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Nuclear Physics A 675 (2000) 531–558

NUCLEAR PHYSICS A

www.elsevier.nl/locate/npe

Damping of multiphonon giant resonances

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Received 19 August 1999; revised 21 February 2000; accepted 21 March 2000

Abstract

The phonon damping model (PDM) is applied to derive the equations that describe the damping of three-, and n-phonon giant resonances. As examples of the application of this approach, the results of numerical calculations for the double giant resonance (DGDR) ($n = 2$) and triple giant dipole resonance (TGDR) ($n = 3$) in ^{90}Zr, ^{120}Sn and ^{208}Pb are discussed and compared with those obtained by folding independent giant dipole resonances (GDRs) (the folding results). For the DGDR in the double magic nucleus ^{208}Pb, we found that these results are very close to the folding results. In the open-shell nuclei ^{90}Zr and ^{120}Sn, a clear deviation from the folding results is observed in calculations in agreement with the experimental trend. The results for the integrated strength and energy of TGDR are found to be much closer to the folding results in all three nuclei. The TGDR widths in the open shell nuclei are found to be larger than the folding results. We also show that the relationship $\mathcal{S}_1^{(2)} = 4\mathcal{S}_1^{(1)}\mathcal{S}_0^{(1)}$, which connects the energy-weighted sum (EWS) $\mathcal{S}_1^{(2)}$ of the DGDR strengths to the EWS $\mathcal{S}_1^{(1)}$ and the non-energy-weighted sum of strengths (NEWS) $\mathcal{S}_1^{(1)}$ of GDR, does not hold in any approximation in which the energy of the two-phonon state is deviated from the sum of energies of the two one-phonon states due to anharmonicity. A small deviation of the two-phonon energy is enough to cause a noticeable change in the DGDR strength compared to the independent-phonon picture. A new sum rule relationship is derived within the PDM.

PACS: 24.30.Cz; 21.60.Jz; 21.10.Pc; 21.60.-n; 23.20.-g
Keywords: Multiphonon giant resonance; Strength function; Energy-weighted sum rule; Anharmonicity

1. Introduction

The recent observation of the double giant dipole resonances (DGDR) in relativistic heavy-ion reactions via Coulomb excitation [1,2] and pion-induced charge exchange

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0375-9474/00/$ – see front matter
PII: S0375-9474(00)00186-X

reactions [3] has sparked intensive studies and hot debates on the issue of multiphonon excitations (see Ref. [4] for a recent review).

In the simplest picture, the giant dipole resonance (GDR), as a collective motion of all protons against neutrons in nuclei, corresponds to the transition of the nucleus from ground state (g.s.) to the first excited state of an idealized dipole vibration that exhausts all the oscillator strength [5]. This first harmonic oscillator quantum phonon serves as the building block of the multiple GDR if the latter is understood as a resonance built on top of other resonances. In such a harmonic picture the energy (frequency) of a n-phonon dipole resonance would equal exactly to n times of the energy of the GDR. In reality, the GDR photoabsorption cross section has a Lorentzian form with a full width at the half maximum (FWHM) Γ, which is particularly small in double magic nuclei ($\simeq 4$ MeV in ^{208}Pb), and a centroid energy close to the energy of the idealized single dipole phonon. The resonance width is understood as follows. The GDR strength is redistributed over a collection of dipole oscillators clustered around the idealized dipole oscillator. This effects is called the Landau splitting (or damping) of the GDR. It is well described within the random-phase-approximation (RPA), where each phonon excitation is composed of a coherent superposition of many particle–hole (ph) configurations. The transfer of the energy of the GDR to other modes of nuclear motions such as $2p2h$ (or $1p1h\otimes$ phonon) ones splits the RPA phonon states further into a dense distribution of many nuclear levels, each of which carries a certain portion of the total GDR strength. The envelope of this distribution can be approximated by a Breit–Wigner shape whose width is called the spreading width $\Gamma^{\downarrow}$. The GDR phonon excitations are also coupled to the continuum region. This coupling leads to an additional contribution called the escaped with $\Gamma^{\uparrow}$. Since the escaped width $\Gamma^{\uparrow}$ is of order of hundreds keV in heavy nuclei, the major contribution to the GDR width comes from the spreading width $\Gamma^{\downarrow}$.

Such a damping mechanism of the GDR makes the whole picture of the multiple GDR more complicated. The first correction to the over simplified harmonic picture of the multiple GDR resonance is to assume that each GDR can be still considered as a single collective phonon but with a definite damping in such a way that it can be approximated by a Lorentzian or Breit–Wigner shape with a FWHM $\Gamma_{\rm GDR} \simeq \Gamma^{\downarrow}$ centered at the experimentally observed GDR energy $E_{\rm GDR}$. The strength distribution of the multiple GDR can be then obtained by folding these noninteracting GDRs [6,7]. As a result the width of the n-phonon GDR is equal to $n\Gamma_{\rm GDR}$. If the Gaussian distribution is used instead of the Lorentzian one to approximate the GDR, the width of the n-independent-phonon GDR comes out as $\sqrt{n}\Gamma_{\rm GDR}$ after the folding [6,7]. Since such harmonic picture neglects the coupling between the GDRs, the fundamental question in the study of the multiphonon resonances is how large the anharmonicity is, or how strongly the coupling between GDR phonons makes the description of the multiphonon resonance deviate from the independent-phonon (harmonic) picture. The reason of raising this question comes from the experimental systematic of the DGDR.

The most representative data for the DGDR have been extracted from the exclusive experiments in heavy-ion collisions using ^{135}Xe and ^{208}Pb projectiles at nearly relativistic kinetic energies [1,2]. These data have shown that the deviation from the harmonic picture

of the DGDR in the closed-shell nucleus (^{208}Pb) is much smaller than in the open-shell one (^{136}Xe). While this deviation from the harmonic picture is not that large in the energy and the width of the DGDR, the effect is dramatic in the DGDR cross section in the Coulomb excitation. In ^{208}Pb, the cross section is about 33% larger than the value obtained by folding the cross-sections of two independent GDRs (the folding results). In ^{136}Xe, the cross section has been observed (215 ± 32 mb) to be strongly enhanced about 2–3 times as compared to the folding results (70–87 mb). The most recent data have shown that there is a certain dependence of the extracted value of the DGDR cross section in ^{136}Xe on the thickness of the target nucleus [8]. However, the large enhancement compared to the folding result remains.

Even though there is a clear indication that anharmonicity and nonlinearity may be the source of the enhancement in the cross section of the DGDR [9–12], it is far beyond a practicable modification to include microscopically the coupling between two GDRs that forms the DGDR. Several attempts have been made in this direction. We mention here only two approaches, which include the configuration mixing explicitly in the wave functions. The second RPA (SRPA) has been extended to the DGDR but the spreading has not been yet treated [13]. Therefore, the DGDR has been obtained to be nearly harmonic. Another example is the calculations in Refs. [15,16] within the quasiparticle-phonon model (QPM) [14]. In Ref. [15] the spreading width of each one-phonon state due to coupling to two-phonon states in ^{136}Xe has been calculated. The DGDR was formed as a direct product of these two damped GDR one-phonon states. The results in Ref. [15] are more or less equivalent to those obtained by folding two independent GDRs with a given width Γ_{GDR} for each of them. The three-phonon terms are needed in the QPM wave function to calculate the fragmentation of the two-phonon states [14]. However, because of the large dimension of the determinant ($\sim 10^3$–10^5 without the three-phonon terms), the QPM equation for the energy is solved practically only with a substantial truncation of the one-phonon basis [14]. In Ref. [16] the five most collective RPA one-phonon GDR states have been selected to form the DGDR states in ^{136}Xe. Moreover, these DGDR states are coupled only to the three-phonon configurations in which two of the three-phonon components coincide with the DGDR states. In this way, the coupling to three-phonon terms gives the width of the two-phonon states similarly to what the coupling to two-phonon states does for the width of the one-phonon states. Once again, this is equivalent to the folding results although there were several thousands three-phonon states included in the calculations. The quantitative effect of the anharmonicity in the DGDR still remains an open question.

Any approximated approach to the giant resonances must fulfill certain sum rules. In the case of the GDR this is the well-known model-independent Thomas–Reiche–Kuhn (TRK) sum rule. Whether similar model-independent sum rules can be found for multiphonon resonances is an interesting question, since such a sum rule would be useful to test the validity of any approximation. In Ref. [17] an identity between the energy-weighted sum (EWS) of dipole strengths and the ground-state expectation value of the double commutator of the Hamiltonian and the dipole operator D has been considered also for the two-body operator D^2. The authors of Ref. [17] have then shown that, if the double commutator $[D,[V,D]]$ between the dipole operator D and the potential part V of the

model Hamiltonian is zero, the EWS $\mathcal{S}_1^{(2)}$ of the DGDR strength is equal to $4\mathcal{S}_1^{(1)}\mathcal{S}_0^{(1)}$, where $\mathcal{S}_1^{(1)}$ and $\mathcal{S}_0^{(1)}$ are the EWS and non-energy weighted sum (NEWS) of strengths of the GDR, respectively. The folding result of the DGDR, which gives $E_{\mathrm{DGDR}} = 2E_{\mathrm{GDR}}$, satisfies this sum rule relationship. If this relationship holds even when the anharmonicity is included in the DGDR, it would mean that there is no way for the DGDR strength to be enhanced as compared to the harmonic limit since both the EWS and NEWS of the GDR strengths are known independently of models. However, the recent study with the complete Hamiltonian of the QPM in Ref. [18] has shown that, in general, the EWS of the DGDR strengths can be enhanced due to the anharmonicity caused by such scattering term since the condition $[D, [V, D]] = 0$ does not hold. We notice that the authors of Refs. [9–11] have also retained the pp and hh terms in the dipole operator. This opens a new route that leads to a certain enhancement of the DGDR cross-section.

This situation requires a simple, yet microscopic approach that can account for the anharmonicity in the n-phonon giant resonance in a transparent way. A step in this direction has been taken in Ref. [12], where the Phonon Damping Model (PDM) [19–21] was applied to derive the equation that describe the damping of the DGDR ($n = 2$). In the present paper, we will extend this approach to the general case of n-phonon giant resonances. The explicit derivation will be present for the TGDR ($n = 3$) while the generalization to an arbitrary n will come as a natural extension from the DGDR and TGDR. We will demonstrate rigorously that, starting from a general many-body Hamiltonian, the relationship $\mathcal{S}_1^{(2)} = 4\mathcal{S}_1^{(1)}\mathcal{S}_0^{(1)}$ [17] is violated within any approximation that uses the (RPA) phonons to build the DGDR and that leads to the DGDR energy different from $2E_{\mathrm{GDR}}$ due to anharmonicity. We will show that this relationship is fulfilled only in the case when the energy shift $\Delta E = E_{\mathrm{DGDR}} - 2E_{\mathrm{GDR}}$ is zero. We will also show that a rather small energy shift ΔE is enough to cause a large change in the EWS of strengths. The results of numerical calculations for the strength function, energy and width of the DGDR and triple GDR (TGDR) in ^{90}Zr, ^{120}Sn, and ^{208}Pb are analyzed in comparison with those obtained by folding independent GDRs. In particular, this is the first time that the numerical results of the damping of three-phonon resonances are presented.

The paper is organized as follows. In Section 2 we derive the equations for the three-phonon resonance within the PDM and propose its extension to the n-phonon resonance. In Section 3, we present our study of the sum rule relationship between EWS of the DGDR strengths and the DGDR strengths. The results of numerical calculations are presented and analyzed in Section 4. The paper is summarized in the last section, where conclusions are drawn.

2. Multiphonon resonances within the phonon damping model (PDM)

The PDM describes the coupling of collective oscillations (phonons) to the field of incoherent nucleon pairs [19–21] making use of a Hamiltonian that is composed of three terms:

$$H = \sum_s E_s a_s^\dagger a_s + \sum_q \omega_q Q_q^\dagger Q_q + \sum_{s,s',q} F_{ss'}^{(q)} a_s^\dagger a_{s'} (Q_q^\dagger + Q_q). \tag{2.1}$$

The first term on the right-hand side (rhs) of Eq. (2.1) corresponds to the field of independent single particles, where $a_s^\dagger$ and a_s are the creation and destruction operators of a particle or hole state with energy $E_s = \epsilon_s - \epsilon_{\rm F}$, ϵ_s the single-particle energy and $\epsilon_{\rm F}$ the Fermi energy. The energy E_s is called the single-particle energy for simplicity whenever there is no confusion with ϵ_s. The second term is the phonon field, where $Q_q^\dagger$ and Q_q are the creation and destruction operators of a phonon with energy ω_q. The last term describes the coupling between the phonon field and the field of all possible ph, pp, and hh pairs. The indices s and s' denote particle (p, $E_p > 0$) or hole (h, $E_h < 0$), while the index q is reserved for the phonon state $q = \{\lambda, i\}$ with multipolarity λ (the projection μ of λ in the phonon index is omitted for simplicity). In general, the sums in the last two terms are carried out over $\lambda \geqslant 1$. The form of PDM Hamiltonian (2.1) is quite common in many microscopic approaches to nuclear collective excitations. The difference between various models is in the way of calculating the single-particle energy E_s, phonon energy ω_q and phonon structure using different effective interactions, which lead to different vertices $F_{ss'}^{(q)}$.

