)

Relativistic fluid generation of Gibbs relation

$$\begin{aligned}\mathcal{S}^\mu_{\rm can} &= \frac{u_\nu}{T}\Theta^{\mu\nu} + \frac{p}{T}u^\mu - \frac{\mu}{T}j^\mu - \frac{1}{T}\omega_{\rho\sigma}S^{\rho\sigma}u^\mu + \mathcal{O}(\partial^2) \\ &= su^\mu + \frac{u_\nu}{T}\Theta^{\mu\nu}_{(1)} - \frac{\mu}{T}j^\mu_{(1)} + \mathcal{O}(\partial^2)\,,\end{aligned}$$

→ $\partial_\mu \mathcal{S}^\mu_{can} \geq 0$

Constrains from entropy principle

$$T^{\mu\nu}_{can} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$$

Leading order ($T^{\mu\nu}_{(0)}$)

Next-to-Leading order ($T^{\mu\nu}_{(1)}$)

$$\begin{aligned} T^{\mu\nu}_{(1s)} &= T^{\mu\nu}_{(1)} + T^{\nu\mu}_{(1)} \\ T^{\mu\nu}_{(1a)} &= T^{\mu\nu}_{(1)} - T^{\nu\mu}_{(1)} \end{aligned}$$

symmetric

anti-symmetric

$$\partial_\mu S^\mu_{can} = \underbrace{T^{\mu\nu}_{(1s)}\partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)}\partial_\mu \frac{\mu}{T}}_{\text{Ordinary terms not related to spin}} + \frac{2T^{\mu\nu}_{(1a)}}{T}\left(\omega_{\mu\nu} + \frac{1}{2}T\omega^{th}_{\mu\nu}\right) \geq 0$$

Thermal vorticity

$$\begin{aligned} \omega^{\mu\nu}_{th} &= (g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta) \\ &\quad \times[\partial_\alpha(u_\beta/T) - \partial_\beta(u_\alpha/T)] \end{aligned}$$

Non-relativistic limit →

$$\epsilon^{ijk}\omega^{ij}_{th} \sim (\nabla \times \frac{\mathbf{v}}{T})^k$$

Global equilibrium

$$\omega^{\mu\nu}_{th} = (g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta) \times[\partial_\alpha(u_\beta/T) - \partial_\beta(u_\alpha/T)]$$

$$\partial_\mu S^\mu_{can} = \overbrace{T^{\mu\nu}_{(1s)}\partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)}\partial_\mu \frac{\mu}{T}}^{\text{Ordinary terms not related to spin}} + \frac{2T^{\mu\nu}_{(1a)}}{T}\underbrace{\left(\omega_{\mu\nu} + \frac{1}{2}T\overset{\substack{\text{Thermal vorticity}\\ \downarrow}}{\omega^{th}_{\mu\nu}}\right)}_{\text{Must vanish!}} = 0$$

Widely proved by many approaches:

F. Becattini, L. Bucciantini, E. Grossi, and L. Tinti, Eur. Phys. J. C 75, 191 (2015)

F. Becattini, W. Florkowski, and E. Speranza, Physics Letters B 789, 419 (2019)

K. Hattori, M. Hongo, X.-G.Huang, M. Matsuo, H. Taya, Phys. Lett. B795 (2019) 100–106.

…

Also see recent reviews:

Y. C. Liu and X. G. Huang, Nucl. Sci. Tech. 31,56 (2020)

J.H. Gao, G.L. Ma, SP, Q. Wang, Nucl. Sci. Tech 31 (2020) 9, 90

Spin chemical potential $\omega^{\mu\nu}$ must be related to thermal vorticity as $-T\omega^{th}_{\mu\nu}/2$ in global equilibrium!

