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The second order anomalous currents from Wigner function approach

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1. arXiv:2003.04517 S.Z. Yang, J.H. Gao, Z.T. Liang, Q. Wang
2. arXiv:2005.08512 R.H. Fang, J.H. Gao, D.F. Hou, C. Zhang
3. arXiv:2002.04800 R.H. Fang, J.H. Gao
4. arXiv:1910.11060 J.H. Gao, Z.T. Liang, Q. Wang
5. arXiv:1810.02028 J.H. Gao, J. Y. Pang, Q. Wang
6. arXiv:1802.06216 J.H. Gao, Z.T. Liang, Q. Wang, X.N. Wang

QCD theory Seminars JP, Jun 8 2020

Outline

- Introduction
- Wigner functions up to 2^{nd} order
- Charge currents and stress tensor up to 2^{nd} order
- The conservation laws and chiral anomaly
- Summary

Chiral Effects in HIC

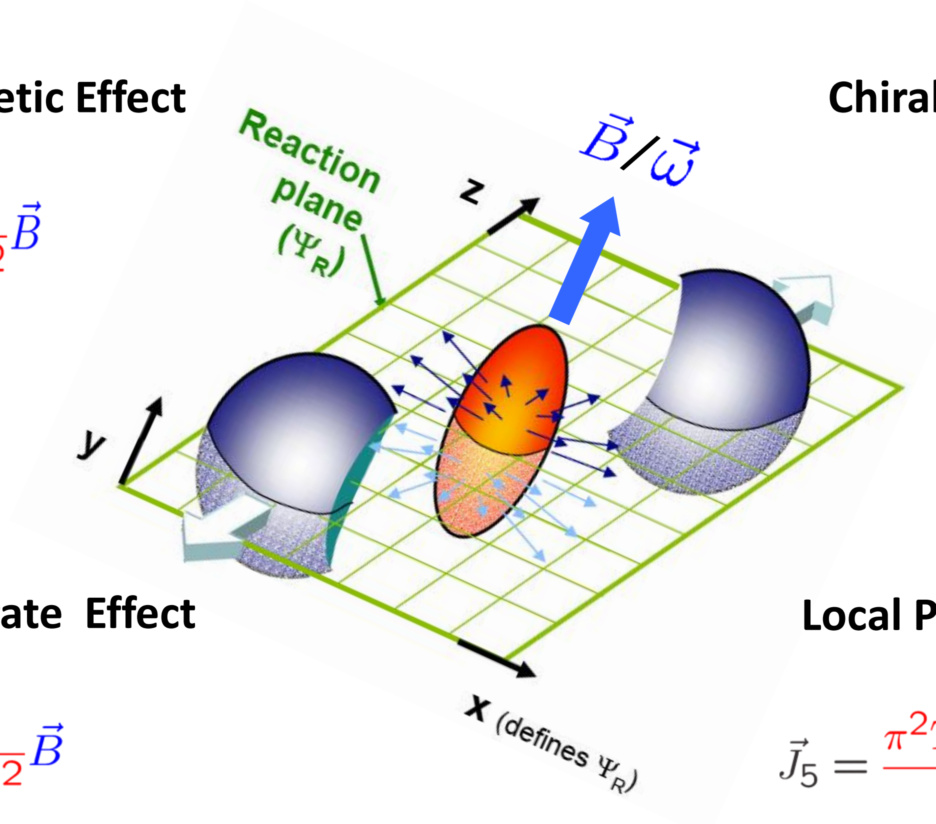
First order currents!

Chiral Magnetic Effect

$$\vec{J} = \frac{\mu_5}{2\pi^2}\vec{B}$$

Chiral Vortical Effect

$$\vec{J} = \frac{\mu\mu_5}{\pi^2}\vec{\omega}$$



Chiral Separate Effect

$$\vec{J}_5 = \frac{\mu}{2\pi^2}\vec{B}$$

Local Polarization Effect

$$\vec{J}_5 = \frac{\pi^2 T^2 + 3(\mu^2 + \mu_5^2)}{6\pi^2}\vec{\omega}$$

Kharzeev, Prog.Part.Nucl. (2014) ; Huang, Rept. Prog. Phys. (2016) ; Kharzeev, Liao, Voloshin Prog.Part.Nucl. (2016);
JHG, Ma, Pu, Wang , 2005.10432 A review for Nucl. Sci. Tech

Theoretical methods

Quantum Field Theory

Kharzeev PRD(2009), Landsteiner PRL(2011),
Fukushima NPA(2010), Hou JHEP(2011) ……

Anomalous Hydrodynamics

Son PRL(2009)
Yee PRC(2014)
Yin PLB(2016)
Hongo PLB(2017) … …

first order:
CME CVE
CSE LPE

Gauge/Gravity Duality

Erdmenger JHEP(2009)
Yee JHEP(2009)
Rebhan JHEP(2010);
Lin PRD(2013) ……

Chiral Kinetic theory

Stephanov PRL(2012)
Son PRD (2013)
Manuel PRD (2014)
Huang PLB(2018)……

Wigner function approach

Gao PRL(2012)
Chen PRL (2013)
Hidaka PRD(2017)
Yang PRD(2018)……

Why Second Order Correction

- **Large vorticity and magnetic fields in heavy ion collisions!**
- **Causal issue in first order relativistic hydrodynamics!**
- **Coupling terms between vorticity and electromagnetic fields!**
- **Check the perturbation formalism!**

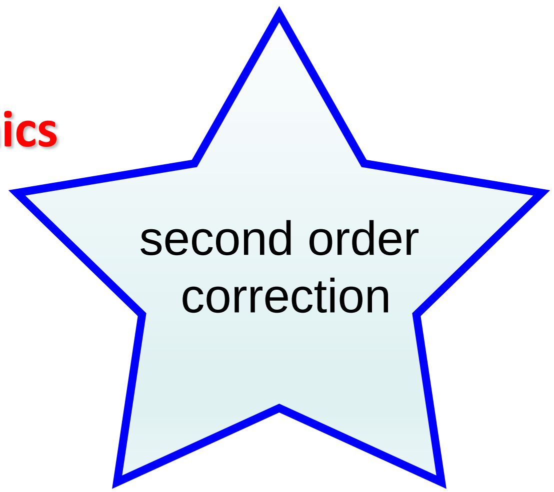
Previous research

Quantum Field Theory:

Jimenzez-Alba PRD(2015) Hattori PRL(2016)

Anomalous Hydrodynamics

Kharzeev PRD(2011)



Gauge/Gravity Duality

Banerjee JHEP(2012)

Bhattacharyya JHEP(2014)

Megias JHEP 2014

Bu 1912.11277

Chiral Kinetic theory

Satow PRD(2014)

Gorbar PRD(2017) PRD(2017)

Abbasi JHEP(2019)

Wigner function approach

Hidaka, Pu, Yang PRD(2018)

Hidaka, Yang PRD(2018)

Yang, JHG, Liang, Wang 2003.04517

Wigner operator in QFT

Density matrix in QED:

$$\rho\left(x+\frac{y}{2},x-\frac{y}{2}\right)=\bar{\psi}\left(x+\frac{y}{2}\right)U\left(x+\frac{y}{2},x-\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)$$