The basic equations for the two-phonon resonance have been derived and discussed thoroughly in Ref. [12]. Following the same line, we will present in this section the formalism for the three-phonon resonance and generalize it to the n-phonon resonance.

The damping of the three-phonon excitation is considered as the result of coupling of three single phonon vibrations, which is damped by coupling to incoherent nucleon pairs. This process can be studied considering the following double-time Green's functions which describe:

(1) the propagation of three free phonons:

$$G_{q_1q_2q_3;q_1'q_2'q_3'}(t-t') = \left\langle\!\left\langle Q_{q_1}(t) Q_{q_2}(t) Q_{q_3}(t);\, Q_{q_1'}^\dagger(t') Q_{q_2'}^\dagger(t') Q_{q_3'}^\dagger(t') \right\rangle\!\right\rangle; \tag{2.2}$$

(2) the transition between "nucleon pair $\otimes$ two-phonon" and three-phonon configurations:

$$\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(t-t') = \left\langle\!\left\langle a_s^\dagger(t) a_{s'}(t) Q_{q_1}(t) Q_{q_2}(t);\, Q_{q_1'}^\dagger(t') Q_{q_2'}^\dagger(t') Q_{q_3'}^\dagger(t') \right\rangle\!\right\rangle. \tag{2.3}$$

In Eqs. (2.2) and (2.3) the standard notation for the retarded double-time Green's function is used [23,24]:

$$\left\langle\!\left\langle A(t);\, B(t') \right\rangle\!\right\rangle = -i\theta(t-t') \left\{ \left\langle A(t) B(t') \right\rangle \mp \left\langle B(t') A(t) \right\rangle \right\}, \tag{2.4}$$

for any operators $A(t)$ and $B(t')$ with $\langle \ldots \rangle$ denoting the average over the grand canonical ensemble at temperature T. The equation of motion for the double-time Green's function $\langle\langle A(t);\, B(t') \rangle\rangle$ with respect to the Hamiltonian H can be derived from

$$i\frac{d}{dt} \left\langle\!\left\langle A(t);\, B(t') \right\rangle\!\right\rangle = \delta(t-t') \left\langle \left[A(t),\, B(t) \right] \right\rangle + \left\langle\!\left\langle \left[A(t),\, H(t) \right];\, B(t') \right\rangle\!\right\rangle \tag{2.5}$$

following the standard procedure in Ref. [24].

Applying Eq. (2.5) to the function (2.2) and Hamiltonian (2.1), we derive the following exact equation, which couples the function (2.2) to the function (2.3):

$$
\begin{aligned}
& i\frac{d}{dt}G_{q_1q_2q_3;q_1'q_2'q_3'}(t-t') \\
&\quad = \delta(t-t')(1+\nu_{q_1}+\nu_{q_2}+\nu_{q_3})\big(\delta_{q_1q_1'}\delta_{q_2q_2'}\delta_{q_3q_3'}+\delta_{q_1q_1'}\delta_{q_2q_3'}\delta_{q_3q_2'} \\
&\qquad + \delta_{q_2q_1'}\delta_{q_1q_2'}\delta_{q_3q_3'}+\delta_{q_2q_1'}\delta_{q_3q_2'}\delta_{q_1q_3'}+\delta_{q_3q_1'}\delta_{q_1q_2'}\delta_{q_2q_3'}+\delta_{q_1q_3'}\delta_{q_2q_2'}\delta_{q_3q_1'}\big) \\
&\qquad + (\omega_{q_1}+\omega_{q_2}+\omega_{q_3})G_{q_1q_2q_3;q_1'q_2'q_3'}(t-t') \\
&\qquad + \sum_{s,s'}\big[F^{(q_1)}_{ss'}\mathcal{G}_{ss'q_2q_3;q_1'q_2'q_3'}(t-t')+F^{(q_2)}_{ss'}\mathcal{G}_{ss'q_1q_3;q_1'q_2'q_3'}(t-t') \\
&\qquad\qquad + F^{(q_3)}_{ss'}\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(t-t')\big]. \qquad (2.6)
\end{aligned}
$$

In Eq. (2.6) the phonon occupation number ν_q occurs as a result of averaging over the grand canonical ensemble $\langle Q^\dagger_q Q_{q'}\rangle = \delta_{qq'}\nu_q$. The equation for the function (2.3), which enters in the rhs of Eq. (2.6), is derived from Eq. (2.5) in the same way. The exact result contains the function (2.3), as well as higher-order Green's functions in the rhs. Confining ourselves to the lowest-order coupling, we can close this hierarchy up to the functions (2.2) and (2.3). This is achieved by applying the decoupling scheme in Ref. [24] to lower the order of the Green's functions, e.g.:

$$
\begin{aligned}
& \big\langle\big\langle a^\dagger_s(t)a_{s''}(t)Q_{q_1}(t)Q_{q_2}(t)Q_{q_3}(t);\, Q^\dagger_{q_1'}(t')Q^\dagger_{q_2'}(t')Q^\dagger_{q_3'}(t')\big\rangle\big\rangle \\
&\quad \approx \delta_{ss''}n_s G_{q_1q_2q_3;q_1'q_2'q_3'}(t-t'), \qquad (2.7)
\end{aligned}
$$

where n_s is the single-particle occupation number $n_s = \langle a^\dagger_s a_s\rangle$. The decoupling is applied whenever there is a sum over the single-particle indices s''. Within this approximation scheme the equation for the function (2.3) can be truncated to the following form:

$$
\begin{aligned}
i\frac{d}{dt}\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(t-t') &= (E_{s'}-E_s+\omega_{q_1}+\omega_{q_{12}})\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(t-t') \\
&\quad + (n_s-n_{s'})\sum_q F^{(q)}_{s's}G_{qq_1q_2;q_1'q_2'q_3'}(t-t'). \qquad (2.8)
\end{aligned}
$$

The Fourier transforms of Eqs. (2.6) and (2.8) provide us with the set of two equations in the energy plane E:

$$
\begin{aligned}
& (E-\omega_{q_1}-\omega_{q_2}-\omega_{q_3})G_{q_1q_2q_3;q_1'q_2'q_3'}(E) \\
&\quad - \sum_{s,s'}\big[F^{(q_1)}_{ss'}\mathcal{G}_{ss'q_2q_3;q_1'q_2'q_3'}(E)+F^{(q_2)}_{ss'}\mathcal{G}_{ss'q_1q_3;q_1'q_2'q_3'}(E)+F^{(q_3)}_{ss'}\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(E)\big] \\
&\quad = \frac{1}{2\pi}(1+\nu_{q_1}+\nu_{q_2}+\nu_{q_3}) \\
&\qquad \times(\delta_{q_1q_1'}\delta_{q_2q_2'}\delta_{q_3q_3'}+\delta_{q_1q_1'}\delta_{q_2q_3'}\delta_{q_3q_2'}+\delta_{q_2q_1'}\delta_{q_1q_2'}\delta_{q_3q_3'} \\
&\qquad + \delta_{q_2q_1'}\delta_{q_3q_2'}\delta_{q_1q_3'}+\delta_{q_3q_1'}\delta_{q_1q_2'}\delta_{q_2q_3'}+\delta_{q_1q_3'}\delta_{q_2q_2'}\delta_{q_3q_1'}), \qquad (2.9) \\
& (E-E_{s'}+E_s-\omega_{q_1}-\omega_{q_2})\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(E) \\
&\quad - (n_s-n_{s'})\sum_q F^{(q)}_{s's}G_{qq_1q_2;q_1'q_2'q_3'}(E)=0. \qquad (2.10)
\end{aligned}
$$

Elimination of $\mathcal{G}_{ss'q_1q_2;q_1'q_2'q_3'}(E)$ by expressing it in terms of $G_{qq_1q_2;q_1'q_2'q_3'}(E)$ using Eq. (2.10) and insertion of the result in Eq. (2.9) produce an equation for $G_{q_1q_2q_3;q_1'q_2'q_3'}(E)$.

For the propagation of a three-phonon configuration $(q_1, q_2, q_3) = (q_1', q_2', q_3')$ this equation takes a simple form as

$$G_{q_1q_2q_3}(E) = \frac{3!}{2\pi}\frac{1+\nu_{q_1}+\nu_{q_2}+\nu_{q_3}}{E-\omega_{q_1}-\omega_{q_2}-\omega_{q_3}-P_{q_1q_2q_3}(E)}, \tag{2.11}$$

where $P_{q_1q_2q_3}(E)$ is the three-phonon polarization operator:

$$P_{q_1q_2q_3}(E) = \sum_{s,s'}(n_s - n_{s'})\left[\frac{F^{(q_1)}_{ss'}F^{(q_1)}_{s's}}{E-E_{s'}+E_s-\omega_{q_2}-\omega_{q_3}} + \frac{F^{(q_2)}_{ss'}F^{(q_2)}_{s's}}{E-E_{s'}+E_s-\omega_{q_1}-\omega_{q_3}} + \frac{F^{(q_3)}_{ss'}F^{(q_3)}_{s's}}{E-E_{s'}+E_s-\omega_{q_1}-\omega_{q_2}}\right]. \tag{2.12}$$

In the lhs of Eqs. (2.9) and (2.12) we omit the subscripts (q_1', q_2', q_3') as they coincide with (q_1, q_2, q_3).

The imaginary and real parts of the analytic continuation of polarization operator $P_{q_1q_2q_3}(E)$ into the complex energy plane $E = \omega \pm i\varepsilon$ (ω is real and $\varepsilon \to 0$) give the damping and the energy shift of the three-phonon excitation, respectively [19–21,24]. The analytical expression of the damping is

$$\gamma_{q_1q_2q_3}(\omega) = \pi\left|\sum_{s,s'}(n_s-n_{s'})\left[F^{(q_1)}_{ss'}F^{(q_1)}_{s's}\delta(\omega-E_{s'}+E_s-\omega_{q_2}-\omega_{q_3}) + F^{(q_2)}_{ss'}F^{(q_2)}_{s's}\delta(\omega-E_{s'}+E_s-\omega_{q_1}-\omega_{q_3}) + F^{(q_3)}_{ss'}F^{(q_3)}_{s's}\delta(\omega-E_{s'}+E_s-\omega_{q_1}-\omega_{q_2})\right]\right|. \tag{2.13}$$

The energy shift has the form

$$P_{q_1q_2q_2}(\omega) = \mathcal{P}\sum_{s,s'}(n_s-n_{s'})\left[\frac{F^{(q_1)}_{ss'}F^{(q_1)}_{s's}}{\omega-E_{s'}+E_s-\omega_{q_2}-\omega_{q_3}} + \frac{F^{(q_2)}_{ss'}F^{(q_2)}_{s's}}{\omega-E_{s'}+E_s-\omega_{q_1}-\omega_{q_3}} + \frac{F^{(q_3)}_{ss'}F^{(q_3)}_{s's}}{\omega-E_{s'}+E_s-\omega_{q_1}-\omega_{q_2}}\right] \tag{2.14}$$

with $\mathcal{P}$ denoting the principal value of the corresponding integral.

The excitation energy of the three-phonon state is defined as the solution $\bar{\omega}$ of the equation for the pole of the Green's function (2.11):

$$\bar{\omega} - (\omega_{q_1}+\omega_{q_2}+\omega_{q_3}) - P_{q_1q_2q_3}(\bar{\omega}) = 0. \tag{2.15}$$

Eqs. (2.13)–(2.15) are the main equations for the study of three-phonon excitations within the PDM.

Recalling now the main equations for the damping of the GDR [19] and DGDR [12] within the PDM, we can easily generalize these equations to the case of n-phonon state by considering the Green's functions

$$G_{q_1\ldots q_n;q_1'\ldots q_n'}(t-t') = \left\langle\!\left\langle Q_{q_1}(t)\ldots Q_{q_n}(t);\, Q^\dagger_{q_1'}(t')\ldots Q^\dagger_{q_n'}(t')\right\rangle\!\right\rangle \quad \text{and}$$

$$\mathcal{G}_{ss'q_1\ldots q_{n-1};q_1'\ldots q_n'}(t-t') = \left\langle\!\left\langle a^\dagger_s(t)a_{s'}(t)Q_{q_1}(t)\ldots Q_{q_{n-1}}(t);\; Q^\dagger_{q_1'}(t')\ldots Q^\dagger_{q_n'}(t')\right\rangle\!\right\rangle.$$

These equations are given below.