Local equilibrium

$$\partial_\mu \mathcal{S}^\mu_{can} = T^{\mu\nu}_{(1s)} \partial_\mu \frac{u_\nu}{T} - j^\mu_{(1)} \partial_\mu \frac{\mu}{T} + \frac{2T^{\mu\nu}_{(1a)}}{T} \left(\omega_{\mu\nu} + \frac{1}{2} T \omega^{th}_{\mu\nu} \right) \geq 0$$

- **Tensor decomposition of energy momentum tensor**

symmetric

$$T^{\mu\nu}_{(1s)} = \underset{\substack{\uparrow \\ \text{heat flow}}}{h^\mu u^\nu} + h^\nu u^\mu + \underset{\substack{\uparrow \\ \text{viscous tensor}}}{\pi^{\mu\nu}}$$

anti-symmetric

$$T^{\mu\nu}_{(1a)} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}$$

$$\left.\begin{aligned} \partial_\mu \mathcal{S}^\mu_{can} = {} & (h^\mu - \frac{e+p}{n} j^\mu_{(1)}) \frac{n}{e+p} (g_{\nu\alpha} - u_\nu u_\alpha) \partial^\nu \frac{\mu}{T} \\ & + \frac{\pi^{\mu\nu}}{T} \partial_{<\mu} u_{\nu>} + \frac{1}{3} \frac{1}{T} \pi^\mu_\mu (\partial \cdot u) \end{aligned}\right\} \text{dissipative terms in ordinary fluids}$$

$$\left.\begin{aligned} & + q^\mu [- \frac{1}{T} (u \cdot \partial) u_\mu + \partial_\mu \frac{1}{T} + 4 \frac{\omega_{\mu\nu} u^\nu}{T}] \\ & + \phi^{\mu\nu} [\omega^{th}_{\mu\nu} + 2\beta \omega_{\mu\nu}] \geq 0 \end{aligned}\right\} \text{new terms related to spin}$$

Entropy principle

$$\left.\begin{aligned}\partial_\mu \mathcal{S}^\mu_{can} = {} & (h^\mu - \frac{e+p}{n} j^\mu_{(1)}) \frac{n}{e+p} (g_{\nu\alpha} - u_\nu u_\alpha) \partial^\nu \frac{\mu}{T} \\ & + \frac{\pi^{\mu\nu}}{T} \partial_{<\mu} u_{\nu>} + \frac{1}{3} \frac{1}{T} \pi^\mu_\mu (\partial \cdot u)\end{aligned}\right\} \text{dissipative terms in ordinary fluids}$$

$$\left.\begin{aligned} & + q^\mu [-\frac{1}{T} (u \cdot \partial) u_\mu + \partial_\mu \frac{1}{T} + 4 \frac{\omega_{\mu\nu} u^\nu}{T}] \\ & + \phi^{\mu\nu} [\omega^{th}_{\mu\nu} + 2\beta \omega_{\mu\nu}] \geq 0\end{aligned}\right\} \text{new terms related to spin}$$

- **To ensure the entropy production rate be always positive, the only possible way is**

$$\begin{aligned} q^\mu &= \lambda[(u \cdot \partial) u^\mu + \frac{1}{T} \Delta^{\mu\nu} \partial_\nu T - 4 \omega^{\mu\nu} u_\nu], \\ \phi^{\mu\nu} &= 2\gamma [T \omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta) \omega_{\alpha\beta}]/T. \end{aligned}$$

$\lambda, \gamma \geq 0$ are new transport coefficients

Brief summary of canonical form

- **Energy momentum tensor has anti-symmetric part**

$$T^{\mu\nu}_{can} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$$

Leading order

Next-to-Leading order

$$T^{\mu\nu}_{(1s)} = T^{\mu\nu}_{(1)} + T^{\nu\mu}_{(1)}$$

symmetric

$$T^{\mu\nu}_{(1a)} = T^{\mu\nu}_{(1)} - T^{\nu\mu}_{(1)}$$

anti-symmetric

- **In global equilibrium, spin chemical potential is related to thermal vorticity**

$$\omega^{\mu\nu} = -\frac{1}{2} T \omega^{\mu\nu}_{th}$$

Power counting: $\omega^{th}_{\mu\nu} \sim O(\partial^1)$

➡ $\omega_{\mu\nu} \sim O(\partial^1)$

- **Symmetric part is as the same as the ordinary fluid. The expression for anti-symmetric part can be derived by entropy principle.**

$$T^{\mu\nu}_{(1a)} = q^\mu u^\nu - q^\nu u^\mu + \phi^{\mu\nu}$$

$$q^\mu = \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu],$$

$$\phi^{\mu\nu} = 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T.$$

Power counting: $T^{\mu\nu}_{(1a)} \sim O(\partial^1)$

$\partial_\rho(u^\rho S^{\mu\nu}) = -2T^{\mu\nu}_{(1a)} \sim O(\partial^1)$

➡ $S^{\mu\nu} \sim O(1)$

$\lambda, \gamma \geq 0$ **are new transport coefficients**

Belinfante form of spin hydrodynamics

Ref.