Gauge link / Wilson line:

$$U\left(A,x+\frac{1}{2}y,x-\frac{1}{2}y\right)\equiv\mathcal{P}\mathrm{Exp}\left(-iey^{\mu}\int_0^1 ds A_{\mu}\left(x-\frac{1}{2}y+sy\right)\right)$$

Wigner operator:

$$\hat{W}(x,p)=\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\rho\left(x+\frac{y}{2},x-\frac{y}{2}\right)$$

Straight line path ⇒

$$\hat{W}(x,p)=\bar{\psi}(x)\delta^4\left(p-\hat{\pi}(x)\right)\psi(x)$$

Particle density at x with kinetic momentum p :

$$\hat{\pi}_{\mu}=\hat{p}_{\mu}-eA_{\mu}(x)$$

Heinz, PRL 1983; Elze NPB 1986

Wigner function and equation

Wigner function

- Unnormal ordered : $W(x,p) = < \hat{W}(x,p) >$
- Normal ordered : $W(x,p) = <: \hat{W}(x,p) :>$

Dirac equation in background electromagnetic field :

$$[i\gamma \cdot D(x) - m]\,\psi(x) = 0 = \bar{\psi}(x)\left[i\gamma \cdot D^{\dagger}(x) + m\right]$$

$$\Downarrow$$

Wigner equation :

$$\left[\gamma_\mu\left(\Pi^\mu + \frac{1}{2} i G^\mu\right) - m\right] W(x,p) = 0$$

Vasak AP1987

$$\Pi^\mu \equiv p^\mu - \frac{1}{2} j_1\left(\frac{1}{2}\Delta\right) F^{\mu\nu}\partial_\nu^p, \quad G^\mu \equiv \partial_x^\mu - j_0\left(\frac{1}{2}\Delta\right) F^{\mu\nu}\partial_\nu^p, \quad \Delta \equiv \partial^p \cdot \partial_x$$

The Wigner function in Wigner equation must be unnormal ordered!

Chiral limit

16 Wigner functions:

$$W = \frac{1}{4}\left[\mathscr{F} + i\gamma^5\mathscr{P} + \gamma^\mu\mathscr{V}_\mu + \gamma^5\gamma^\mu\mathscr{A}_\mu + \frac{1}{2}\sigma^{\mu\nu}\mathscr{S}_{\mu\nu}\right]$$

scalar (𝓕), pseudo (𝓟), vector (𝓥), axial (𝓐), tensor (𝓢)

32 Wigner equations:

Real parts

$$\begin{aligned}
\Pi^\mu\mathscr{V}_\mu &= m\mathscr{F}, \\
-G^\mu\mathscr{A}_\mu &= 2m\mathscr{P}, \\
2\Pi_\mu\mathscr{F} + G^\nu\mathscr{S}_{\mu\nu} &= 2m\mathscr{V}_\mu, \\
G_\mu\mathscr{P} - \epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathscr{S}^{\rho\sigma} &= 2m\mathscr{A}_\mu, \\
(G_\mu\mathscr{V}_\nu - G_\nu\mathscr{V}_\mu) - 2\epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathscr{A}^\sigma &= 2m\mathscr{S}_{\mu\nu}.
\end{aligned}$$

Imaginary parts

$$\begin{aligned}
0 &= G^\mu\mathscr{V}_\mu, \\
0 &= \Pi^\mu\mathscr{A}_\mu, \\
0 &= G_\mu\mathscr{F} - 2\Pi^\nu\mathscr{S}_{\mu\nu}, \\
0 &= 4\Pi_\mu\mathscr{P} + \epsilon_{\mu\nu\rho\sigma}G^\nu\mathscr{S}^{\rho\sigma}, \\
0 &= 2(\Pi_\mu\mathscr{V}_\nu - \Pi_\nu\mathscr{V}_\mu) + \epsilon_{\mu\nu\rho\sigma}G^\rho\mathscr{A}^\sigma.
\end{aligned}$$

Chiral limit m=0

$$\begin{aligned}
\Pi^\mu\mathscr{V}_\mu &= 0, \quad \Pi^\mu\mathscr{A}_\mu = 0, \\
G^\mu\mathscr{V}_\mu &= 0, \quad G^\mu\mathscr{A}_\mu = 0, \\
\epsilon_{\mu\nu\rho\sigma}G^\rho\mathscr{A}^\sigma &= -2(\Pi_\mu\mathscr{V}_\nu - \Pi_\nu\mathscr{V}_\mu), \\
\epsilon_{\mu\nu\rho\sigma}G^\rho\mathscr{V}^\sigma &= -2(\Pi_\mu\mathscr{A}_\nu - \Pi_\nu\mathscr{A}_\mu).
\end{aligned}$$

8 functions +16 equations

⟺

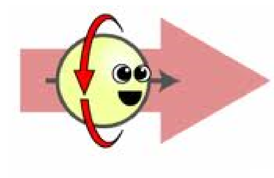
$$\begin{aligned}
\Pi_\mu\mathscr{F} + \frac{1}{2}G^\nu\mathscr{S}_{\mu\nu} &= 0, \\
\frac{1}{2}G_\mu\mathscr{F} - \Pi^\nu\mathscr{S}_{\mu\nu} &= 0, \\
-G_\mu\mathscr{P} + \epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathscr{S}^{\rho\sigma} &= 0, \\
\Pi_\mu\mathscr{P} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}G^\nu\mathscr{S}^{\rho\sigma} &= 0.
\end{aligned}$$

8 functions +16 equations

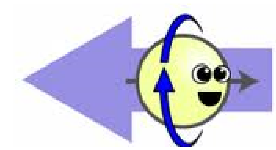
Right/Left-handed Basis

Chirality basis: $\mathscr{J}_s^\mu \equiv \frac{1}{2}(\mathscr{V}_\mu + s\mathscr{A}_\mu)$

Right: $s = +1$



Left: $s = -1$



$$G^\mu \mathscr{V}_\mu = 0, \quad \Pi^\mu \mathscr{V}_\mu = 0, \quad 2(\Pi_\mu \mathscr{V}_\nu - \Pi_\nu \mathscr{V}_\mu) = -\epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{A}^\sigma$$

$$G^\mu \mathscr{A}_\mu = 0, \quad \Pi^\mu \mathscr{A}_\mu = 0, \quad 2(\Pi_\mu \mathscr{A}_\nu - \Pi_\nu \mathscr{A}_\mu) = -\epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{V}^\sigma$$

$$G_\mu \mathscr{J}_s^\mu = 0, \quad \Pi_\mu \mathscr{J}_s^\mu = 0, \quad 2(\Pi_\mu \mathscr{J}_{s\nu} - \Pi_\nu \mathscr{J}_{s\mu}) = -s\epsilon_{\mu\nu\rho\sigma} G^\rho \mathscr{J}_s^\sigma$$

4 independent functions + 8 coupled equations

Disentanglement Theorem

Component decomposition : $\mathscr{J}_s^\mu = \mathscr{J}_{sn} n_\mu + \bar{\mathscr{J}}_s^\mu$ $\quad n^2 = 1, \ \mathscr{J}_n = n \cdot \mathscr{J}, \ \ n \cdot \bar{\mathscr{J}} = 0$

Auxiliary n_μ can be identified as the 4-velocity of reference frame !

