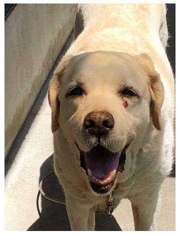
Range correction in the weak-binding relation for unstable states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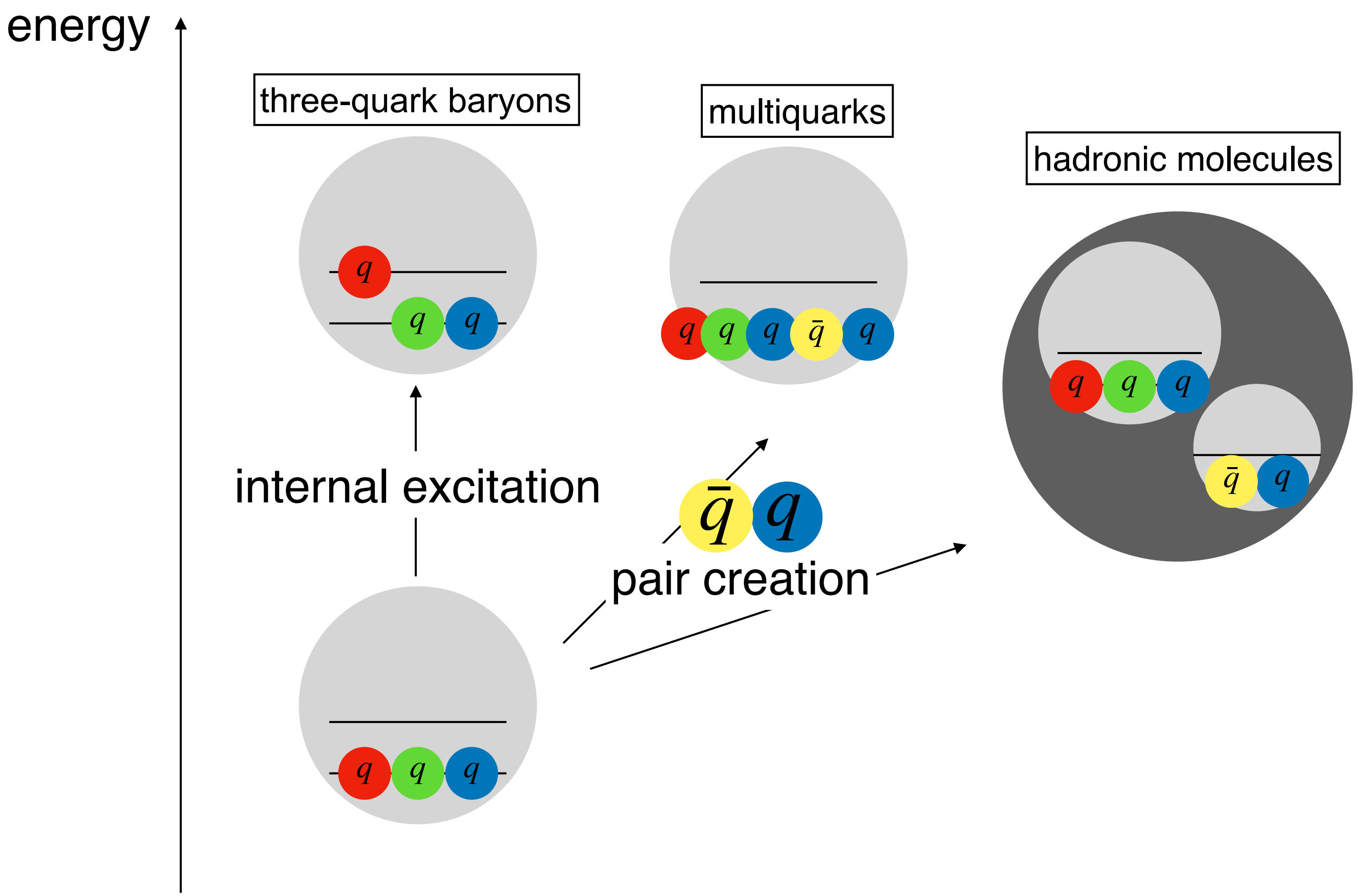
Background