K. Fukushima, SP,

"Spin Hydrodynamics and Symmetric Energy - Momentum Tensors

– A current induced by the spin vorticity", 2010.01608

Basic conservation equations

- **Total angular momentum conservation**

$$\begin{aligned} J_{\mathrm{Bel}}^{\mu\nu\alpha} &= J^{\mu\nu\alpha} + \partial_\rho (x^\nu K_{\mathrm{Bel}}^{\rho\mu\alpha} - x^\alpha K_{\mathrm{Bel}}^{\rho\mu\nu}), \\ &= x^\nu T_{\mathrm{Bel}}^{\mu\alpha} - x^\alpha T_{\mathrm{Bel}}^{\mu\nu}. \end{aligned}$$

$$\boxed{\partial_\mu J_{\mathrm{Bel}}^{\mu\nu\alpha} \quad = \quad 0,}$$

These two equations are equivalent!

- **Energy momentum conservation**

✓ $$\boxed{\partial_\mu T_{\mathrm{Bel}}^{\mu\nu} \quad = \quad 0.}$$

- **Current conservation**

✓ $$\boxed{\partial_\mu j^\mu = 0}$$

No spin in Belinfante form?

$$\boxed{\partial_\mu J_{\text{Bel}}^{\mu\nu\alpha} = 0,} \quad \xleftrightarrow{\text{equivalent}} \quad \boxed{\partial_\mu T_{\text{Bel}}^{\mu\nu} = 0.}$$

The common argument:

- There is no equation for spin.
- There is no degree of freedom for spin.
- We cannot observe spin effect in Belinfante form.

Recalling what we discussed in introduction.

- Belinfante energy momentum tensor is connected to canonical one by pseudo gauge transformation.

If a physical (spin) effect disappears after a physical transformation, then, did that mean this "physical effect" is unphysical? Or, should it be that this physical effect will appear somewhere?

Belinfante energy momentum tensor

- **We take the pseudo gauge transformation**

$$T^{\mu\nu}_{Bel} = T^{\mu\nu}_{can} + \partial_\lambda K^{\lambda\mu\nu}_{Bel} = T^{\mu\nu}_0 + T^{\mu\nu}_{(1)}$$

Leading order

Next-to-Leading order

$$T^{\mu\nu}_{(1)} = h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})$$

heat flow

viscous tensor

spin density tensor

spin corrections to the energy momentum tensor

Spin corrections

$$T^{\mu\nu}_{(1)} = h^{\mu}u^{\nu} + h^{\nu}u^{\mu} + \pi^{\mu\nu} + \frac{1}{2}\partial_{\lambda}(u^{\mu}S^{\nu\lambda} + u^{\nu}S^{\mu\lambda})$$

heat flow **viscous tensor** **spin density tensor**

spin corrections to the energy momentum tensor

Using standard tensor decomposition, we have

$$T^{\mu\nu}_{(1)} = (\delta e_s)u^{\mu}u^{\nu} + (h^{\mu} + \delta h^{\mu}_{s})u^{\nu} + (h^{\nu} + \delta h^{\nu}_{s})u^{\mu} + \pi^{\mu\nu} + \delta\pi^{\mu\nu}_{s}$$

$$\delta e_s = u_{\rho}\partial_{\sigma}S^{\rho\sigma},$$

↔ **Spin correction to energy density**

$$\delta h^{\mu}_{s} = \frac{1}{2}\Delta^{\mu}_{\sigma}\partial_{\lambda}S^{\sigma\lambda} + \frac{1}{2}u_{\rho}S^{\rho\lambda}\partial_{\lambda}u^{\mu}$$

↔ **Spin correction to heat flow**

$$\delta\pi^{\mu\nu}_{s} = \partial_{\lambda}(u^{<\mu}S^{\nu>\lambda}) + \frac{1}{3}\partial_{\lambda}(u^{\sigma}S^{\rho\lambda})\Delta_{\rho\sigma}.$$

↔ **Spin correction to viscous tensor**

Spin will appear as corrections to the ordinary dissipative terms!