Semiclassical expansion: $\mathscr{J}_s^\mu = \sum_{k=0}^{\infty} \hbar^k \mathscr{J}_s^{(k)\mu} \quad G^\mu = \sum_{k=0}^{\infty} \hbar^k G^{\mu(k)} \quad \Pi^\mu = \sum_{k=0}^{\infty} \hbar^k \Pi^{\mu(k)}$

Only $\mathscr{J}_{sn}$ is independent :

$$
\begin{aligned}
\bar{\mathscr{J}}_s^{(0)\mu} &= \frac{\bar{p}^\mu}{p_n} \mathscr{J}_{sn}^{(0)\mu}, \\
\bar{\mathscr{J}}_s^{\mu(1)} &= \frac{\bar{p}^\mu}{p_n} \mathscr{J}_{sn}^{(1)} - \frac{s}{2p_n} \epsilon^{\mu\nu\rho\sigma} n_\nu \nabla_\sigma \left(\frac{p_\rho}{p_n} \mathscr{J}_{sn}^{(0)\mu} \right), \\
\cdots \quad & \cdots\cdots
\end{aligned}
$$

1 function + 1 equation

It has been proved as a theorem up to any order of $\hbar$!

arXiv:1802.06216 JHG, Z.T. Liang, Q. Wang, X.N. Wang Phys.Rev. D98 (2018)

Distribution function in different frames

Transformation rule of distribution function in different frames n and n':

$$F_{sn} \equiv \frac{\mathscr{J}_{sn}}{p_n}$$

$$\delta F_{sn}^{(0)} = F_{sn'}^{(0)} - F_{sn}^{(0)} = 0, \quad \delta F_{sn}^{(1)} = F_{sn'}^{(1)} - F_{sn}^{(1)} - \frac{s\epsilon^{\lambda\nu\rho\sigma} n_\lambda n'_\nu \nabla_\rho (p_\sigma F_{sn}^{(0)})}{2\,(n'\cdot p)\,(n\cdot p)}$$

Side jump

Non-trivial transformation and chiral vortical effect:

$$\mathscr{J}_s^{(1)\mu} = \frac{s p^\mu \epsilon^{\nu\sigma\rho\lambda} p_\nu n_\sigma \partial_\rho^x \beta_\lambda}{4 n\cdot p} F_{sn}^{(0)\prime} + \frac{s\epsilon^{\mu\nu\rho\sigma} n_\nu p_\sigma p^\lambda \partial_\rho^x \beta_\lambda}{2 n\cdot p} F_{sn}^{(0)\prime}$$

$$= -\frac{s}{4}\epsilon^{\mu\nu\rho\sigma} p_\nu \partial_\rho^x \beta_\sigma F_{sn}^{(0)\prime}$$

$n^\mu = u^\mu$ $\quad$ $\beta^\mu = u^\mu / T$

Spin-vorticity coupling $\frac{1}{3} j_\mu^{\mathrm{CVE}}$	Magnetization $\frac{2}{3} j_\mu^{\mathrm{CVE}}$	$1 j_\mu^{\mathrm{CVE}}$

arXiv:1810.02028 JHG, J. Y. Pang, Q. Wang Phys. Rev. D 100 (2019)

Covariant perturbation expansion

Wigner equation in static and uniform EM field: $G^\mu = \nabla^\mu = \partial_x^\mu - F^{\mu\nu}\partial_\nu^p, \quad \Pi^\mu = p^\mu$

$$\nabla_\mu \mathscr{J}_s^\mu = 0, \quad p_\mu \mathscr{J}_s^\mu = 0, \quad 2(p_\mu \mathscr{J}_{s\nu} - p_\nu \mathscr{J}_{s\mu}) = -s\epsilon_{\mu\nu\rho\sigma}\hbar\nabla^\rho \mathscr{J}_s^\sigma$$

$\hbar$ semiclassical expansion = $\nabla^\mu = \partial_x^\mu - F^{\mu\nu}\partial_\nu^p$ ∂_x^μ & $F^{\mu\nu}$ expansion

Wigner equation order by order: Iterative equation

$$\nabla_\mu \mathscr{J}_s^{(n)\mu} = 0, \quad p_\mu \mathscr{J}_s^{(n)\mu} = 0, \quad 2\left(p_\mu \mathscr{J}_{s\nu}^{(n)} - p_\nu \mathscr{J}_{s\mu}^{(n)}\right) = -s\epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathscr{J}_s^{(n-1)\sigma}$$

$$\mathcal{J}_{s\mu}^{(n)} = p_\mu f_s^{(n)} + X_{s\mu}^{(n)}$$

$$\mathscr{J}_{s\mu}^{(n)} = \mathcal{J}_{s\mu}^{(n)}\delta(p^2) + \frac{s}{2p^2}\epsilon_{\mu\nu\rho\sigma}p^\nu\nabla^\rho \mathscr{J}_s^{(n-1)\sigma}$$

The zeroth order solution

The 0th order equations:

$$p^{\mu}\mathscr{J}_{s\mu}^{(0)} = 0,$$
$$p_{\mu}\mathscr{J}_{s\nu}^{(0)} - p_{\nu}\mathscr{J}_{s\mu}^{(0)} = 0,$$

⇒

The 0th order solution:

$$\mathscr{J}_{s\mu}^{(0)} = p_{\mu}\delta\left(p^2\right)f_s$$

Fermi-Dirac distribution: $\beta = 1/T, \quad \bar{\mu}_s = \mu_s/T, \quad \beta^{\mu} = \beta u^{\mu}$

$$f_s = \frac{1}{4\pi^3}\left[\theta\left(p_0\right)\frac{1}{e^{(\beta\cdot p-\bar{\mu}_s)}+1} + \theta\left(-p_0\right)\left(\frac{1}{e^{-(\beta\cdot p-\bar{\mu}_s)}+1} - 1\right)\right]$$

Impose transport equation:

$$\nabla_{\mu}\mathscr{J}_s^{(0)\mu} = 0$$

⇒

Vlasov equation

$$\delta(p^2)p^{\mu}\nabla_{\mu}f_s^{(0)} = 0$$

Constraint conditions

Global equilibrium condition:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0, \quad \partial_\mu \bar{\mu}_s + F_{\mu\nu}\beta^\nu = 0;$$

$$\Rightarrow \quad \delta(p^2) p^\mu \nabla_\mu f_s^{(0)} = 0$$

Integrability condition

$$\partial_\nu \partial_\mu \bar{\mu}_s = \partial_\mu \partial_\nu \bar{\mu}_s$$

$$\Updownarrow$$

$$F_\lambda{}^\mu \Omega^{\nu\lambda} - F_\lambda{}^\nu \Omega^{\mu\lambda} = 0$$

$$\beta_\mu = -\Omega_{\mu\nu} x^\nu,$$

$$\Omega_{\mu\nu} = \left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)/2$$

$\Omega_{\mu\nu}$: constant

$$\bar{\mu}_s = -\frac{1}{2} F^{\mu\lambda} x_\lambda \Omega_{\mu\nu} x^\nu + c_s,$$ c_s : constant

Find the solution under global equilibrium with constant $F^{\mu\nu}$ and $\Omega^{\mu\nu}$!