(i) The Green's function for the propagation of n-phonon state:

$$G_{q_1\ldots q_n}(E) = \frac{n!}{2\pi}\frac{1+\sum_{i=1}^{n}\nu_{q_i}}{E-\sum_{i=1}^{n}\omega_{q_i}-P_{q_1\ldots q_n}(E)}. \tag{2.16}$$

(ii) The damping of n-phonon state:

$$\gamma_{q_1\ldots q_n}(\omega) = \pi\left|\sum_{s,s'}(n_s-n_{s'})\sum_{i=1}^{n}F_{s's}^{(q_i)}\delta(\omega-E_{s'}+E_s-\sum_{j\neq i}^{n}\omega_{q_j})\right|. \tag{2.17}$$

(iii) The energy shift of n-phonon state compared to the sum of energies of n single phonon states:

$$P_{q_1\ldots q_n}(\omega) = \mathcal{P}\sum_{s,s'}(n_s-n_{s'})\sum_{i=1}^{n}\frac{F_{ss'}^{(q_i)}F_{s's}^{(q_i)}}{\omega-E_{s'}+E_s-\sum_{j\neq i}^{n}\omega_{q_j}}. \tag{2.18}$$

(iv) The energy of n-phonon state:

$$\bar{\omega}-\sum_{i=1}^{n}\omega_{q_i}-P_{q_1\ldots q_n}(\bar{\omega}) = 0. \tag{2.19}$$

The FWHM Γ_n of the n-phonon resonance is defined as twice the value of the damping (2.17) taken at $\omega=\bar{\omega}$ as $\Gamma_n = 2\gamma_{q_1\ldots q_n}(\bar{\omega})$.

The strength function $S_{q_1\ldots q_n}(\omega)$ is derived from the spectral intensity $J_{q_1\ldots q_n}(\omega)$ in a similar way as in the case of GDR [19–21] and DGDR [12]. The approximated form of the strength function is

$$S_{q_1\ldots q_n}(\omega) = \frac{n!}{\pi}\frac{\gamma_{q_1\ldots q_n}(\omega)}{(\omega-\bar{\omega})^2+\gamma_{q_1\ldots q_n}^2(\omega)}, \tag{2.20}$$

where it is assumed that the spectral intensity has a steep maximum at $\omega=\bar{\omega}$.

The k-moment of the multiphonon resonance is defined in the standard way as

$$m_k^{(n)} = \int_{E_1}^{E_2} S_{q_1\ldots q_n}(\omega)\omega^k d\omega, \tag{2.21}$$

where the cases with $k=0$ and 1 correspond to the NEWS $\mathcal{S}_0^{(n)}$ and EWS $\mathcal{S}_1^{(n)}$ of strengths, respectively. Putting $q_i=q$ for all i in Eqs. (2.16)–(2.21), we obtain the equations for n identical phonon resonance from which we recover the equations for the damping of the GDR, DGDR, and TGDR with $n=1$, 2, and 3, respectively. The phonon occupation number ν_q and single-particle occupation number n_s in Eqs. (2.6)–(2.19) can be approximated by the Bose–Einstein and Fermi–Dirac distributions at temperature T, respectively [20]. As we consider $T=0$ in the present paper, we have $\nu_q=0$, $n_p=0$, and $n_h=1$.

We notice that in the derivation of the equations for the multiphonon resonance we always neglected the terms that can be decoupled to be proportional to

$$\left\langle\!\left\langle a_s^\dagger(t)a_{s'}(t)Q_{q_1}(t)\ldots Q_{q_{n-2}}(t);\, Q_{q_1'}^\dagger(t')\ldots Q_{q_n'}^\dagger(t')\right\rangle\!\right\rangle \quad (n\geqslant 2).$$

As has been discussed in the cases of GDR [19] and DGDR [12], such terms are related to the single-particle damping caused by the mutual coupling to the phonon field. Their contribution to the damping of the collective phonon such as GDR can be omitted in the first order. This is tantamount to omitting the function similar to (2.3) in the hierarchy of higher-order Green functions, but in which the number of phonon operators at the time $\tau = t$ is smaller than $n - 1$. In Appendix A, we demonstrate that the contribution of such terms is really negligible for DGDR and TGDR.

It is also worth noticing that a nucleon pair $a_s^\dagger a_{s'}$ can be expanded in an infinite boson expansion series, e.g., of the Belyaev–Zelevinsky type [25]. Therefore, the derived damping of (multiple) phonon excitations caused by the last term at the rhs of Eq. (2.1) in fact describes not only the Landau damping, which is the spreading of a collection of harmonic oscillators (phonons), but also damping due to the coupling of multiphonon states with the *ph* pairs as clearly shown in Eq. (2.17).

3. On the sum rule relationship for the DGDR

A general sum rule relationship for the n-phonon resonance is absent at present. Moreover, the complexity of all possible transitions between the n-phonon resonance and the group of $(n-1)$-phonon resonances on which it is built makes such a relationship, if any, unlikely to be model-independent already for $n \geqslant 3$. In the present section we study only the EWS and NEWS of the DGDR strengths.

It is well known that for a given Hamiltonian with a two-body interaction the following identity takes place [25,26]:

$$\mathcal{S}_1 \equiv \sum_\nu (E_\nu - E_0)\left|\langle\nu|\,\widehat{\mathcal{O}}\,|0\rangle\right|^2 = \tfrac{1}{2}\langle 0|\big[\widehat{\mathcal{O}},\big[H,\widehat{\mathcal{O}}\big]\big]|0\rangle, \tag{3.1}$$

where $\widehat{\mathcal{O}}$ is a Hermitian operator and $\{|\nu\rangle\}$ the complete set of exact eigenstates with eigenvalues (energies) E_ν of the Hamiltonian H. The lhs of Eq. (3.1) is the EWS of strengths of transitions from the ground state $|0\rangle$ with energy E_0 to the excited states $|\nu\rangle$ generated by the operator $\widehat{\mathcal{O}}$. In the case of the dipole operator $\widehat{\mathcal{O}} = D$, if the potential V in the Hamiltonian were local and had no charge exchange part, it would commute with D. Hence the rhs of Eq. (3.1) would be equal to $NZ/(2MA)$ independently of models and of the structure of the ground state $|0\rangle$. Equation (3.1) becomes then $\mathcal{S}_1^{(1)} = NZ/(2MA)$, which is the well known TRK sum rule for the GDR [25,26].

The authors of Ref. [17] have extended the identity (3.1) to the two-phonon excitations to derive a model-independent sum rule for the DGDR. Putting $\widehat{\mathcal{O}} = D^2$ instead of D and evaluating the rhs of Eq. (3.1) in a similar way, it is easily to obtain that

$$\mathcal{S}_1^{(2)} = 4\frac{NZ}{2MA}\,\langle 0|\,D^2\,|0\rangle \equiv 4\mathcal{S}_1^{(1)}\mathcal{S}_0^{(1)}, \tag{3.2}$$

provided the following condition holds:

$$\big[D,[V,D]\big] = 0. \tag{3.3}$$

The authors of Ref. [17] concluded that Eq. (3.2) is a model-independent relationship and, since the EWS $\mathcal{S}_1^{(1)}$ and NEWS $\mathcal{S}_0^{(1)}$ of the GDR on the rhs of Eq. (3.2) are known, the unknown EWS $\mathcal{S}_1^{(2)}$ of the DGDR strength on the lhs of Eq. (3.2) cannot exceed the value in its rhs. Hence, there is no way to get any enhancement of the DGDR strength compared to the folding result, as the latter satisfies the rhs of Eq. (3.2) [17]. It is easy to see that, starting from a general many-body Hamiltonian, the condition (3.3) does not hold in general. For instance, neither the QPM Hamiltonian [14] nor the PDM Hamiltonian (2.1) satisfies the condition (3.3). Indeed, as the dipole operator D can be represented in the second quantization as a superposition of $a_s^\dagger a_{s'}$ with $ss' = ph$, pp' or hh' (see Eq. (3.13) for $\lambda = 1$ below), the commutator between D and the last term $V \equiv \sum_{ss'q} F_{ss'}^{(q)} a_s^\dagger a_{s'}(Q_q^\dagger + Q_q)$ of (2.1) is not zero as can be verified by a simple check using the exact commutation relations:

$$[a_s^\dagger a_{s'}, a_{s_1}^\dagger a_{s'_1}] = \delta_{s's_1} a_s^\dagger a_{s'_1} - \delta_{s'_1 s} a_{s_1}^\dagger a_{s'}. \tag{3.4}$$

As a results, we obtain for the commutator $[V, D]$ the following expression:

$$[V, D] = \sum_{ss'q} \sum_{s_1} F_{ss'}^{(q)} \big[\langle s_1 | \mathcal{M}(E1) | s' \rangle a_s^\dagger a_{s_1} - \langle s_1 | \mathcal{M}(E1) | s \rangle a_{s_1}^\dagger a_{s'} \big] (Q_q^\dagger + Q_q). \tag{3.5}$$

Similarly, one obtains for the double commutator $[D, [V, D]]$ the following expression:

$$\begin{aligned} \big[D, [V, D]\big] = \sum_{ss'q} \sum_{s_1 s'_1} F_{ss'}^{(q)} \big[& 2 \langle s'_1 | \mathcal{M}(E1) | s' \rangle \langle s | \mathcal{M}(E1) | s_1 \rangle a_{s_1}^\dagger a_{s'_1} \\ & - \langle s_1 | \mathcal{M}(E1) | s' \rangle \langle s'_1 | \mathcal{M}(E1) | s_1 \rangle a_s^\dagger a_{s'_1} \\ & - \langle s_1 | \mathcal{M}(E1) | s'_1 \rangle \langle s | \mathcal{M}(E1) | s_1 \rangle a_{s'_1}^\dagger a_{s'} \big] (Q_q^\dagger + Q_q). \end{aligned} \tag{3.6}$$

Neither the rhs of Eqs. (3.5) nor the rhs of Eq. (3.6) is a number. They contain operators, that do not cancel each other in general case. Therefore, the condition (3.3) does not hold for (3.6). This means that the relationship (3.2) cannot take place.

The nonzero value of the commutator $[V, D]$ in (3.5) also leads to the violation of the TKR sum rule. However, under a certain approximation, the expectation value of this commutator in the ground state can be considered to be equal zero, conserving the TKR sum rule. Within the PDM, the decoupling approximation of type (2.7) is made to close the hierarchy of the Green functions. We show below that this kind of decoupling approximation eventually makes the ground-state average $\langle 0 | [V, D] | 0 \rangle$ vanish, while it is not the case for $\langle 0 | [D, [V, D]] | 0 \rangle$.

Applying (2.7) to ground-state average values of Eqs. (3.5) and (3.6), we obtain:

$$\begin{aligned} \langle 0 | [V, D] | 0 \rangle \simeq \sum_{ss'q} \sum_{s_1} F_{ss'}^{(q)} \big[& \langle s_1 | \mathcal{M}(E1) | s' \rangle \delta_{ss_1} n_s \\ & - \langle s | \mathcal{M}(E1) | s_1 \rangle \delta_{s_1 s'} n_{s'} \big] \langle 0 | Q_q^\dagger + Q_q | 0 \rangle = 0 \end{aligned} \tag{3.7}$$

because $\langle 0 | Q_q^\dagger + Q_q | 0 \rangle$ is zero. It is important to emphasize that the decoupling scheme of type (2.7) (see also Ref. [20]) only allows the single-particle index s_1 (or s_2) of the dipole-operator matrix element $\langle s_2 | \mathcal{M}(E1) | s_1 \rangle$ to coincide with the index s (or s') of the

coupling matrix element $F^{(q)}_{ss'}$. Taking now the same level of decoupling into account, we obtain, in the same way, the following expression for the ground-state expectation value K of Eq. (3.6):

$$K \equiv \langle 0|\big[D,[V,D]\big]|0\rangle \simeq 2\sum_{ss'q}\sum_{s_1s'_1} F^{(q)}_{ss'}\,\langle s'_1|\mathcal{M}(E1)|s'\rangle\,\langle s|\mathcal{M}(E1)|s_1\rangle\langle 0|\,a^\dagger_{s_1}a_{s'_1}(Q^\dagger_q+Q_q)\,|0\rangle \tag{3.8}$$

because the decoupling scheme allows the ground-state average values of the last two terms at the rhs of (3.6) vanish as in (3.7), but not the one of the first term, as has been mentioned above.