Frame dependence

- In ordinary relativistic fluid, we have Landau (energy) frame and Particle frame.
- The heat flow depends on frame. In Landau frame, there is no heat flow,

$$u_L^\mu = u^\mu + \frac{1}{e+p}(h^\mu + \delta h^\mu),$$

but the dissipative current will be modified by spin correction!

$$j_{\mathrm{L}(1)}^\mu = \left(j_{(1)}^\mu - \frac{n}{e+p} h^\mu \right) + \delta j_{(1)}^\mu \qquad \delta j_{(1)}^\mu = -\frac{n}{e+p}\delta h^\mu .$$

Non-relativistic limit →

$$\delta \boldsymbol{j}_{(1)} = -\frac{n}{2(e+p)}\left[\boldsymbol{\nabla}\times\boldsymbol{S} + \dot{\boldsymbol{v}}\times\boldsymbol{S} + (\boldsymbol{\nabla}\cdot\boldsymbol{v})\mathfrak{s} - 2(\mathfrak{s}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \dot{\mathfrak{s}}\right].$$

Quantum spin vorticity

- **We have derived**

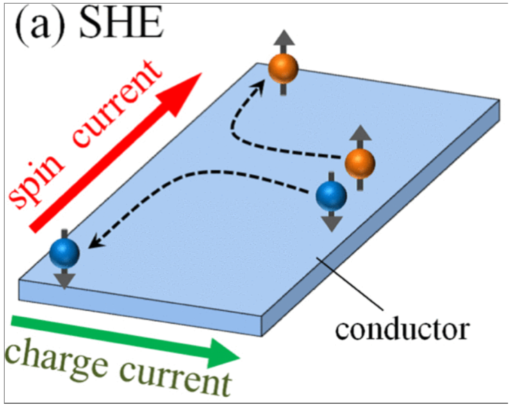
$$\delta \mathbf{j}_{(1)} \propto -(\nabla \times \mathbf{S})$$

$S^i = \epsilon^{ijk} S^{ij}$

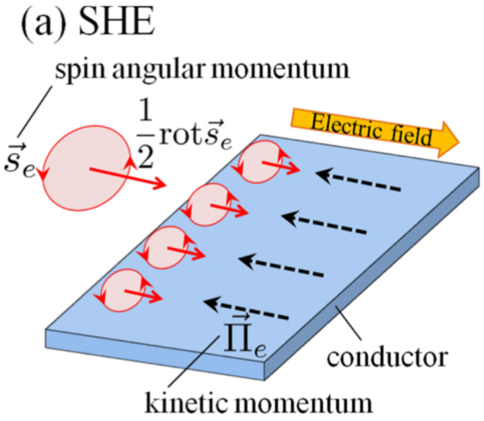
spin density alone

i-th direction

Curl of spin will induce a current



Standard Spin Hall Effect (SHE)



Spin Hall Effect (SHE) understood by quantum spin vorticity

M. Fukuda, K. Ichikawa, M. Senami, and A. Tachibana, AIP Advances 6, 025108 (2016).

Entropy principle (1)

- **Using the same method, we get the entropy production rate**

$$\partial_\mu \mathcal{S}^\mu = \left(\frac{n}{e+p}h^\mu - j^\mu_{(1)}\right)\Delta_{\mu\nu}\partial^\nu \frac{\mu}{T} + \frac{1}{T}\pi^{\mu\nu}\partial_\mu u_\nu + \Delta$$

Spin corrections

$$\Delta \equiv \frac{1}{2}\left[\partial_\lambda (u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]\partial_\mu \frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda (u^\lambda S^{\rho\sigma}) .$$

It is not in a squared form at all!

- **The simplest way to ensure entropy principle is to let**

$$\boldsymbol{S^{\mu\nu} = 0, \Delta = 0}$$

???

That is the way to "get" the common argument "No degree of freedom for spin in Belinfante form".

Of course, it is a (trivial) solution. But, is it the only solution?