The first order solution

General form for the 1st order solution:

$$\mathscr{J}_{s\mu}^{(1)} = p_\mu f_s^{(1)}\delta(p^2) + X_{s\mu}^{(1)}\delta(p^2) + \frac{s}{2p^2}\epsilon_{\mu\nu\rho\sigma}p^\nu\nabla^\rho \mathscr{J}_s^{(0)\sigma}$$

Further determine $f_s^{(1)}$ and $X_{s\mu}^{(1)}$:

$$2\left(p_\mu \mathscr{J}_{s\nu}^{(1)} - p_\nu \mathscr{J}_{s\mu}^{(1)}\right) = -s\epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathscr{J}_s^{(0)\sigma} \quad \Rightarrow \quad X_{s\mu}^{(1)} = -\frac{s}{2}\tilde{\Omega}_{\mu\lambda}p^\lambda f_s'$$

$$\nabla_\mu \mathscr{J}_s^{(1)\mu} = 0 \quad \Rightarrow \quad \delta(p^2)p^\mu\nabla_\mu f_s^{(1)} = 0 \quad \xrightarrow{\text{set}} \quad f_s^{(1)} = 0$$

The 1st order solution:

$$\mathscr{J}_{s\mu}^{(1)} = -\frac{s}{2}\tilde{\Omega}_{\mu\lambda}p^\lambda f_s' - s\tilde{F}_{\mu\lambda}p^\lambda f_s\delta'(p^2)$$

$$\tilde{\Omega}_{\mu\lambda} = \epsilon_{\mu\lambda\alpha\beta}\Omega^{\alpha\beta}/2, \quad \tilde{F}_{\mu\lambda} = \epsilon_{\mu\lambda\alpha\beta}F^{\alpha\beta}/2$$

The second order solution

General form for the 2nd order solution:

$$\mathscr{J}_{s\mu}^{(2)} = p_\mu f_s^{(2)} \delta(p^2) + X_{s\mu}^{(2)} \delta(p^2) + \frac{s}{2p^2} \epsilon_{\mu\nu\rho\sigma} p^\nu \nabla^\rho \mathscr{J}_s^{(1)\sigma}$$

Similar to 1st order, we can determine $X_{s\mu}^{(2)} = 0$ and set $f_s^{(2)} = 0$

The 2nd order solution:

$$\begin{aligned} \mathscr{J}_{s\mu}^{(2)} = & -\frac{1}{4}\Omega_{\gamma\mu}\Omega^{\gamma\lambda} p_\lambda f_s'' \delta(p^2) - \frac{1}{4} p_\mu \Omega_{\gamma\beta} p^\beta \Omega^{\gamma\lambda} p_\lambda f_s'' \delta'(p^2) \\ & + F_{\gamma\mu}\Omega^{\gamma\lambda} p_\lambda f_s' \delta'(p^2) + \frac{1}{2} p_\mu F_{\gamma\beta} p^\beta \Omega^{\gamma\lambda} p_\lambda f_s' \delta''(p^2) \\ & - F_{\gamma\mu} F^{\gamma\lambda} p_\lambda f_s \delta''(p^2) - \frac{1}{3} p_\mu F_{\gamma\beta} p^\beta F^{\gamma\lambda} p_\lambda f_s \delta'''(p^2) \end{aligned}$$

Solution up to 2nd order

The solution under global equilibrium with constant $F^{\mu\nu}$ and $\Omega^{\mu\nu}$:

$$\mathscr{J}_{s\mu}^{(0)} = p_\mu \delta\left(p^2\right) f_s$$

$$\mathscr{J}_{s\mu}^{(1)} = -\frac{s}{2}\tilde{\Omega}_{\mu\lambda} p^\lambda f_s' \delta(p^2) - s\tilde{F}_{\mu\lambda} p^\lambda f_s \delta'(p^2)$$

$$\begin{aligned}
\mathscr{J}_{s\mu}^{(2)} = & -\frac{1}{4}\Omega_{\gamma\mu}\Omega^{\gamma\lambda} p_\lambda f_s'' \delta(p^2) - \frac{1}{4} p_\mu \Omega_{\gamma\beta} p^\beta \Omega^{\gamma\lambda} p_\lambda f_s'' \delta'(p^2) \\
& + F_{\gamma\mu}\Omega^{\gamma\lambda} p_\lambda f_s' \delta'(p^2) + \frac{1}{2} p_\mu F_{\gamma\beta} p^\beta \Omega^{\gamma\lambda} p_\lambda f_s' \delta''(p^2) \\
& - F_{\gamma\mu} F^{\gamma\lambda} p_\lambda f_s \delta''(p^2) - \frac{1}{3} p_\mu F_{\gamma\beta} p^\beta F^{\gamma\lambda} p_\lambda f_s \delta'''(p^2)
\end{aligned}$$

Charge currents at 0th order

Left-handed or right-handed current:

$$j_s^\mu = \int d^4p \, \mathscr{J}_s^\mu$$

Vector and axial currents:

$$j^\mu = j_{+1}^\mu + j_{-1}^\mu, \quad j_5^\mu = j_{+1}^\mu - j_{-1}^\mu$$

Charge currents at 0th order:

$$\mu_s = \mu + s\mu_5$$

Left/right:

$$j_s^{(0)\mu} = n_s u^\mu \qquad n_s = \frac{\mu_s}{6\pi^2}\left(\pi^2 T^2 + \mu_s^2\right)$$

Vector:

$$j^{(0)\mu} = n u^\mu \qquad n = \frac{\mu}{3\pi^2}\left(\pi^2 T^2 + \mu^2 + 3\mu_5^2\right)$$

Axial:

$$j_5^{(0)\mu} = n_5 u^\mu \qquad n_5 = \frac{\mu_5}{3\pi^2}\left(\pi^2 T^2 + 3\mu^2 + \mu_5^2\right)$$

Charge currents at 1st order

Charge currents at 1st order:

Left/right:

$$j_s^{(1)\mu} = \xi_s \omega^\mu + \xi_{Bs} B^\mu \qquad \xi_s = \frac{s(\pi^2 T^2 + 3\mu_s^2)}{12\pi^2}, \quad \xi_{Bs} = \frac{s}{4\pi^2}\mu_s$$

Vector:

$$j^{(1)\mu} = \xi \omega^\mu + \xi_B B^\mu \qquad \xi = \frac{\mu\mu_5}{\pi^2}, \quad \xi_B = \frac{\mu_5}{2\pi^2}$$

Axial:

$$j_5^{(1)\mu} = \xi_5 \omega^\mu + \xi_{B5} B^\mu \qquad \xi_5 = \frac{\pi^2 T^2 + 3(\mu^2 + \mu_5^2)}{6\pi^2}, \quad \xi_{B5} = \frac{\mu}{2\pi^2}$$

Electric part and magnetic part decomposition:

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma$$

$$T\Omega_{\mu\nu} = \varepsilon_\mu u_\nu - \varepsilon_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma} u^\rho \omega^\sigma$$

Charge currents at 2nd order

Left-handed/Right-handed currents at 2nd order:

Anomalous magneto-vorticity coupling K. Hattori, Y. Yin PRL2016

$$j_s^{(2)\mu} = -\frac{\mu_s}{4\pi^2}(\varepsilon^2+\omega^2)u^\mu - \frac{1}{8\pi^2}(\varepsilon\cdot E+\omega\cdot B)u^\mu - \frac{C_s}{24\pi^2}(E^2+B^2)u^\mu$$