Let us now derive the modified sum rule relationship within the approximation (3.7) and (3.8). Applying the identity (3.1) with $\widehat{\mathcal{O}} \equiv D^2$, we calculate the EWS of the DGDR strength $\mathcal{S}^{(2)}_1$ as

$$\begin{aligned}\mathcal{S}^{(2)}_1 &= \tfrac{1}{2}\langle 0|\big[D^2,\big[T+V,D^2\big]\big]|0\rangle = 4\mathcal{S}^{(1)}_1\mathcal{S}^{(1)}_0 + \tfrac{1}{2}\langle 0|\big[D^2,\big[V,D^2\big]\big]|0\rangle\\ &= 4\mathcal{S}^{(1)}_1\mathcal{S}^{(1)}_0 + \tfrac{1}{2}\big[\langle 0|\,D^2\big[D,[V,D]\big]|0\rangle + 2\langle 0|\,D\big[D,[V,D]\big]D\,|0\rangle\\ &\quad + \langle 0|\big[D,[V,D]\big]D^2\,|0\rangle\big].\end{aligned} \tag{3.9}$$

The term $4\mathcal{S}^{(1)}_1\mathcal{S}^{(1)}_0$ in this derivation comes from the double commutator between the double-dipole operator D^2 and the kinetic part T of the Hamiltonian [17]. The EWS $\mathcal{S}^{(1)}_1$ is the TKR sum rule, while the NEWS $\mathcal{S}^{(1)}_0$ is equal to $\langle 0|D^2|0\rangle$. Replacing the double commutator $[D,[V,D]]$ in the last term at the rhs of (3.9) by its ground-state expectation value K from (3.8), we obtain the new sum rule relationship:

$$\mathcal{S}^{(2)}_1 \simeq 4\big(\mathcal{S}^{(1)}_1 + \tfrac{1}{2}K\big)\mathcal{S}^{(1)}_0. \tag{3.10}$$

We have just shown that the PDM, which uses the decoupling scheme (2.7), conserves the TKR sum rule (in average), but violates the relationship (3.2). Instead of (3.2), the new relationship (3.10) holds, taking into account the nonzero ground-state expectation value of $[D,[V,D]]$. It is worth noticing that this modification is not quite the same as in the case when the Hamiltonian includes nonlocal and/or exchange-current forces such as isospin-dependent, Majorana exchange or one-pion exchange forces, etc. These interactions lead to an additional violation of TKR sum rule with an enhancement factor up to 30–40% [27].

In view of several popular microscopic theories that use the RPA phonon operators to generate the giant resonances and to study their damping by mixing these single RPA phonons with two-, three-phonon configurations, we will show below how anharmonicities between the RPA phonons really make the EWS of the DGDR strengths deviate from the relationship (3.2).

The RPA one-phonon state $|i\rangle^{(\lambda)}$ with number i and multipolarity λ is define as

$$|i\rangle^{(\lambda)} = Q^\dagger_{\lambda i}\,|\mathrm{RPA}\rangle, \tag{3.11}$$

where $|\mathrm{RPA}\rangle$ is the correlated RPA ground state (phonon vacuum), i.e., $Q_{\lambda i}\,|\mathrm{RPA}\rangle = 0$, and the phonon operator $Q^\dagger_{\lambda i}$ is defined as a linear superposition of the ph pair operator $B^\dagger_{ph} = a^\dagger_p a_h$ as

$$Q^{\dagger}_{\lambda i} = \sum_{ph} \left[X^{(\lambda i)}_{ph} B^{\dagger}_{ph} - Y^{(\lambda i)}_{ph} B_{ph} \right]. \tag{3.12}$$

The equation of motion applied to a model Hamiltonian with two-body interaction can be linearized in the space of one-phonon states (3.11) assuming the quasiboson approximation (QBA) of the phonon operator (3.12). The $X^{(\lambda i)}_{ph}$ and $Y^{(\lambda i)}_{ph}$ amplitudes satisfy then the well-known orthogonality and closure relationships within the QBA. This RPA equation defines the energies $\omega^{(\lambda)}_i = E^{(\lambda)}_i - E_0$, and the X, Y amplitudes of the one-phonon state (3.11) [25,26,28].

The electric multipole operator $\widehat{\mathcal{O}}_\lambda$ can be expressed in terms of the RPA phonon operators $Q^{\dagger}_{\lambda i}$ and $Q_{\lambda i}$ as

$$\widehat{\mathcal{O}}_\lambda = \sum_i \mathcal{F}^{(\lambda)}_i \left(Q^{\dagger}_{\lambda i} + Q_{\lambda i} \right) + \sum_{ss'=pp',\, hh'} \langle s' | \mathcal{M}(E\lambda) | s \rangle \, a^{\dagger}_s a_{s'}, \quad \text{where} \tag{3.13}$$

$$\mathcal{F}^{(\lambda)}_i = \sum_{ph} \langle h | \mathcal{M}(E\lambda) | p \rangle \left(X^{(\lambda i)}_{ph} + Y^{(\lambda i)}_{ph} \right), \tag{3.14}$$

with the matrix elements of the Eλ-transitions $\langle h | \mathcal{M}(\mathrm{E}\lambda) | p \rangle$. The EWS $\mathcal{S}^{(\lambda)}_1$ and NEWS $\mathcal{S}^{(\lambda)}_0$ of strengths of Eλ-transitions from the ground state $|\mathrm{RPA}\rangle$ to the one-phonon state $|i\rangle^{(\lambda)}$ (3.11) can be easily calculated as

$$\begin{aligned} \mathcal{S}^{(\lambda)}_1 &= \sum_i \left| {}^{(\lambda)}\langle i | \widehat{\mathcal{O}}_\lambda | \mathrm{RPA} \rangle \right|^2 \omega^{(\lambda)}_i = \sum_i \left[\mathcal{F}^{(\lambda)}_i \right]^2 \omega^{(\lambda)}_i, \\ \mathcal{S}^{(\lambda)}_0 &= \sum_i \left| {}^{(\lambda)}\langle i | \widehat{\mathcal{O}}_\lambda | \mathrm{RPA} \rangle \right|^2 = \sum_i \left[\mathcal{F}^{(\lambda)}_i \right]^2. \end{aligned} \tag{3.15}$$

The product of the type at the rhs of (3.2) can be now evaluated using (3.15) as

$$\begin{aligned} 4\mathcal{S}^{(\lambda)}_0 \mathcal{S}^{(\lambda)}_1 &= 2 \left\{ \sum_{i_1} \left(\mathcal{F}^{(\lambda)}_{i_1} \right)^2 \omega^{(\lambda)}_{i_1} \sum_{i_2} \left(\mathcal{F}^{(\lambda)}_{i_2} \right)^2 + \sum_{i_1} \left(\mathcal{F}^{(\lambda)}_{i_1} \right)^2 \sum_{i_2} \left(\mathcal{F}^{(\lambda)}_{i_2} \right)^2 \omega^{(\lambda)}_{i_2} \right\} \\ &= 2 \sum_{i_1 i_2} \left[\mathcal{F}^{(\lambda)}_{i_1} \right]^2 \left[\mathcal{F}^{(\lambda)}_{i_2} \right]^2 \left(\omega^{(\lambda)}_{i_1} + \omega^{(\lambda)}_{i_2} \right). \end{aligned} \tag{3.16}$$

Let us now define the two-phonon state that is constructed from two RPA phonon operators (3.12) as

$$\left| i^{(\lambda_1)}_1 i^{(\lambda_2)}_2 \right\rangle = \frac{1}{\sqrt{\delta_{\lambda_1 \lambda_2} \delta_{i_1 i_2} + 1}} \, Q^{\dagger}_{\lambda_1 i_1} Q^{\dagger}_{\lambda_2 i_2} \, |\mathrm{RPA}\rangle, \tag{3.17}$$

where the factor $(\delta_{\lambda_1 \lambda_2} \delta_{i_1 i_2} + 1)^{-1/2}$ is introduced to avoid double counting when $\lambda_1 = \lambda_2$ and $i_1 = i_2$. To be precise, the introduction of two-phonon excitation also modifies the ground-state $|\mathrm{RPA}\rangle$ to $|\tilde{0}\rangle$ due to phonon correlations in the ground state, whose effect can be neglected so long as $\langle \tilde{0} | Q^{\dagger} Q | \tilde{0} \rangle \simeq 0$. In the giant resonance region at zero temperature we can safely put $|\tilde{0}\rangle = |\mathrm{RPA}\rangle$. The matrix element of the transition from the ground state $|\mathrm{RPA}\rangle$ to the two-phonon state (3.17) that is caused by the two-body operator $\widehat{\mathcal{O}}_{\lambda_1 \lambda_2} = \widehat{\mathcal{O}}_{\lambda_1} \widehat{\mathcal{O}}_{\lambda_2}$ can be now easily calculated using Eqs. (3.13) and (3.17). It is equal to

$$\left\langle i^{(\lambda_1)}_1 i^{(\lambda_2)}_2 \right| \widehat{\mathcal{O}}_{\lambda_1 \lambda_2} \, |\mathrm{RPA}\rangle = \frac{1}{\sqrt{\delta_{\lambda_1 \lambda_2} \delta_{i_1 i_2} + 1}} \left(\mathcal{F}^{(\lambda_1)}_{i_1} \mathcal{F}^{(\lambda_2)}_{i_2} + \delta_{\lambda_1 \lambda_2} \mathcal{F}^{(\lambda_1)}_{i_2} \mathcal{F}^{(\lambda_2)}_{i_1} \right). \tag{3.18}$$

Putting $\lambda_1 = \lambda_2 = \lambda$, we get from Eq. (3.18) the matrix element of the double-phonon transition:

$$\langle i_1^{(\lambda)} i_2^{(\lambda)} | \, \widehat{\mathcal{O}}_{\lambda\lambda} \, |\mathrm{RPA}\rangle = \frac{2}{\sqrt{\delta_{i_1 i_2} + 1}} \mathcal{F}_{i_1}^{(\lambda)} \mathcal{F}_{i_2}^{(\lambda)}. \tag{3.19}$$

The EWS $\mathcal{S}_1^{(\lambda\lambda)}$ of the double-phonon strength is

$$\mathcal{S}_1^{(\lambda\lambda)} \equiv \sum_{i_1 i_2} \left| \langle i_1^{(\lambda)} i_2^{(\lambda)} | \, \widehat{\mathcal{O}}_{\lambda\lambda} |\mathrm{RPA}\rangle \right|^2 \omega_{i_1 i_2}^{(\lambda\lambda)} = 4 \sum_{i_1 i_2} \frac{1}{\delta_{i_1 i_2} + 1} \left[\mathcal{F}_{i_1}^{(\lambda)}\right]^2 \left[\mathcal{F}_{i_2}^{(\lambda)}\right]^2 \omega_{i_1 i_2}^{(\lambda\lambda)}, \tag{3.20}$$

where $\omega_{i_1 i_2}^{(\lambda\lambda)}$ is the energy of the two-phonon state $|i_1^{(\lambda)} i_2^{(\lambda)}\rangle$, which, in general, is not equal to $\omega_{i_1}^{(\lambda)} + \omega_{i_2}^{(\lambda)}$. Comparing now the lhs of Eq. (3.20) and the lhs of Eq. (3.16), we see that they would be equal to each other if their rhs coincided. This would take place only if the following conditions were fulfilled:

$$\text{(a)} \quad i_1 = i_2 = i, \qquad \text{(b)} \quad \omega_{i_1 i_2}^{(\lambda\lambda)} = \omega_{i_1}^{(\lambda)} + \omega_{i_2}^{(\lambda)} = 2\omega_i^{(\lambda)}.$$

Indeed, if (a) is fulfilled, Eq. (3.20) becomes

$$\mathcal{S}_1^{(\lambda\lambda)}\big|_{i_1 = i_2 = i} = 2 \sum_i \left[\mathcal{F}_i^{(\lambda)}\right]^4 \omega_{ii}^{(\lambda\lambda)}, \tag{3.21}$$

while Eq. (3.16) becomes

$$4\mathcal{S}_1^{(\lambda)} \mathcal{S}_0^{\lambda}\big|_{i_1 = i_2 = i} = 4 \sum_i \left[\mathcal{F}_i^{(\lambda)}\right]^4 \omega_i^{(\lambda)}. \tag{3.22}$$

The lhs of Eqs. (3.21) and (3.22) would be equal to each other if their rhs were equal, i.e., if

$$2 \sum_i \left[\mathcal{F}_i^{(\lambda)}\right]^4 \left(\omega_{ii}^{(\lambda\lambda)} - 2\omega_i^{(\lambda)}\right) = 0. \tag{3.23}$$

Equation (3.23), with arbitrary $\mathcal{F}_i^{(\lambda)}$, takes place only if $\omega_{ii}^{(\lambda\lambda)} = 2\omega_i^{(\lambda)}$, i.e., if condition (b) holds. If only (a) is fulfilled, we can estimate the shift $\Delta\omega_{ii}^{(\lambda\lambda)}$ of the two-phonon energy $\omega_{ii}^{(\lambda\lambda)}$ from $2\omega_i^{\lambda}$ from the difference $\Delta\mathcal{S}_1^{(\lambda\lambda)}$ between the EWS of the double-phonon strengths and $4\mathcal{S}_1^{(\lambda)}\mathcal{S}_0^{(\lambda)}$

$$\Delta\mathcal{S}_1^{(\lambda\lambda)} \equiv \mathcal{S}_1^{(\lambda\lambda)}\big|_{i_1 = i_2 = i} - 4\mathcal{S}_1^{(\lambda)} S_0^{(\lambda)} = 2 \sum_i \left[\mathcal{F}_i^{(\lambda)}\right]^4 \Delta\omega_{ii}^{(\lambda\lambda)}. \tag{3.24}$$

Equation (3.24) shows that a small energy shift $\Delta\omega_{ii}^{(\lambda\lambda)}$ in the two-phonon energy $\omega_{ii}^{(\lambda\lambda)}$ as compared to $2\omega_i^{(\lambda)}$ is sufficient to cause a large difference in the EWS of the two-phonon strengths compared to $4S_1^{(\lambda)}S_0^{(\lambda)}$ because of the sum with the multipliers $\mathcal{F}_i^{(\lambda)}$ in front of $\Delta\omega_{ii}^{(\lambda\lambda)}$. This means that a small anharmonicity in the DGDR energy leads to the big difference in its EWS of strengths compared to the harmonic limit. It is worth noticing that the derivation discussed above is applied for any multipolarity λ. In particular, for the dipole case $\lambda = 1$, it does not depend on whether the DGDR is formed in a one- or two-step process.