Entropy principle (2)

- **Using the same method, we get the entropy production rate**

$$\partial_\mu \mathcal{S}^\mu = \left(\frac{n}{e+p}h^\mu - j^\mu_{(1)}\right)\Delta_{\mu\nu}\partial^\nu\frac{\mu}{T} + \frac{1}{T}\pi^{\mu\nu}\partial_\mu u_\nu + \Delta$$

Spin corrections

$$\Delta \equiv \frac{1}{2}\left[\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\right]\partial_\mu\frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma}) .$$

It is not in a squared form at all!

$$\Longrightarrow \quad \begin{aligned} \Delta = & \boxed{\frac{1}{2}\partial_\mu\left[\partial_\lambda(u^\lambda S^{\mu\nu} + u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})\frac{u_\nu}{T}\right]} \Longrightarrow \partial_\mu \delta\mathcal{S}^\mu \\ & - \frac{1}{2}\left[\partial_\lambda(u^\lambda S^{\mu\nu})\right]\partial_\mu\frac{u_\nu}{T} - \frac{\omega_{\rho\sigma}}{T}\partial_\lambda(u^\lambda S^{\rho\sigma}) . \end{aligned}$$

- **We move the total derivatives to the entropy flow (redefine the entropy flow)**

$$\partial_\mu(\mathcal{S}^\mu + \delta\mathcal{S}^\mu) = \ldots + \Delta' \geq 0$$

Similar to the anomalous fluid ***D.T. Son, P. Surowka, PRL. 103, 191601 (2009).***

Entropy principle (3)

$$\partial_\mu(\mathcal{S}^\mu + \delta\mathcal{S}^\mu) = \ldots + \Delta' \geq 0 \quad \Delta' = -\partial_\lambda(u^\lambda S^{\mu\nu})\left(\frac{1}{2}\partial_\mu \frac{u_\nu}{T} + \frac{\omega_{\mu\nu}}{T}\right)$$

It reproduces the results in canonical form!

- **In global equilibrium, $\omega^{\mu\nu} = -T\omega^{th}_{\mu\nu}/2$.**
- **In local equilibrium, by the tensor decomposition,**

$$\partial_\lambda(u^\lambda S^{\rho\sigma}) = q^\rho u^\sigma - q^\sigma u^\rho + \phi^{\rho\sigma}$$

we can get the same results as in canonical form.

$$\begin{aligned} q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\ \phi^{\mu\nu} &= 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T. \end{aligned}$$

We have re-discovered the equation of motion for spin by entropy principle!

Main equations for Belinfante form

- **Energy momentum conservation**

$$\boxed{\partial_\mu T^{\mu\nu}_{\text{Bel}} \quad = \quad 0.} \xleftrightarrow{\text{equivalent}} \boxed{\partial_\mu J^{\mu\nu\alpha}_{\text{Bel}} \quad = \quad 0,}$$

- **Current conservation**

$$\boxed{\partial_\mu j^\mu = 0}$$

- **Equations from entropy principle**

$$\partial_\lambda(u^\lambda S^{\rho\sigma}) = q^\rho u^\sigma - q^\sigma u^\rho + \phi^{\rho\sigma}$$

$$\begin{aligned} q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\ \phi^{\mu\nu} &= 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T. \end{aligned}$$

- **Number of equations: 4+1+6= 11**
- **Variables: $T, \mu, S^{\mu\nu}, u^i$, 1+1+6+3=11**

- **Equation of state: $e = e(T, \mu, S^{\mu\nu})$ + Gibbs relation** $\omega^{\mu\nu} = \left.\frac{de}{dS^{\mu\nu}}\right|_{n,s}$

Puzzles, summary and outlook

Recalling Belinfante energy momentum tensor

$T_{Bel}^{\mu\nu}$ is symmetric, gauge invariant (with gluons).

✓ Ordinary relativistic hydrodynamics formulism are symmetric.
(Relatively easy to be extended to spin hydro ?)

✓ Anomalous (magneto-) hydrodynamics (including the spin current for massless fermions) are symmetric.
(Relatively easy to be checked in massless limit)

✓ Maybe, we eventually need to add the gluons' contributions.
(A gauge invariant macroscopic theory may be more acceptable.)

Not prefect ! Unsolved problem

- **Energy momentum conservation**

$$\boxed{\partial_\mu T^{\mu\nu}_{\mathrm{Bel}} \quad = \quad 0.} \quad \xleftrightarrow{\text{equivalent}} \quad \boxed{\partial_\mu J^{\mu\nu\alpha}_{\mathrm{Bel}} \quad = \quad 0,}$$

- **Current conservation**

$$\boxed{\partial_\mu j^\mu = 0}$$

- **Equations from entropy principle**

$$\partial_\lambda(u^\lambda S^{\rho\sigma}) = q^\rho u^\sigma - q^\sigma u^\rho + \phi^{\rho\sigma}$$

$$\begin{aligned} q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\ \phi^{\mu\nu} &= 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T. \end{aligned}$$

This is not a fundamental equation!