→ Charge density

$$-\frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho \omega_\sigma - \frac{C_s}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma$$

→ Hall current from $F^{\mu\nu}$

Hall current from the coupling of $F^{\mu\nu}$ and $\Omega^{\mu\nu}$

$$C_s = \frac{1}{T}\int_0^\infty \frac{dy}{y}\left[\frac{e^{y-\bar{\mu}_s}}{\left(e^{y-\bar{\mu}_s}+1\right)^2} - \frac{e^{y+\bar{\mu}_s}}{\left(e^{y+\bar{\mu}_s}+1\right)^2}\right]$$

$$F_\lambda{}^\mu \Omega^{\nu\lambda} - F_\lambda{}^\nu \Omega^{\mu\lambda} = 0 \Rightarrow \epsilon_{\mu\nu\rho\sigma}E^\rho \omega^\sigma = \epsilon_{\mu\nu\rho\sigma}\varepsilon^\rho B^\sigma$$

Charge currents at 2nd order

Charge currents at 2nd order:

Charge density

Vector:

$$j^{(2)\mu} = -\frac{\mu}{4\pi^2}(\varepsilon^2+\omega^2)u^\mu - \frac{1}{4\pi^2}(\varepsilon\cdot E+\omega\cdot B)u^\mu - \frac{C}{12\pi^2}(E^2+B^2)u^\mu$$

$$-\frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho \omega_\sigma - \frac{C}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma$$

Hall current

$$C = \frac{1}{2}\left(C_{+1}+C_{-1}\right), \quad C_5 = \frac{1}{2}\left(C_{+1}-C_{-1}\right)$$

Axial:

$$j_5^{(2)\mu} = -\frac{\mu_5}{4\pi^2}(\varepsilon^2+\omega^2)u^\mu - \frac{C_5}{12\pi^2}(E^2+B^2)u^\mu$$

Charge density

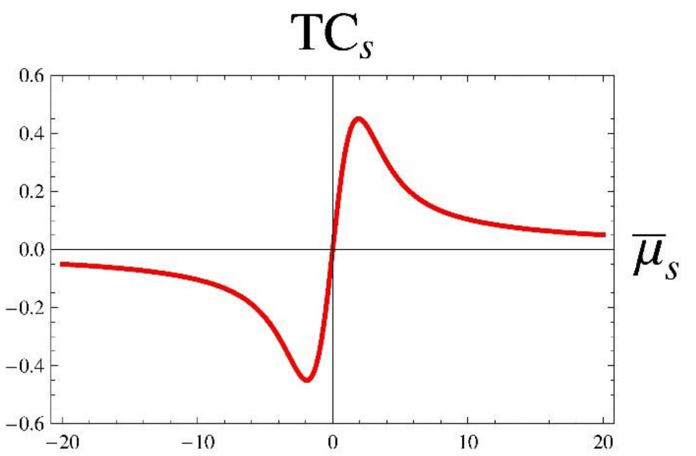
$$-\frac{C_5}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma$$

Hall current

Hall currents from EM field

LH / RH Hall coefficient from EM field:

$$C_s = \frac{1}{T}\int_0^\infty \frac{dy}{y}\left[\frac{e^{y-\bar{\mu}_s}}{\left(e^{y-\bar{\mu}_s}+1\right)^2} - \frac{e^{y+\bar{\mu}_s}}{\left(e^{y+\bar{\mu}_s}+1\right)^2}\right]$$



$|\mu_s \ll T|$: $TC_s \approx 0.4263\bar{\mu}_s$

$|\mu_s \gg T|$: $TC_s = 1/\bar{\mu}_s$

Vector and axial Hall coefficient: $C = \frac{1}{2}\left(C_{+1} + C_{-1}\right)$, $C_5 = \frac{1}{2}\left(C_{+1} - C_{-1}\right)$

$|\mu_s \ll T|$

$$TC \approx 0.4263\bar{\mu}$$
$$TC_5 \approx 0.4263\bar{\mu}_5$$

$|\mu_s \gg T|$:

$$TC = \bar{\mu}/(\bar{\mu}^2 - \bar{\mu}_5^2)\ ,$$
$$TC_5 = -\bar{\mu}_5/(\bar{\mu}^2 - \bar{\mu}_5^2)$$

Stress tensor at 0th and 1st order

Canonical stress tensor:

$$T_s^{\mu\nu} = \int d^4p\, \mathscr{J}_s^{\mu} p^{\nu} \qquad T^{\mu\nu} = T_{+1}^{\mu\nu} + T_{-1}^{\mu\nu}$$

Total stress tensor up to 1st order:

$$T^{(0)\mu\nu} = \rho u^{\mu} u^{\nu} - \frac{1}{3}\rho \Delta^{\mu\nu}$$

$$T^{(1)\mu\nu} = n_5\left(u^{\mu}\omega^{\nu} + u^{\nu}\omega^{\mu}\right) + \frac{\xi}{2}\left(u^{\mu}B^{\nu} + u^{\nu}B^{\mu}\right)$$ Symmetric

$$-\frac{n_5}{2}\left(u^{\mu}\omega^{\nu} - u^{\nu}\omega^{\mu} + \epsilon^{\mu\nu\alpha\beta}u_{\alpha}\varepsilon_{\beta}\right) - \frac{\xi}{2}\epsilon^{\mu\nu\alpha\beta}u_{\alpha}E_{\beta}$$ Antisymmetric

Energy density:

$$\rho = \frac{T^4}{4\pi^2}\left[\frac{7}{15}\pi^4 + 2\pi^2(\bar{\mu}^2 + \bar{\mu}_5^2) + \bar{\mu}^4 + 6\bar{\mu}^2\bar{\mu}_5^2 + \bar{\mu}_5^4\right]$$

Stress tensor at 2nd order

Decomposition：

$$T_s^{(2)\mu\nu} = T_{s,\mathrm{vv}}^{(2)\mu\nu} + T_{s,\mathrm{ve}}^{(2)\mu\nu} + T_{s,\mathrm{ee}}^{(2)\mu\nu}$$

v: vorticity tensor

e: electromagnetic tensor

$$T_{s,\mathrm{vv}}^{(2)\mu\nu} = -\frac{1}{4}\Omega^{\gamma}{}_{\beta}\Omega_{\gamma\lambda}\int d^4p\; p^\mu p^\nu p^\beta p^\lambda f_s''\delta'(p^2) - \frac{1}{4}\Omega^{\gamma\mu}\Omega_{\gamma\lambda}\int d^4p\; p^\nu p^\lambda f_s''\delta(p^2)\,,$$

$$T_{s,\mathrm{ve}}^{(2)\mu\nu} = \frac{1}{2}F^{\gamma}{}_{\beta}\Omega_{\gamma\lambda}\int d^4p\; p^\mu p^\nu p^\beta p^\lambda f_s'\delta''(p^2) + F^{\gamma\mu}\Omega_{\gamma\lambda}\int d^4p\; p^\nu p^\lambda f_s'\delta'(p^2)\,,$$

$$T_{s,\mathrm{ee}}^{(2)\mu\nu} = -\frac{1}{3}F^{\gamma}{}_{\beta}F_{\gamma\lambda}\int d^4p\; p^\mu p^\nu p^\beta p^\lambda f_s\delta'''(p^2) - F^{\gamma\mu}F_{\gamma\lambda}\int d^4p\; p^\nu p^\lambda f_s\delta''(p^2)$$