We have shown how the relationship (3.2) should be replaced with the new relationship (3.10) because the condition (3.3) required for (3.2) does not hold for the PDM Hamiltonian. We have also shown that the deviation from the relationship (3.2) is ultimately related to the shift of the two-phonon energy from its value obtained within the harmonic limit, which is equal to the sum of the energies of two single phonon excitations. In the next section, we will demonstrate that, by removing the energy shift so that the DGDR energy is equal to twice the GDR energy, i.e., its value in the harmonic limit, one can restore the relationship (3.2). An evaluation of the energy shift $\Delta\omega_{ii}^{(\lambda\lambda)}$ (3.24) is given in Appendix B making use of a general model Hamiltonian with two-body force. Of course, this does not exclude a possibility that anharmonicity, local terms in the Hamiltonian, etc., acting together may eventually lead to the zero value of the commutator $[V, D]$, and therefore, to the relationship (3.2). In order to demonstrate this, at least numerically, the consistent inclusion of ground-state correlations, anharmonicities of the one phonon state and corrections to the transition operator, and other effects, which are left out in this work should be explicitly taken into account. The anharmonicity in the energy and the transition strength of two-phonon giant resonances have also been studied in detail by several authors within different models in recent papers [29–31].

4. Numerical results

In this section we present the results of the calculations of the energy, FWHM, strength function and first moment of the DGDR and TGDR in ^{90}Zr, ^{120}Sn, and ^{208}Pb within the formalism developed in Section 2. In these calculations we assume that the properties of the GDR are known and can be well described within a microscopic model such as the QPM [14]. Therefore, in order to maintain the simplicity and transparency, we employ the same scenario, which has been successfully used to describe the damping of the hot GDR within the PDM-1 [20]. According to this scenario, the GDR is generated by a single collective and structureless phonon with energy ω_q that is close to the energy $\omega_{\rm GDR}$ of the ground-state GDR and exhausts all the oscillator strength.

As has been pointed out in Section 3, the PDM conserves the TKR sum rule for the GDR in average since $\langle 0|\,[V, D]\,|0\rangle$ is zero (Eq. (3.7)). In this case, the EWS $\mathcal{S}_1^{(n)}$ and NEWS $\mathcal{S}_0^{(n)}$ are just proportional to the moments $m_0^{(n)}$ and $m_1^{(n)}$ (2.21), respectively, with a factor $N^{(n)} = N^n$, where N is the strength normalization factor for the dipole case. The value of the latter is 84.4 mb for ^{90}Zr, 136.9 mb for ^{120}Sn, and 262.8 mb for ^{208}Pb. Multiplying these values by those of $m_1^{(1)}$ from Table 3 of Ref. [12], one easily obtains the absolute values of the EWS $\mathcal{S}_1^{(1)}$ as 1334.36 mb MeV for ^{90}Zr, 1870.05 mb MeV for ^{120}Sn, and 3221.93 mb MeV for ^{208}Pb. The ratios between these values and the corresponding TRK sum rules values 60 NZ/A (mb MeV) are 1, 1.07, and 1.08, respectively. They are in good agreement with the experimental values of the averaged integrated cross section, which are equal to 1.05 ± 0.07 TRK sum rule units for average upper integration limit $E\gamma = 28.2$ MeV [32]. With the integration extended to higher energy, the experimental

integrated cross sections may reach the values of 1.4 times the TRK sum rule due to the contribution of exchange forces.

The absolute values of the EWS and NEWS strengths for the DGDR can also be obtained in the same way, multiplying respectively $m_1^{(2)}$ and $m_0^{(2)}$ in Table 3 of Ref. [12] by N^2. It should be noticed, however, that when $n > 1$, as in the case of DGDR (e.g., $n = 2$) the absolute values of the EWS of strengths of the DGDR cannot be compared directly with the experimental integrated cross-section $\sigma_{\rm C}$ extracted from the Coulomb excitations because the latter is related to the DGDR photoabsorption cross section $\sigma(E)$ as

$$\sigma_{\rm C} = \int_{E_{\rm min}}^{\infty} \mathcal{N}(E)\sigma(E)\,dE. \tag{4.1}$$

The function $\mathcal{N}(E)$ is the photon spectral function which, according to the Weisäker–Williams method, appears after the integration of the spectrum of virtual photon $\mathcal{N}(E, b)$ over the impact parameter b. Since in the present work we are interested in the anharmonic effects in the strength functions of the multiphonon resonances compared to the values obtained within the independent phonon pictures, we prefer to consider the relative differences rather than the absolute values to avoid complication related to reaction mechanisms.[2] Therefore, we are going to check in the present section the relationship

$$m_1^{(2)} = 4m_1^{(1)}m_0^{(1)} \tag{4.2}$$

which is equivalent to (3.2), based on the discussion above.

We use the realistic single-particle energies obtained in the Woods–Saxon potentials at $T = 0$ for the nuclei under consideration. These discrete spectra cover an energy interval from around -35 to 25 MeV, i.e., including the continuum region. In the PDM-1 reasonable agreement between theory and data has been achieved via coupling of GDR phonon to all ph, pp, and hh configurations [19,20]. The phonon energy ω_q and the matrix elements of the coupling to ph and pp or hh $F_{ph}^{(q)} = F_1$ for $(s, s') = (p, h)$, and $F_{pp}^{(q)} = F_{hh}^{(q)} = F_2$ for $(s, s') = (p, p')$ or (h, h') are introduced as parameters of the model. Even though the higher-order graphs were not included explicitly in the equations within the PDM-1, this procedure implies that their effects are incorporated effectively in the parameters F_1 and F_2. The PDM-2, which includes these coupling explicitly up to two-phonon terms [21], gives the similar results for the hot GDR. The energy ω_q and the temperature-independent parameters F_1 and F_2 for the couplings have been chosen so that the experimental width and energy of the GDR at $T = 0$ are reproduced, and that the GDR energy does not vary appreciably when T changes (see the details in Ref. [20]). The same parameters ω_q and F_1 are used for the calculations of the DGDR and TGDR in the present paper. The experimental values for the GDR energy reproduced by these parameters are $E_{\rm GDR} = 16.8$ MeV for ^{90}Zr, 15.4 MeV for ^{120}Sn, and 13.5 MeV for ^{208}Pb. The calculated

[2] A systematic study of the cross sections $\sigma_{\rm C}$ using the PDM strength functions for the DGDR in ^{136}Xe and ^{208}Pb is now being carried out. The results will be compared directly with the experimental data and will be reported in a subsequent publication [33].

values of the GDR FWHM are $\Gamma_{\rm GDR} = 4.2$ MeV for ^{90}Zr, 4.9 MeV for ^{120}Sn, and 4.0 MeV for ^{208}Pb, which are the same as the empirical values. The parameter F_2 is not needed at zero-temperature.

For open-shell nuclei at nonzero temperature, superfluid pairing contributes at $T < 1$ MeV because the pairing gap decreases as T increases up to a critical temperature $T_{\rm cr}$ where the gap vanishes as in the finite-temperature BCS or remains small if thermal fluctuations due to nuclear finiteness are taken into account. Since the increase of the GDR width at low temperatures is caused by the coupling of the GDR phonon to pp and hh configuration [20], the decrease of the paring gap, which is also due to pp and hh interaction near the Fermi surface, slows down the increase of the hot GDR width at $T < T_{\rm cr}$. The GDR strength functions in ^{120}Sn obtained within the PDM-1 at $T = 0.1$, 0.4, and 0.6 MeV are shown in Fig. 1. The dashed curves have been obtained including a temperature-dependent neutron pairing gap $\Delta(T)$ with $\Delta(0) = 1.4$ MeV within the approximation mentioned in

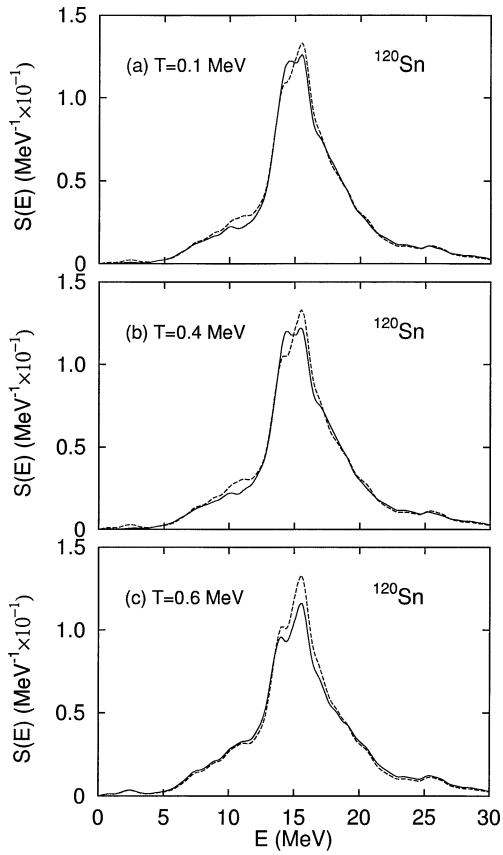

Fig. 1. Strength function of the GDR in ^{120}Sn obtained in PDM-1 at several temperatures $T < 0.8$ MeV. The dashed curves are results when superfluid pairing is included, while solid curves are those obtained neglecting the pairing.

Ref. [34]. It is seen from this figure that the effect of pairing starts to show up clearly only at temperatures close to $T_{\rm cr} \simeq 0.8$ MeV. At $T = 0.6$ MeV ($\Delta(T) \simeq 0.9$ MeV) e.g., the GDR width becomes slightly smaller than the value obtained neglecting pairing (Fig. 1(c)). At very low temperatures, e.g., $T = 0.1$ MeV, the strength functions calculated with and without pairing look almost the same, except for some slight difference around 10 MeV and at the GDR peak (Fig. 1(a)). When the average shape of the hot GDR was generated in the CASCADE calculations [34], the contribution of small changes in the strength function due to pairing at each temperature enters coherently in the sum from $T = 3.2$ MeV down to $T = 0.1$ MeV. This leads to a better agreement between the PDM prediction and the data, especially around $E = 10$ MeV. However, for the ground-state GDR ($T = 0$) considered in the present work, the effect can be safely omitted by the reason discussed above (Fig. 1(a)). Moreover, we compare here the DGDR and TGDR calculated within the present model with the results obtained in the folding model. As we employ for the calculations in both models the same ground-state GDR obtained within the PDM-1, the inclusion or omission of pairing should not modify substantially the relative difference between the results obtained in these two models for the open-shell nuclei.

The energy-weighted (EW) strength functions $S(E) \times E$ for the DGDR in ^{90}Zr, ^{120}Sn, and ^{208}Pb calculated using Eq. (2.20) for $n = 2$ are shown in Fig. 2. The left panels (Fig. 2(a)–(c)) are the results obtained using a smearing parameter $\varepsilon = 0.5$ MeV in the δ-function, while a value $\varepsilon = 0.05$ MeV has been used for the results displayed on the right panels (Fig. 2(d)–(f)). We will call these results "results with anharmonicity" to distinguish with the "folding results" obtained by folding the strength functions of two independent GDRs [6,35]. The latter have been calculated in PDM-1, i.e., based on Eqs. (2.16)–(2.20) with $n = 1$. The folding results are shown as dashed curves on the left panels. From the dense fine structure of the strength functions on the right panels, it is seen that the damping of the DGDR due to coupling to incoherent nucleon pairs is really complex. The gross structure of the DGDR obtained with the larger ε on the left panels, however, is very close to a single Breit–Wigner shape. The largest difference between the results with anharmonicity and the folding results of the DGDR is seen in ^{120}Sn (Fig. 2(b)). The magnitude of the DGDR peak is about two times larger that the folding result, i.e., about the same order of the difference reported for the DGDR in ^{136}Xe [1,4,8]. In ^{208}Pb the DGDR looks quite harmonic as this difference is rather small.