- **Number of equations: 4+1+6= 11**
- **Variables: $T, \mu, S^{\mu\nu}, u^i$, 1+1+6+3=11**

- **Equation of state: $e = e(T, \mu, S^{\mu\nu})$ + Gibbs relation** $\omega^{\mu\nu} = \left.\frac{de}{dS^{\mu\nu}}\right|_{n,s}$

Summary and puzzles

Summary

- We have discussed the Belinfante energy momentum tensor, which is symmetric and gauge invariant.
- We have found the spin corrections to the dissipative terms, including quantum spin vorticity.
- By redefining the entropy flow,
  - we can reproduce the well-known results "in global equilibrium the spin chemical potential is related to thermal vorticity $\omega^{\mu\nu} = -T\omega_{\mu\nu}^{th}/2$."
  - In Local equilibrium, we can rediscover the evolution equations for the spin effects, which is consistent with the one derived in canonical form.

Puzzles

- We do not know how to get the evolution equations for spin in a more fundamental way.

Thank you for your time!

Any comments are welcome!

Backup

Symmetric energy-momentum tensor

$$T^{\mu\nu}_{A,\mathrm{Bel}} \equiv -F^{\mu}_{\ \alpha}F^{\nu\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\,, \tag{21}$$

$$T^{\mu\nu}_{\psi,\mathrm{Bel}} \equiv \bar{\psi}i\gamma^{\mu}\overleftrightarrow{D}^{\nu}\psi + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\partial_{\lambda}(\bar{\psi}\gamma_5\gamma_{\rho}\psi)\,. \tag{22}$$

These are very desirable expressions and all the terms are manifestly gauge invariant, thus corresponding to physical observables in principle. At this point, one might have thought that $T^{\mu\nu}_{\psi,\mathrm{Bel}}$ does not look symmetric with respect to μ and ν. In a quite non-trivial way one can prove that the above fermionic part is alternatively expressed as $T^{\mu\nu}_{\psi,\mathrm{Bel}} = \bar{\psi}i\gamma^{(\mu}\overleftrightarrow{D}^{\nu)}\psi$, which is obviously symmetric.

$$\frac{1}{2} = J_q + J_g \tag{23}$$

Coming back to the angular momentum, we can introduce the Belinfante "improved" form for the angular momentum, i.e.,

$$J^{\lambda\mu\nu}_{\mathrm{Bel}} \equiv J^{\lambda\mu\nu} + \partial_{\rho}\left(x^{\mu}K^{\rho\lambda\nu}_{\mathrm{Bel}} - x^{\nu}K^{\rho\lambda\mu}_{\mathrm{Bel}}\right)\,. \tag{24}$$

Main equations

$$\boxed{\partial_\mu T^{\mu\nu}_{\mathrm{Bel}} \;=\; 0.} \qquad T^{\mu\nu}_{Bel} \;=\; T^{\mu\nu}_{can} + \partial_\lambda K^{\lambda\mu\nu}_{Bel} = T^{\mu\nu}_{0} + T^{\mu\nu}_{(1)}$$

$$T^{\mu\nu}_{(1)} = h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} + \frac{1}{2}\partial_\lambda(u^\mu S^{\nu\lambda} + u^\nu S^{\mu\lambda})$$

heat flow

viscous tensor

spin density tensor

spin corrections to the energy momentum tensor

$$\partial_\lambda(u^\lambda S^{\rho\sigma}) = q^\rho u^\sigma - q^\sigma u^\rho + \phi^{\rho\sigma}$$

$$\begin{aligned} q^\mu &= \lambda[(u\cdot\partial)u^\mu + \frac{1}{T}\Delta^{\mu\nu}\partial_\nu T - 4\omega^{\mu\nu}u_\nu], \\ \phi^{\mu\nu} &= 2\gamma[T\omega^{\mu\nu}_{th} + 2(g^{\mu\alpha} - u^\mu u^\alpha)(g^{\nu\beta} - u^\nu u^\beta)\omega_{\alpha\beta}]/T. \end{aligned}$$