The “vv” and “ve” contribution

Moments expansion:

$$\int d^4p\, p_\nu p_\lambda Y = u_\nu u_\lambda \int d^4p\,(u\cdot p)^2 Y + \frac{1}{3}\Delta_{\mu\nu}\int d^4p\,\bar{p}^2 Y,$$

$$\int d^4p\, p_\mu p_\nu p_\beta p_\lambda Y = u_\mu u_\nu u_\beta u_\lambda \int d^4p\,(u\cdot p)^4 Y + \frac{1}{15}\Delta_{(\mu\nu}\Delta_{\beta\lambda)}\int d^4p\,\bar{p}^4 Y + \frac{1}{3}u_{(\mu}u_\nu\Delta_{\beta\lambda)}\int d^4p(u\cdot p)^2\bar{p}^2 Y$$

Scalar: $Y(u,p)$

The “vv” and “ve” contributions:

$$T^{(2)\mu\nu}_{s,\mathrm{vv}} = -\frac{s}{2}\xi_s\Big[3u^\mu u^\nu(\omega^2+\varepsilon^2) - \Delta^{\mu\nu}(\omega^2+\varepsilon^2) - 2(u^\mu\epsilon^{\nu\alpha\beta\gamma}+u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\varepsilon_\beta\omega_\gamma$$
$$-2(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\varepsilon_\beta\omega_\gamma\Big],$$

$$T^{(2)\mu\nu}_{s,\mathrm{ve}} = -\frac{s}{2}\xi_{Bs}\Big[u^\mu u^\nu(\omega\cdot B+\varepsilon\cdot E) - (\omega^\mu B^\nu + E^\mu\varepsilon^\nu) - (u^\mu\epsilon^{\nu\alpha\beta\gamma}+u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha E_\beta\omega_\gamma$$
$$-2(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha E_\beta\omega_\gamma\Big]$$

The "ee" contribution

Dimensional regularization:

$$\int d^d p\, p_\nu p_\lambda Y = u_\nu u_\lambda \int d^d p\,(u\cdot p)^2 Y + \frac{1}{d-1}\Delta_{\mu\nu}\int d^d p\, \bar{p}^2 Y, \qquad d = 4-\epsilon$$

$$\int d^d p\, p_\mu p_\nu p_\beta p_\lambda Y = u_\mu u_\nu u_\beta u_\lambda \int d^d p\,(u\cdot p)^4 Y + \frac{1}{d^2-1}\Delta_{(\mu\nu}\Delta_{\beta\lambda)}\int d^d p\, \bar{p}^4 Y + \frac{1}{d-1}u_{(\mu}u_\nu\Delta_{\beta\lambda)}\int d^d p (u\cdot p)^2 \bar{p}^2 Y$$

Electromagnetic field contributions:

$$T^{(2)\mu\nu}_{s,\mathrm{ee}} = -\frac{1}{12}\kappa_s^\epsilon\left(\frac{1}{4}g^{\mu\nu}F_{\gamma\beta}F^{\gamma\beta} - F^{\gamma\mu}F_\gamma{}^\nu\right) + \frac{1}{48\pi^2}u^\mu u^\nu E^2 - \frac{1}{48\pi^2}\Delta^{\mu\nu}\left(E^2+2B^2\right)$$
$$+\frac{1}{12\pi^2}\left(E^\mu E^\nu + B^\mu B^\nu\right) + \frac{1}{16\pi^2}\left(u^\mu\epsilon^{\nu\alpha\beta\gamma} + u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma$$
$$+\frac{1}{16\pi^2}\left(u^\mu\epsilon^{\nu\alpha\beta\gamma} - u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma$$

$$\kappa_s^\epsilon = \frac{4\pi^{\frac{3-\epsilon}{2}}T^{-\epsilon}}{\Gamma\left(\frac{3-\epsilon}{2}\right)(2\pi)^{3-\epsilon}}\int_0^\infty \frac{dy}{y^{1+\epsilon}}\left[\frac{1}{e^{(y-\bar{\mu}_s)}+1} + \frac{1}{e^{(y+\bar{\mu}_s)}+1} - 1\right]$$

Ultraviolet divergence

Expand κ_s^ϵ around $\epsilon = 0$:

$$\kappa_s^\epsilon = -\frac{1}{\pi^2}\left[\frac{1}{\epsilon} + \ln 2 + \frac{1}{2}\ln\pi + \frac{1}{2}\psi\left(\frac{3}{2}\right) - \ln T + \hat{\kappa}_s\right]$$

Ultraviolet logarithmic divergence

$$\hat{\kappa}_s = \int_0^\infty dy \ln y \frac{d}{dy}\left[\frac{1}{e^{(y-\bar{\mu}_s)}+1} + \frac{1}{e^{(y+\bar{\mu}_s)}+1}\right]$$

Total stress tensor by summing RH and LH:

$$\begin{aligned}
T_{\rm vv}^{(2)\mu\nu} &= -\frac{1}{2}\xi_5\left[3u^\mu u^\nu(\omega^2+\varepsilon^2) - \Delta^{\mu\nu}(\omega^2+\varepsilon^2) - 2(u^\mu\epsilon^{\nu\alpha\beta\gamma} + u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\varepsilon_\beta\omega_\gamma\right.\\
&\left.\quad -2(u^\mu\epsilon^{\nu\alpha\beta\gamma} - u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\varepsilon_\beta\omega_\gamma\right],\\
T_{\rm ve}^{(2)\mu\nu} &= -\frac{1}{2}\xi_{B5}\left[u^\mu u^\nu(\omega\cdot B + \varepsilon\cdot E) - (\omega^\mu B^\nu + E^\mu\varepsilon^\nu) - (u^\mu\epsilon^{\nu\alpha\beta\gamma} + u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha E_\beta\omega_\gamma\right.\\
&\left.\quad -2(u^\mu\epsilon^{\nu\alpha\beta\gamma} - u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha E_\beta\omega_\gamma\right],\\
T_{\rm ee}^{(2)\mu\nu} &= -\frac{1}{6}\kappa^\epsilon\left(\frac{1}{4}g^{\mu\nu}F_{\gamma\beta}F^{\gamma\beta} - F^{\gamma\mu}F_\gamma{}^\nu\right) + \frac{1}{24\pi^2}\left[u^\mu u^\nu E^2 - \Delta^{\mu\nu}\left(E^2+2B^2\right)\right.\\
&\quad +4\left(E^\mu E^\nu + B^\mu B^\nu\right) + 3\left(u^\mu\epsilon^{\nu\alpha\beta\gamma} + u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma\\
&\left.\quad +3\left(u^\mu\epsilon^{\nu\alpha\beta\gamma} - u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma\right]
\end{aligned}$$

$$\kappa^\epsilon \equiv (\kappa_{+1}^\epsilon + \kappa_{-1}^\epsilon)/2$$

Trace of the stress tensor

Traceless stress tensor order by order:

$$g^{\mu\nu}T^{(0)}_{\mu\nu} = 0, \quad g^{\mu\nu}T^{(1)}_{\mu\nu} = 0, \quad g^{\mu\nu}T^{(2)}_{\mu\nu} = 0$$