Since the interaction between phonon and nucleon pairs is put to be the same F_1 for all levels, the difference between the DGDRs in open-shell and closed-shell nuclei can be understood based on the properties of their single-particle spectra alone. In the double magic nucleus ^{208}Pb the Fermi surface lies just between two shells with a gap between shells of about 3–4 MeV. The $1\hbar\omega_0$ excitation, which corresponds to the GDR, can be generated by a group of 5–6 single-particle levels just below the Fermi surface to a group of 6–7 levels of the next shell situated just above the Fermi surface. The GDR in ^{208}Pb, therefore, is very collective with a FWHM of about 4 MeV. In the open-shell nuclei the distant between the occupied and unoccupied levels that belong to neighbor shells from two sides of the Fermi surface is about 2 MeV between $2p_{1/2}$ and $1g_{9/2}$ proton levels in ^{90}Zr, and particularly small (less than 1 MeV) between $3s_{1/2}$ and $2d_{3/2}$ neutron levels in

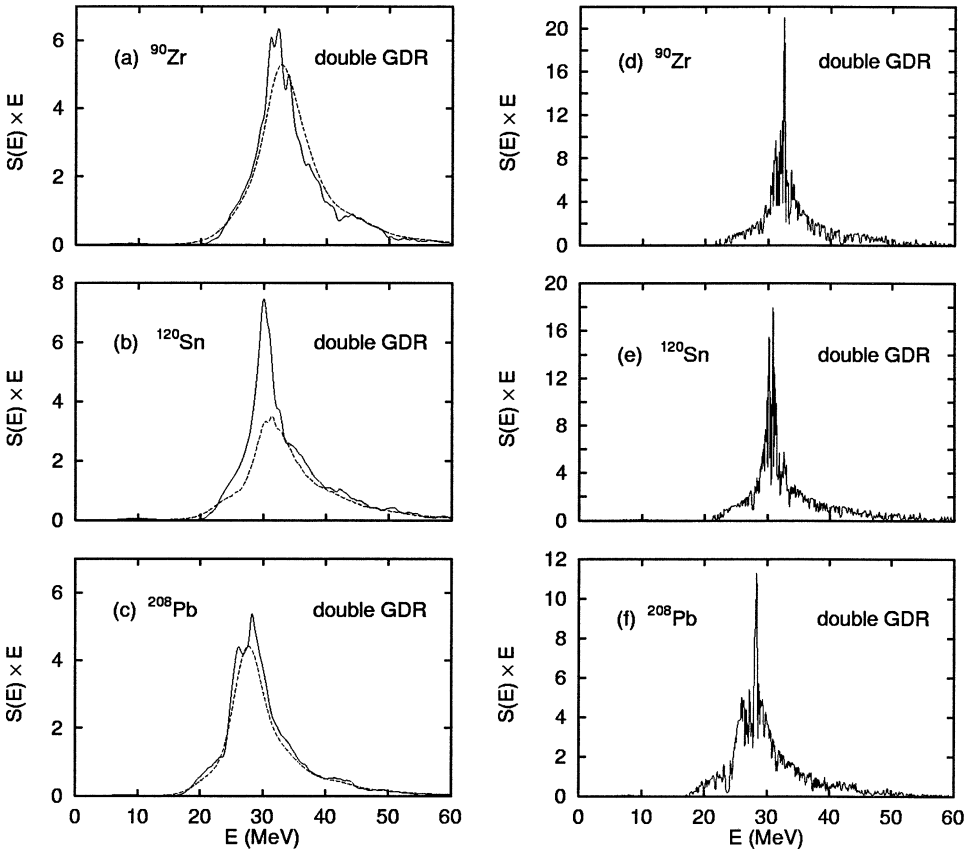

Fig. 2. EW strength function ($S_{qq}(E) \times E$) of the DGDR. The solid curves are results with anharmonicity. The dashed curves corresponds to the results obtained by folding the strength functions of two independent GDRs within PDM-1. Results in panels (a)–(c) have been obtained using the smearing parameter $\varepsilon = 0.5$ MeV, while the value $\varepsilon = 0.05$ MeV was used to calculate those in (d)–(f).

^{120}Sn. In ^{90}Zr there are only four occupied levels in the shell just below the Fermi surface, while in ^{120}Sn this number is only three. The strength of $1\hbar\omega_0$ transition is also shared to $1g_{9/2}$ level in ^{90}Zr, and to $2d_{3/2}$ and $1h_{11/2}$ in ^{120}Sn, while going across them to reach the next shell above above the Fermi surface. As a result the GDR in ^{120}Sn is less collective with the largest FWHM of 4.9 MeV. This is the reason why the effect of anharmonicity is expected to be strongest in ^{120}Sn and weakest in ^{208}Pb. In Ref. [12] it has been also found that the harmonicity in the DGDR seems to be restored in all three nuclei at temperature $T \geqslant 1.5$ MeV. This is another evident in favor of the analysis discussed above since shell effects and other quantal effects disappear starting from around this value of temperature.

The calculated DGDR centroid energy $\bar{E}_{\mathrm{DGDR}} \equiv \bar{E}_2$ and the energy of the DGDR peak E_2 are listed in Table 1. The values show that the difference between the results with anharmonicity and the folding results is quite small (around 1–2%). The ratio between the energy of the DGDR peak E_2 and twice the GDR energy $2E_1$ varies from 0.96 in ^{90}Zr to

Table 1
Centroid energies $\bar{E}_2$ and the energy E_2 of the main peak of DGDR in comparison with the folding result $E_2(f)$ and twice the GDR energy $2E_1$. The values of energy are in MeV

	$\bar{E}_2$	E_2	$E_2(f)$	$\bar{E}_2/E_2(f)$	$E_2/(2E_1)$	$\bar{E}_2/(2E_1)$
^{90}Zr	32.2	32.2	32.6	0.99	0.96	0.96
^{120}Sn	30.8	30.0	30.5	1.01	0.97	1.00
^{208}Pb	27.6	28.2	27.5	1.00	1.04	1.02

Table 2
FWHM Γ_2 of the DGDR in comparison with the folding result $\Gamma_2(f)$ and twice the DGDR width $2\Gamma_1$. The values of width are in MeV

	Γ_2	$\Gamma_2(f)$	$\Gamma_2/\Gamma_2(f)$	$\Gamma_2/(2\Gamma_1)$
^{90}Zr	6.49	8.3	0.78	0.77
^{120}Sn	6.05	9.4	0.64	0.62
^{208}Pb	7.95	6.4	1.24	0.99

Table 3
Moments $m_1^{(2)}$ (EWS) of the DGDR in comparison with the folding results $m_1^{(2)}(f)$ and the product $4m_1^{(1)}m_0^{(1)}$ of the EWS and NEWS of GDR. The moments have been calculated within $0 \leqslant \omega \leqslant$ 80 MeV

	$m_1^{(2)}$	$m_1^{(2)}(f)$	$4m_1^{(1)}m_0^{(1)}$	$m_1^{(2)}/m_1^{(2)}(f)$
^{90}Zr	57.69	59.14	59.45	0.98
^{120}Sn	58.37	45.46	45.51	1.284
^{208}Pb	47.57	41.88	41.27	1.136

1.04 in ^{208}Pb. The energy shift $\Delta\omega \equiv \bar{E}_2 - 2E_1$ is -1.4, 0.3, and 1.2 MeV for ^{90}Zr, ^{120}Sn and ^{208}Pb, respectively.

The calculated values of the DGDR FWHM Γ_{DGDR} are shown in Table 2. The results with anharmonicity Γ_2 are smaller than the folding results $\Gamma_2(f)$ by around 22% in ^{90}Zr, and 36% in ^{120}Sn. In ^{208}Pb, on the contrary, the width obtained with anharmonicity is larger by 24% than the folding result in reasonable agreement with the data [2,8]. It is also interesting to notice that the folding results give the FWHM that is quite close to $\sqrt{2}\Gamma_{\mathrm{GDR}}$ in ^{90}Zr and ^{120}Sn, but not in ^{208}Pb where it is almost equal to $2\Gamma_{\mathrm{GDR}}$ (indicated in the Table 2 as $2\Gamma_1$).

The values of the moments $m_k^{(n)}$ ($k = 0, 1$ and $n = 1, 2$) obtained in the calculations with anharmonicity and by folding two independent GDRs are presented in Table 3. Comparing the folding results for $m_1^{(2)}$ (i.e., $m_1^{(2)}(f)$ in the third column) with the values of $4m_1^{(1)}m_0^{(1)}$ in the fourth column, it is seen that the relationship (4.2) holds reasonably well with the

folding results. However, it is violated when the results with anharmonicity in the second column ($m_1^{(2)}$) are compared with $4m_1^{(1)}m_0^{(1)}$. The violation is strongest for the DGDR in ^{120}Sn, where the EWS of the DGDR strengths with anharmonicity is larger than the folding result by almost 30%. In ^{208}Pb the result with anharmonicity is larger than the folding result by 13.6%, while in ^{90}Zr it is smaller by 2%.

In Appendix B we have shown that the origin of the deviation from the harmonic limit of the DGDR lies in a four-boson term of the boson expansion of the PDM Hamiltonian (2.1) as well as of any Hamiltonian with two body interaction. In order to illustrate this numerically, we show here that by introducing artificially a four-boson term that removes the energy shift $\Delta\omega$, we can restore the relationship (4.2) in open shell nuclei. Indeed, the inclusion of a term of type (B.13) leads to the shift in the pole of Eq. (2.19). For the DGDR ($n = 2$) this ultimately causes some additional shift $\Delta\omega_q$ of the parameter ω_q. Therefore, varying the parameter ω_q would be somewhat equivalent to this effect. Shown in Fig. 3 and Table 4 are the results for the DGDR in ^{120}Sn obtained using two parameters $\omega_{q_1} =$ 11.2 MeV and $\omega_{q_2} = 13.0$ MeV in the equations for DGDR (see Eqs. (2.13)–(2.17) of Ref. [12]) instead of one parameter $\omega_q = 17$ MeV to cancel artificially the energy shift of the DGDR centroid energy. The results obtained are quite close to the folding ones and, therefore, the relationship (4.2) are satisfactorily reproduced. We notice, however, that the

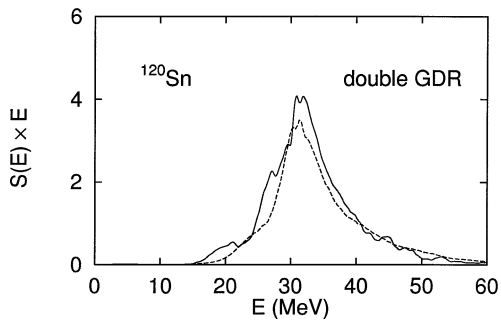

Fig. 3. EW strength function of the DGDR in ^{120}Sn. The solid curve has been obtained using new parameters ω_{q_1} and ω_{q_2} instead of ω_q to cancel the shift of the DGDR centroid energy (see text). The dashed curve is the same as in Fig. 2(b).

Table 4
Moment $\widetilde{m}_1^{(2)}$ (EWS), energy $\widetilde{E}_2$ and FHWM $\widetilde{\Gamma}_2$ of DGDR in ^{120}Sn obtained using parameters ω_{q_1} and ω_{q_2} instead of ω_q to cancel artificially the shift of the centroid energy (see text). The values are compared with the folding results indicated by (f). The moments have been calculated within $0 \leqslant \omega \leqslant 80$ MeV

	EWS		FWHM		Energy	
	$\widetilde{m}_1^{(2)}$	$\widetilde{m}_1^{(2)}/m_1^{(2)}(f)$	$\widetilde{\Gamma}_2$	$\widetilde{\Gamma}_2/\Gamma_2(f)$	$\widetilde{E}_2$	$\widetilde{E}_2/E_2(f)$
^{120}Sn	48.46	1.07	10.2	1.09	30.5	1.00

good description of the GDR (using ω_q) no longer holds using these new parameters.