⇐

$$g^{\mu\nu}T_{\mu\nu} = \int d^4p\, p_\mu \mathscr{V}^\mu = 0$$

Separate contribution from pure electromagnetic field:

$$g_{\mu\nu}g^{\mu\nu} = 4 - \epsilon$$

$$g_{\mu\nu}T^{(2)\mu\nu}_{\text{ee}} = -\frac{1}{6}\kappa^\epsilon g_{\mu\nu}\left(\frac{1}{4}g^{\mu\nu}F_{\gamma\beta}F^{\gamma\beta} - F^{\gamma\mu}F_\gamma{}^\nu\right) = -\frac{1}{24\pi^2}\cdot\frac{1}{\epsilon}\cdot\epsilon F_{\mu\nu}F^{\mu\nu} = -\frac{e^2}{24\pi^2}F_{\mu\nu}F^{\mu\nu}$$

$$+\frac{1}{24\pi^2}g_{\mu\nu}\left[u^\mu u^\nu E^2 - \Delta^{\mu\nu}\left(E^2 + 2B^2\right) + 4\left(E^\mu E^\nu + B^\mu B^\nu\right)\right] = \frac{e^2}{24\pi^2}F_{\mu\nu}F^{\mu\nu}$$

$$+\frac{1}{8\pi^2}g_{\mu\nu}\left[\left(u^\mu\epsilon^{\nu\alpha\beta\gamma} + u^\nu\epsilon^{\mu\alpha\beta\gamma}\right) + \left(u^\mu\epsilon^{\nu\alpha\beta\gamma} - u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)\right]u_\alpha E_\beta B_\gamma = 0$$

EM field contribution

Revisit divergence part:

$$g_{\mu\nu}T^{(2)\mu\nu}_{\text{ee}} = -\frac{1}{24\pi^2}\cdot\frac{1}{\epsilon}\cdot\epsilon\cdot F_{\mu\nu}F^{\mu\nu} = -\frac{\beta}{2e}F_{\mu\nu}F^{\mu\nu}$$

Trace anomaly for QED:

$$g_{\mu\nu}T^{\mu\nu}_{\text{QED}} = \frac{e^2}{24\pi^2}\cdot\frac{1}{\epsilon}\cdot\epsilon\cdot F_{\mu\nu}F^{\mu\nu} = +\frac{\beta}{2e}F_{\mu\nu}F^{\mu\nu}$$

The total stress tensor by including the quantum correction from gauge field:

$$\begin{aligned} T^{(2)\mu\nu}_{\text{ee}} = & \frac{1}{6\pi^2}\left(\hat{\kappa}+\ln\frac{\Lambda}{T}\right)\left(\frac{1}{4}g^{\mu\nu}F_{\gamma\beta}F^{\gamma\beta}-F^{\gamma\mu}F_{\gamma}{}^{\nu}\right) \\ & +\frac{1}{24\pi^2}\left[u^\mu u^\nu E^2-\Delta^{\mu\nu}\left(E^2+2B^2\right)+4\left(E^\mu E^\nu+B^\mu B^\nu\right)\right] \\ & +\frac{1}{8\pi^2}\left[\left(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma+\left(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma\right] \end{aligned}$$

Trace anomaly:

$$g^{\mu\nu}T_{\mu\nu} = \frac{e^2}{24\pi^2}F_{\mu\nu}F^{\mu\nu}$$

The conservation Law

Constraint conditions

$$\begin{aligned}\partial_\mu \beta_\nu + \partial_\nu \beta_\mu &= 0,\\ \partial_\mu \bar{\mu} + F_{\mu\nu}\beta^\nu &= 0,\\ \partial_\mu \bar{\mu}_5 &= 0,\end{aligned}$$

Conservation Laws

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= F^{\nu\mu} j_\mu,\\ \partial_\mu j^\mu &= 0,\\ \partial_\mu j_5^\mu &= -\frac{1}{2\pi^2} E\cdot B\end{aligned}$$

Symmetric and antisymmetric part of stress tensor:

$$T^{\mu\nu} = T_S^{\mu\nu} + T_A^{\mu\nu}$$

$$\partial_\mu T_S^{\mu\nu} = F^{\nu\mu} j_\mu, \qquad \partial_\mu T_A^{\mu\nu} = 0$$

Chiral anomaly:

$$\partial_\mu j_5^\mu = \partial_\mu j_5^{(1)\mu} = T B^\mu \partial_\mu \frac{\xi_{B5}}{T} = \frac{T}{2\pi^2} B^\mu \partial_\mu \bar{\mu} = -\frac{1}{2\pi^2} E\cdot B$$

How does the chiral anomaly emerges from quantum kinetic theory?

Chiral anomaly in QKT

Chiral anomaly from chiral kinetic theory: f particle $\bar{f}$ antiparticle

$$\partial_\mu j_5^\mu = \hbar E\cdot B\int\frac{d^3\mathbf{p}}{4\pi^3}\mathbf{\Omega}\cdot\nabla_{\mathbf{p}}(f+\bar{f}) = -\hbar E\cdot B\int\frac{d^3\mathbf{p}}{4\pi^3}(f+\bar{f})\nabla_{\mathbf{p}}\cdot\mathbf{\Omega}$$

Berry curvature: $\mathbf{\Omega} = \frac{\mathbf{p}}{2|\mathbf{p}|^3}$ Berry monopole: $\nabla_p\cdot\mathbf{\Omega} = 2\pi\delta^3(\mathbf{p})$

$$\partial_\mu j_5^\mu = -\frac{\hbar E\cdot B}{2\pi^2}\left[f(\mathbf{p}=0)+\bar{f}(\mathbf{p}=0)\right]$$

$f(\mathrm{p}=0)+\bar{f}(\mathrm{p}=0)=1$

$$= -\frac{\hbar}{2\pi^2}E\cdot B$$

Stephanov & Yin PRL 109,(2012)162001, Son & Yamamoto PRD 87 (2013) 8, 085016

Chiral anomaly in QKT

Fujikawa et al PRA2005,PRD2005,PRD2006

Berry's phase and chiral anomaly basically different

Mueller et al PRD2017,PRD2018,PRD2019

Berry's phase and chiral anomaly arise from different part

Hidaka et al PRD2018

Chiral anomaly from the non-trivial boundary condition

JHG et al PRD2018

New possible source term contributing to chiral anomaly

Yee et al PRD2020

Chiral anomaly and Berry connection from Feynman diagram

Chiral anomaly from Dirac sea

Wigner equation: $\nabla_\mu \mathscr{J}_s^{(1)\mu} = 0$ $\Rightarrow$ $\partial^x_\mu \mathscr{J}_s^{(1)\mu} = F_{\mu\nu}\partial_p^\nu \mathscr{J}_s^{(1)\mu}$

Take difference and integrate over p: $\partial^x_\mu j_5^\mu = F_{\mu\nu}\int d^4p\, \partial_p^\nu \mathscr{J}_s^{(1)\mu}$

$$\mathscr{J}^{(1)}_{s\mu} = -\frac{s}{2}\tilde{\Omega}_{\mu\lambda}p^\lambda f_s'\delta(p^2) - s\tilde{F}_{\mu\lambda}p^\lambda f_s\delta'(p^2) \qquad f_s = \frac{1}{4\pi^3}\left[\theta(p_0)\frac{1}{e^{(\beta\cdot p-\bar{\mu}_s)}+1} + \theta(-p_0)\left(\frac{1}{e^{-(\beta\cdot p-\bar{\mu}_s)}+1}-1\right)\right]$$

Chen, Pu, Q. Wang & X.N. Wang PRL2013

$$\partial^x_\mu j_5^\mu = -\frac{E\cdot B}{4\pi^3}\int d^4p\,\partial^\mu[p_\mu\delta'(p^2)] = \left\{\begin{array}{l} -\frac{E\cdot B}{2\pi^2}\int\frac{d^4p_E}{2\pi^2}\partial_\mu\left(\frac{p_E^\mu}{p_E^4}\right) \\ -\frac{E\cdot B}{2\pi^2}\int\frac{d^3\mathbf{p}}{2\pi}\partial_{\mathbf{p}}\cdot\left(\frac{\mathbf{p}}{2|\mathbf{p}|^3}\right)\end{array}\right\} = -\frac{E\cdot B}{2\pi^2}$$

4d Berry monopole

3d Berry monopole

ArXiv:1910.11060 JHG, Z.T. Liang, Q. Wang; ArXiv:2002.04800 R.H. Fang, JHG

CKE Particle vs Antiparticle

CKE for particle by $\int_0^\infty dp_0$

$$\left(1+s\hbar\vec{B}\cdot\vec{\Omega}_p\right)\partial_t f_p + \left[\vec{v}+s\hbar\vec{E}\times\vec{\Omega}_p + s\hbar\left(\hat{\vec{p}}\cdot\vec{\Omega}_p\right)\vec{B}\right]\cdot\vec{\nabla}_x f_p + \left[\tilde{\vec{E}}+\vec{v}\times\vec{B}+s\hbar\vec{E}\cdot\vec{B}\vec{\Omega}_p\right]\cdot\vec{\nabla}_p f_p + s\hbar\vec{E}\cdot\vec{B}(\vec{\nabla}_p\cdot\vec{\Omega}_p)f_p = 0$$

CKE for antiparticle by $\int_{-\infty}^0 dp_0$

$$\left(1-s\hbar\vec{B}\cdot\vec{\Omega}_p\right)\partial_t \bar{f}_p + \left[\vec{v}-s\hbar\vec{E}\times\vec{\Omega}_p - s\hbar\left(\hat{\vec{p}}\cdot\vec{\Omega}_p\right)\vec{B}\right]\cdot\vec{\nabla}_x \bar{f}_p - \left[\tilde{\vec{E}}+\vec{v}\times\vec{B}-s\hbar\vec{E}\cdot\vec{B}\vec{\Omega}_p\right]\cdot\vec{\nabla}_p \bar{f}_p + s\hbar\vec{E}\cdot\vec{B}(\vec{\nabla}_p\cdot\vec{\Omega}_p)(\bar{f}_p-1) = 0$$

Null normal contribution

Chiral anomaly:

$$\partial_\mu j_s^\mu = -s\hbar\vec{E}\cdot\vec{B}\int\frac{d^3\vec{p}}{(2\pi)^3}\vec{\nabla}_p\cdot\left[\vec{\Omega}(f_p+\bar{f}_p)\right] + s\hbar\vec{E}\cdot\vec{B}\int\frac{d^3\vec{p}}{(2\pi)^3}\vec{\nabla}_p\cdot\vec{\Omega}$$

Dirac sea contribution

Chiral anomaly for massive fermion

Chiral anomaly for massive fermion:

$$\partial_\mu j_5^\mu = -2mj_5 - \frac{1}{2\pi^2} E\cdot B \qquad j_5 = \int d^4p \mathscr{P}$$

Chiral anomaly for massive fermion from Wigner equation:

$$\partial^x_\mu j_5^\mu = -2mj_5 - \frac{E\cdot B}{4\pi^3}\int d^4p \partial^\mu [p_\mu \delta'(p^2 - m^2)]$$

$$= \left\{ \begin{array}{l} -2mj_5 - \frac{E\cdot B}{2\pi^2}\int \frac{d^4p_E}{2\pi^2}\partial_\mu \left(\frac{p_E^\mu}{(p_E + m^2)^2}\right) \\ -2mj_5 - \frac{E\cdot B}{2\pi^2}\int \frac{d^3\mathbf{p}}{2\pi}\partial_{\mathbf{p}}\cdot\left(\frac{\mathbf{p}}{2(\mathbf{p}^2+m^2)^{3/2}}\right) \end{array}\right\} = -2mj_5 - \frac{E\cdot B}{2\pi^2}$$

Modified Berry curvature: $\boldsymbol{\Omega} = \frac{\mathbf{p}}{2(\mathbf{p}^2 + m^2)^{3/2}}$ No exact Berry monopole:

ArXiv:1910.11060 JHG, Z.T. Liang, Q. Wang; ArXiv:2002.04800 R.H. Fang, JHG

Nonperturbative calculation

Chiral fermion in uniform magnetic field. The summation over Landau levels can be transformed into integration by Abel-Plana formula

$$\frac{1}{2}\mathcal{F}(0)+\sum_{n=1}^{\infty}\mathcal{F}(n)=\int_0^{\infty}dt\mathcal{F}(t)+i\int_0^{\infty}dt\frac{\mathcal{F}(it)-\mathcal{F}(-it)}{e^{2\pi t}-1}.$$

Normal ordered energy density for the righthand: $b=2eB\beta^2=2eB/T^2$

$$T^{00} \;=\; \ldots+\frac{b^2\ln b^2}{384\pi^2\beta^4}+\frac{b^2}{96\pi^2\beta^4}C_1(\bar{\mu}_s)+\frac{1}{2\pi^2\beta^4}\sum_{n=1}^{\infty}K_{2n+2}C_{2n+1}(\bar{\mu}_s)b^{2n+2},$$

Unnormal ordered energy density for the righthand:

$$T^{00} \;=\; \ldots-\varepsilon_0+\frac{b^2}{96\pi^2}\left(C_1(\bar{\mu}_s)+\ln\frac{\Lambda}{T}\right)+\frac{1}{2\pi^2\beta^4}\sum_{n=1}^{\infty}K_{2n+2}C_{2n+1}(\bar{\mu}_s)b^{2n+2},$$

$$C_{2n+1}(\bar{\mu}_s)=-\delta_{n,0}+\frac{1}{(4n+1)!}\int_0^{\infty}dy\ln y\frac{d^{4n+1}}{dy^{4n+1}}\left(\frac{1}{e^{y+\bar{\mu}_s}+1}+\frac{1}{e^{y-\bar{\mu}_s}+1}\right)$$

ArXiv:2005.08512 R.H. Fang, JHG, D.F. Hou, C. Zhang

Summary

- **The charge currents and stress tensor up to second order $\hbar$ have been obtained from Wigner function approach.**
- **The charge and energy densities and the pressure have contributions from the vorticity and electromagnetic field at the second order.**
- **The vector and axial Hall currents can be induced along the direction orthogonal to the vorticity and electromagnetic field at the second order.**
- **Chiral anomaly in quantum kinetic theory can be derived from the Dirac sea or the vacuum contribution in the un-normal-ordered Wigner function.**

Thanks for your attention!