The EW strength functions of the TGDR in ^{90}Zr, ^{120}Sn, and ^{208}Pb calculated with anharmonicity are presented in Fig. 4 (solid curves) in comparison with the folding results (dashed curves). The results for the FWHM, energy, and the EWS are shown in Table 5. We can see that the values of the EWS and energy obtained with anharmonicity are very

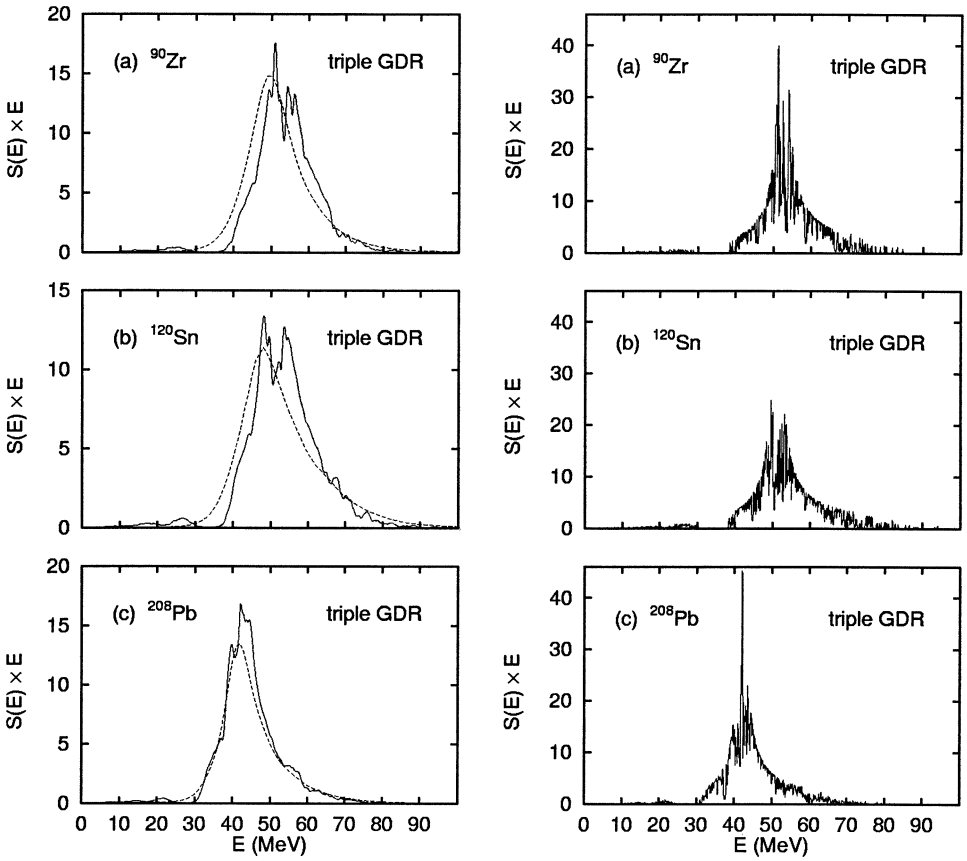

Fig. 4. EW strength function of the TGDR. The notation is the same as in Fig. 2.

Table 5
First moment $m_1^{(3)}$, FWHM (in MeV) and energy (in MeV) of TGDR. The moments have been calculated within $0 \leqslant \omega \leqslant 100$ MeV

	EWS			FWHM				Energy			
Nucleus	$m_1^{(3)}$	$m_1^{(3)}(f)$	$m_1^{(3)}/m_1^{(3)}(f)$	Γ_3	$\Gamma_3(f)$	$\Gamma_3/\Gamma_3(f)$	$\Gamma_3/(3\Gamma_1)$	$\bar{E}_3$	$\bar{E}_3(f)$	$\bar{E}_3/\bar{E}_3(f)$	$\bar{E}_3/(3E_1)$
^{90}Zr	237.15	253.24	0.94	15.7	13.5	1.16	1.25	52.4	49.5	1.06	1.04
^{120}Sn	231.07	229.68	1.01	20.2	16.5	1.22	1.37	51.2	48.0	1.07	1.11
^{208}Pb	210.18	189.09	1.11	9.6	10.0	0.96	0.80	42.2	42.0	1.01	1.04

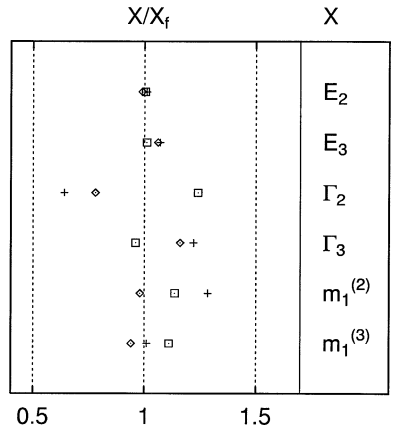

Fig. 5. Comparison of quantities X for the DGDR and TGDR with those (X_f) obtained by folding independent GDRs. Diamonds denote results in ^{90}Zr; crosses, results in ^{120}Sn; squares, results in ^{208}Pb.

close to the folding results. Similarly to the DGDR, the TGDR in the double magic nucleus ^{208}Pb is found to be very harmonic. The main peaks of the TGDR with anharmonicity are located at slightly higher energies than the folding results. However, their shapes are more symmetric as compared to the folding results. Therefore the resulting centroid energies are nearly the same as given by the folding results, i.e., $3E_1$. From Fig. 4 it is also seen that the TGDR peak in open shell nuclei splits slightly. This leads to a FWHM that is larger than the prediction of the folding results $3\Gamma_1$ by 25% in ^{90}Zr, and by 37% in ^{120}Sn. On the contrary, the TGDR width in ^{208}Pb is smaller than $3\Gamma_1$ by 20%.

A systematic comparison between the results with anharmonicity and the folding results for the DGDR and TGDR in all three nuclei is displayed in Fig. 5 as the ratio X/X_f where $X = E_n, \Gamma_n$ and $\mathcal{S}_1^{(n)}$ with $n = 2$ (DGDR) and 3 (TGDR) and the subscript f denoting the folding results. From this systematic we can predict that the TGDR looks more harmonic than the DGDR. The experimental data for the triple resonances are not yet available at present. Therefore, the measurements of triple giant resonances like those under consideration at GANIL are highly desirable to test the capability of the present model.

5. Conclusions

In this work we have extended the Phonon Damping Model (PDM) to the description of multiphonon giant resonances. The equations for the damping of the three-phonon and n-phonon giant resonances have been derived using the double-time Green's function method. The numerical calculations have been carried out for the first time for TGDR in ^{90}Zr, ^{120}Sn, and ^{208}Pb. The results for DGDR and TGDR are compared with the prediction by folding independent GDRs. We have also performed a detailed study of the sum rule

relationship (3.2) between the EWS of the DGDR strengths and four times of the product of the EWS and NEWS of GDR strengths.

The analysis of the results obtained allows us to make the following conclusions.

(i) Relationship (3.2) is violated if there is a nonzero shift of the DGDR energy from twice the GDR energy due to anharmonicities. A small anharmonicity in the DGDR energy can lead to a large deviation of the EWS of the DGDR strengths from the results within the independent-phonon picture. A new sum rule relationship (3.10) is derived within the PDM.

(ii) It is observed, for the first time in the calculated strength functions, that there is a noticeable difference between the DGDRs in open-shell and double-magic nuclei. In (neutron) open-shell nuclei, such as ^{120}Sn, the difference between the results with anharmonicity and the folding results is dramatic with the EWS of the DGDR strengths exceeding the harmonic prediction by nearly 30%. The magnitude of the resonance peak is about two times larger than the one obtained in the folding model. In the double magic nucleus ^{208}Pb the DGDR is rather harmonic. This observation is in agreement with the experimental trend for the DGDR in ^{136}Xe and ^{208}Pb.

(iii) The TGDR is more harmonic than the DGDR in all three nuclei. The TGDR energy is well described by $3E_{\mathrm{GDR}}$. The TGDR width, however, is larger than $3\Gamma_{\mathrm{GDR}}$ by 25–40% in open shell nuclei. In ^{208}Pb it is smaller than $3\Gamma_{\mathrm{GDR}}$ by around 20%.

The present work has been dedicated solely to the study of the anharmonicity in the multiphonon resonances. We have not yet performed the calculations of the cross section in the Coulomb excitation of the DGDR, which has been observed experimentally to be enhanced strongly compared to the prediction by the folding results. There is a number of other factors that are left out in this study and that may also contribute to this enhancement such as the charge-exchange part in the residual interaction, the effects of higher-order configuration mixing, the nonlinearity of the external electromagnetic field, etc. It is also true that, while the two components of the DGDR with the total momentum $J^{\pi} = 0^{+}$ and 2^{+} are degenerate [16], the explicit introduction of the angular momentum coupling as well as isospin may also contribute for n-phonon resonances with $n > 2$. At the same time, further experimental measurements are also required to reduce the large error bars in the present data as well as to establish a systematic dependence of the cross section on the thickness of the targets. Nonetheless, the results in the present work has definitely shown that anharmonicity plays a significant role in the deviation of the DGDR from the harmonic picture especially in the open shell nuclei.

Acknowledgements

Numerical calculations were carried out using a 64-bit Alpha AXP work-station running Digital UNIX (OSF/1) at the Computer Science Laboratory of RIKEN. One of us (N.D.D.) thanks V. Zelevinsky for fruitful discussions regarding the sum rule of the DGDR.

Appendix A. Contribution of the single-particle damping

As has been discussed thoroughly within the PDM-1 [20], we consider here only the part of the single-particle damping that is caused by the mutual coupling to the phonon field. The interparticle interactions, etc., may also contribute to increase the total single-particle width, but they do not affect the damping of the collective phonon because there is no direct coupling between them and the phonon field or the coupling enters in higher-order in the boson expansion. In this appendix we will show that the contribution of the Green's function

$$\mathcal{G}^{(n)}_{ss';q_1\ldots q_{n-2};q'_1\ldots q'_n}(t-t') = \langle\langle a^\dagger_s(t)a_{s'}(t)\,Q_{q_1}(t)\ldots Q_{q_{n-2}}(t);\, Q^\dagger_{q'_1}(t')\ldots Q^\dagger_{q'_n}(t')\rangle\rangle, \quad n\geqslant 2, \tag{A.1}$$

to the damping of the multiple GDR is negligible. Below we present in brief the derivation of the equation for the Green's function (A.1) with $n=2$ and make the generalization to an arbitrary n. Following the standard procedure of Ref. [24] we found, for $n=2$:

$$\begin{aligned} i\frac{d}{dt}\mathcal{G}^{(2)}_{ss';q'_1q'_2}(t-t') &= (E_{s'}-E_s)\mathcal{G}_{ss';q'_1q'_2}(t-t') \\ &\quad + \sum_{qs_1}\big[F^{(q)}_{s's_1}\langle\langle a^\dagger_s(t)a_{s_1}(t)\big(Q^\dagger_q(t)+Q_q(t)\big);\, Q^\dagger_{q'_1}(t')Q^\dagger_{q'_2}(t')\rangle\rangle \\ &\quad - F^{(q)}_{s_1s'}\langle\langle a^\dagger_{s_1}(t)a_{s'}(t)(Q^\dagger_q(t)+Q_q(t));\, Q^\dagger_{q'_1}(t')Q^\dagger_{q'_2}(t')\rangle\rangle\big]. \end{aligned} \tag{A.2}$$

The Fourier transform of Eq. (A.2) in the energy plane E yields

$$\mathcal{G}^{(2)}_{ss';q'_1q'_2}(E) = \frac{1}{E-E_{s'}+E_s}\sum_{qs_1}\big[F^{(q)}_{s's_1}\mathcal{G}_{ss'q;q'_1q'_2}(E) - F^{(q)}_{s_1s}\mathcal{G}_{s_1sq;q'_1q'_2}(E)\big]. \tag{A.3}$$

In the spirit of the perturbation theory, the lhs of Eq. (A.3) can be estimated by replacing functions $\mathcal{G}_{s_\alpha s_\beta q;q'_1q'_2}(E)$ in the rhs with their expressions obtained in Ref. [12] where function $\mathcal{G}^{(2)}_{ss';q'_1q'_2}(E)$ was omitted. The result is

$$\begin{aligned} \mathcal{G}^{(2)}_{ss';q'_1q'_2}(E) = \frac{1}{E-E_{s'}+E_s}\sum_{q_1q_2s_1}\Bigg[&\frac{(n_s-n_{s_1})F^{(q_1)}_{s's_1}F^{(q_2)}_{s_1s}}{E-E_{s_1}+E_s-\omega_{q_1}} \\ &- \frac{(n_{s_1}-n_{s'})F^{(q_1)}_{s_1s}F^{(q_2)}_{s's_1}}{E-E_{s'}+E_{s_1}-\omega_{q_1}}\Bigg]G_{q_1q_2;q'_1q'_2}(E). \end{aligned} \tag{A.4}$$

For the DGDR, Eq. (A.4) has a simple form as

$$\mathcal{G}^{(2)}_{ss';qq}(E) = G_{qq}(E)\Delta S^{(2)}_{ss'}(E), \tag{A.5}$$

where the correction factor to the DGDR strength is

$$\begin{aligned} \Delta S^{(2)}_{ss'}(E) = &\frac{1}{E-E_{s'}+E_s} \\ &\times\sum_{s_1}F^{(q)}_{s's_1}F^{(q)}_{s_1s}\Big[\frac{n_s-n_{s_1}}{E-E_{s_1}+E_s-\omega_q} - \frac{n_{s_1}-n_{s'}}{E-E_{s'}+E_{s_1}-\omega_q}\Big]. \end{aligned} \tag{A.6}$$

After an average over all the single-particle levels, this factor yields an overall correction to the EWS in the form

$$\Delta S^{(2)} = \frac{\sum_{jj'} \Omega_j \Omega_{j'} \int \Delta S_{jj'}(\omega)\, d\omega}{(\sum_j \Omega_j)^2}, \quad \Omega_j = 2j+1. \tag{A.7}$$

The similar derivation can be done for $n = 3$, etc. The general expression of the correction due to the function (A.1) to n-phonon resonance is

$$\Delta S^{(n)}_{ss'} = \frac{1}{E - E_{s'} + E_s - (n-2)\omega_q} \sum_{s_1} F^{(q)}_{s's_1} F^{(q)}_{s_1 s} \Big[\frac{n_s - n_{s_1}}{E - E_{s_1} + E_s - (n-1)\omega_q} - \frac{n_{s_1} - n_{s'}}{E - E_{s'} + E_{s_1} - (n-1)\omega_q} \Big]. \tag{A.8}$$

The numerical estimation of $\Delta S^{(n)}$ for DGDR ($n = 2$) and TGDR (n=3) has been carried out for all three nuclei. In ^{120}Sn, where the enhancement compared to the folding results of the DGDR is the largest, $\Delta S^{(2)}$ reduces the EWS by less than 0.03%. A similar result has been obtained for TDGDR. We conclude that the contribution of the terms of type (A.1) can be safely omitted.

Appendix B. Derivation of the shift of the DGDR energy due to anharmonicity

In this appendix we derive the shift due to anharmonicity of the two-phonon energy from the sum of two one-phonon energies. If the two-phonon state (3.17) are the eigenstates of the Hamiltonian H with the eigenenergy $\omega^{(\lambda_1\lambda_2)}_{i_1 i_2}$, we have the equation

$$H \left| i_1^{(\lambda_1)} i_2^{(\lambda_2)} \right\rangle = \omega^{(\lambda_1\lambda_2)}_{i_1 i_2} \left| i_1^{(\lambda_1)} i_2^{(\lambda_2)} \right\rangle. \tag{B.1}$$

The lhs of (B.1) can be calculated using the explicit form of $|i_1^{(\lambda_1)} i_2^{(\lambda_2)}\rangle$ in the rhs of Eq. (3.17) assuming that the one-phonon state $Q^\dagger_{\lambda i}|0\rangle$ is also an eigenstate of H with the eigenenergy $\omega_i^{(\lambda)}$. The ground state (phonon vacuum) $|0\rangle$ is equal to $|\mathrm{RPA}\rangle$ if the structure of the one-phonon states are determined within the RPA. A simple calculation yields for the lhs of (B.1):

$$H \left| i_1^{(\lambda_1)} i_2^{(\lambda_2)} \right\rangle = \left(\omega_{i_1}^{(\lambda_1)} + \omega_{i_2}^{(\lambda_2)}\right) \left| i_1^{(\lambda_1)} i_2^{(\lambda_2)} \right\rangle + \frac{1}{\sqrt{\delta_{\lambda_1\lambda_2}\delta_{i_1 i_2} + 1}} \left[[H, Q_{\lambda_1}]^\dagger, Q^\dagger_{\lambda_2} \right] |0\rangle. \tag{B.2}$$

Equalizing the rhs of (B.2) and the rhs of (B.1), we obtain:

$$\Delta\omega^{(\lambda_1\lambda_2)}_{i_1 i_2} \left| i_1^{(\lambda_1)} i_2^{(\lambda_2)} \right\rangle \equiv \left[\omega^{(\lambda_1\lambda_2)}_{i_1 i_2} - \left(\omega_{i_1}^{(\lambda_1)} + \omega_{i_2}^{(\lambda_2)}\right) \right] \left| i_1^{(\lambda_1)} i_2^{(\lambda_2)} \right\rangle = \frac{1}{\sqrt{\delta_{\lambda_1\lambda_2}\delta_{i_1 i_2} + 1}} \left[[H, Q_{\lambda_1}]^\dagger, Q^\dagger_{\lambda_2} \right] |0\rangle. \tag{B.3}$$

Acting $\langle i_1^{(\lambda_1)} i_2^{(\lambda_2)}|$ on the left of the lhs and rhs of Eq. (B.3), we come to

$$\Delta\omega^{(\lambda_1\lambda_2)}_{i_1 i_2} = \frac{1}{\sqrt{\delta_{\lambda_1\lambda_2}\delta_{i_1 i_2} + 1}} \langle 0 | Q_{\lambda_2 i_2} Q_{\lambda_1 i_1} \left[[H, Q^\dagger_{\lambda_1}], Q^\dagger_{\lambda_2} \right] |0\rangle. \tag{B.4}$$

In the case of two identical phonons $\lambda_1 = \lambda_2 = \lambda$, $i_1 = i_2 = i$, Eq. (B.4) yields:

$$\Delta\omega_{ii}^{(\lambda\lambda)} = \tfrac{1}{2}\,\langle 0|\, Q_{\lambda i}\, Q_{\lambda i}\big[[H,\, Q_{\lambda i}^{\dagger}],\, Q_{\lambda i}^{\dagger}\big]\,|0\rangle, \tag{B.5}$$

which is not zero with a Hamiltonian containing a two-body interaction as we shall see below.

Without a loss of generality, let us consider a model Hamiltonian with a separable two-body force:

$$H_M = \sum_s E_s\, B_{ss}^{\dagger} + H_V, \qquad H_V = \sum_{s_1 s_1' s_2 s_2'} f_{s_1 s_1'} f_{s_2 s_2'} B_{s_1 s_1'}^{\dagger} B_{s_2' s_2}, \tag{B.6}$$

where $B_{ss'}^{\dagger} = a_s^{\dagger} a_{s'}$ ($ss' = ph,\, pp',\, hh'$). The interaction part H_V can be represented as the sum of

$$H_V = H_{phph} + (H_{phpp} + \text{h.c.}) + (H_{phhh} + \text{h.c.}) + H_{pppp} + H_{hhhh}, \tag{B.7}$$

where the subscripts in the rhs indicate the summations carried over $(s_1, s_1', s_2, s_2') = (phph)-$, $(phpp)-$, $(phhh)-$, $(pppp)-$, and $(hhhh)-$ indices, respectively. Expressing H_{phph} in terms of RPA phonon operators $Q^{\dagger}$ and Q using Eq. (3.12), it is simple to show that the expectation value at the rhs of Eq. (B.5) yields zero with respect to H_{phph}. The pp (hh) pair operators $B_{pp'}$ ($B_{hh'}$) in the other terms at the rhs of (B7) can also be expressed in terms of the RPA phonons using the boson mapping [36]:

$$B_{pp'}^{\dagger} \to \sum_h B_{ph}^{\dagger} B_{p'h}, \qquad B_{hh'}^{\dagger} \to \delta_{hh'} - \sum_p B_{ph'}^{\dagger} B_{ph}. \tag{B.8}$$

Applying the mapping (B.8), it is easy to see that the terms H_{phpp} and H_{phhh} at the rhs of (B.7) also yield zero in the expectation value at the rhs of Eq. (B.5) because they contains odd number of phonon operators of type $\sim Q^{\dagger}Q^{\dagger}Q$, $Q^{\dagger}QQ$, $Q^{\dagger}Q^{\dagger}Q^{\dagger}$ and QQQ. Those are the terms that lead to the spreading width of the single-phonon giant resonance due to two-phonon or three-phonon configuration mixing. This result shows that the part causing the major spreading of the GDR does not contribute to the shift of the DGDR energy in Eq. (B.5).

The last two terms H_{pppp} and H_{hhhh} at the rhs of Eq. (B.7) are those whose contribution to the rhs of (B.5) gives a nonzero value. Applying the mapping (B.8) in combination with the definition (3.12), we obtain, after a simple derivation, the remaining parts of H_{pppp} and H_{hhhh}, whose expectation values in the RPA ground-state are not zero. They are:

$$H'_{pppp} = \sum_{\nu_1\nu_2\nu_3\nu_4} V^{(1)}_{\nu_1\nu_3\nu_2\nu_4}\, Q^{\dagger}_{\nu_1}\, Q^{\dagger}_{\nu_3}\, Q_{\nu_2}\, Q_{\nu_4}, \quad \nu_j = (\lambda_j i_j), \quad \text{where} \tag{B.9}$$

$$\begin{aligned} V^{(1)}_{\nu_1\nu_3\nu_2\nu_4} = \sum_{p_1 p_1' p_2 p_2' hh'} f_{p_1 p_1'} f_{p_2 p_2'} \Big(& X^{(\nu_1)}_{p_1 h} X^{(\nu_2)}_{p_1' h} X^{(\nu_3)}_{p_2' h} X^{(\nu_4)}_{p_2 h} + Y^{(\nu_2)}_{p_1 h} Y^{(\nu_1)}_{p_1' h} Y^{(\nu_4)}_{p_2' h} Y^{(\nu_3)}_{p_2 h} \\ & + X^{(\nu_1)}_{p_1 h} Y^{(\nu_3)}_{p_1' h} Y^{(\nu_2)}_{p_2' h} X^{(\nu_4)}_{p_2 h} + Y^{(\nu_4)}_{p_1 h} X^{(\nu_2)}_{p_1' h} X^{(\nu_3)}_{p_2' h} Y^{(\nu_1)}_{p_2 h} \Big), \end{aligned} \tag{B.10}$$

$$H'_{hhhh} = \sum_{\nu_1\nu_2\nu_3\nu_4} V^{(2)}_{\nu_1\nu_3\nu_2\nu_4}\, Q^{\dagger}_{\nu_1}\, Q^{\dagger}_{\nu_3}\, Q_{\nu_2}\, Q_{\nu_4}, \quad \text{where} \tag{B.11}$$

$$V^{(2)}_{\nu_1\nu_3\nu_2\nu_4} = \sum_{h_1h'_1h_2h'_2pp'} f_{h_1h'_1} f_{h_2h'_2}\Big(X^{(\nu_1)}_{ph'_1}X^{(\nu_2)}_{ph_1}X^{(\nu_3)}_{p'h_2}X^{(\nu_4)}_{p'h'_2} + Y^{(\nu_2)}_{ph'_1}Y^{(\nu_1)}_{ph_1}Y^{(\nu_4)}_{p'h_2}Y^{(\nu_3)}_{p'h'_2} + X^{(\nu_1)}_{ph'_1}Y^{(\nu_3)}_{ph_1}Y^{(\nu_2)}_{p'h_2}X^{(\nu_4)}_{p'h'_2} + Y^{(\nu_4)}_{ph'_1}X^{(\nu_2)}_{ph_1}X^{(\nu_3)}_{p'h_2}Y^{(\nu_1)}_{p'h'_2}\Big). \quad \text{(B.12)}$$

Therefore, the part of the Hamiltonian that yields nonzero expectation value for the rhs of (B.5) is

$$H' = \sum_{\nu_1\nu_2\nu_3\nu_4} V_{\nu_1\nu_2\nu_3\nu_4} Q^\dagger_{\nu_1} Q^\dagger_{\nu_3} Q_{\nu_2} Q_{\nu_4}, \quad V_{\nu_1\nu_2\nu_3\nu_4} = V^{(1)}_{\nu_1\nu_2\nu_3\nu_4} + V^{(2)}_{\nu_1\nu_2\nu_3\nu_4}. \quad \text{(B.13)}$$

Using Eqs. (B.5) and (B.13) we obtain for the energy shift:

$$\Delta\omega^{(\lambda_1\lambda_2)}_{i_1i_2} = \frac{1}{\delta_{\lambda_1\lambda_2}\delta_{i_1i_2}+1}(V_{1221} + V_{1212} + V_{2121} + V_{2112}), \quad \text{(B.14)}$$

where $V_{1221} = V_{\lambda_1 i_1,\lambda_2 i_2,\lambda_2 i_2,\lambda_1 i_1}$, etc. If the phonons are identical ($i_1 = i_2 = i$, $\lambda_1 = \lambda_2 = \lambda$) the shift of two-phonon energy from twice the one-phonon energy is

$$\Delta\omega^{\lambda\lambda}_{(ii)} = 2V_{\lambda i,\lambda i,\lambda i,\lambda i}. \quad \text{(B.15)}$$

As has been mentioned at the end of Section 2, a nucleon pair $a^\dagger_s a_{s'}$ can be expressed in term of an infinite boson expansion series of Belyaev–Zelevinsky type [25]. Therefore, the last term of the Hamiltonian (2.1) contains also the four-boson terms of type (B.13), which causes the energy shift.

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