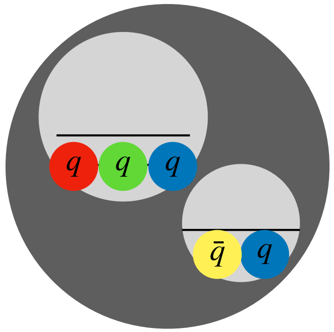
candidates for exotic hadrons
$\Lambda(1405)$, XYZ meson etc… → multiquarks, hadronic molecules

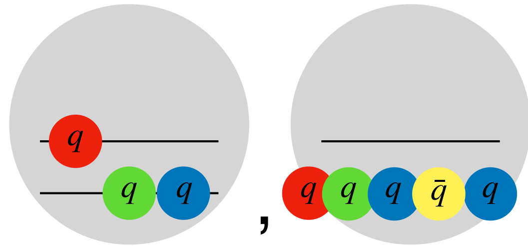


structure of hadrons ← model independent — observable

Previous work

Hadron wave function



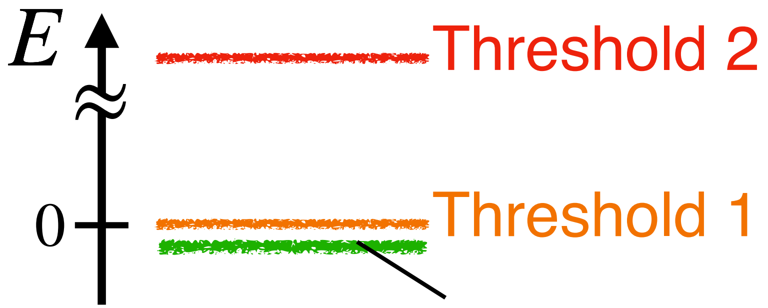


$$|\Psi\rangle = \sqrt{X}\,|\text{hadronic molecule}\rangle + \sqrt{1-X}\,|\text{others}\rangle$$

Compositeness (weight of hadronic molecule)

Weak-binding relation for bound state

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$$



a_0 (scattering length) $R_{\rm typ}$(interaction range)

$R \equiv (2\mu B)^{-1/2}$, B (binding energy)

When $R \gg R_{\rm typ}$: observable(a_0, B) ⟶ compositeness(X)

S. Weinberg, Phys. Rev. 137, B672 (1965); Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

Motivation

Weak-binding relation $a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$

Low-energy universality → $a_0 = R \ (R \to \infty)$

-Deviation by contributions from other channels ← $X \neq 1$

-Deviation by interaction range ← $R_{\rm typ} \neq 0$

→ We study the range correction in the weak-binding relation by introducing the effective range r_e.

Effective range model

E. Braaten, M. Kusunoki, and D. Zhang, Annals Phys. 323, 1770 (2008), 0709.0499.

Single channel scattering of identical bosons with mass m :

$$\mathscr{H}_{\text{int}} = \frac{1}{4}\lambda_0(\psi^\dagger\psi)^2 + \frac{1}{4}\rho_0 \nabla(\psi^\dagger\psi)\cdot\nabla(\psi^\dagger\psi)$$

Off-shell T-matrix: $T(E,k,k') = T_1(E) + T_2(E)(k^2+k'^2) + T_3(E)k^2k'^2,$

$$\begin{pmatrix} T_1 & T_2 \\ T_2 & T_3 \end{pmatrix} = -i\begin{pmatrix} \lambda_0 & \rho_0 \\ \rho_0 & 0 \end{pmatrix} - i\begin{pmatrix} \lambda_0 & \rho_0 \\ \rho_0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{i}{E-q^2/m+i0^+} & \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{iq^2}{E-q^2/m+i0^+} \\ \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{iq^2}{E-q^2/m+i0^+} & \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{iq^4}{E-q^2/m+i0^+} \end{pmatrix}\begin{pmatrix} T_1 & T_2 \\ T_2 & T_3 \end{pmatrix}.$$

cut off at Λ, typical range $R_{\text{typ}} \sim 1/\Lambda$

On-shell scattering amplitude:

$$f(k) = \left[-\frac{8\pi}{m}\frac{\left(1+\frac{m}{12\pi^2}\Lambda^3\rho_0\right)^2}{N(k)} - \frac{2}{\pi}\Lambda - ik\right]^{-1} \quad N(k) = \left[\lambda_0 - \frac{m}{20\pi^2}\Lambda^5\rho_0^2\right] + 2\rho_0\left(\frac{m}{24\pi^2}\Lambda^3\rho_0 + 1\right)k^2$$

$$= \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 + \mathcal{O}\left(\frac{1}{\Lambda}\right) - ik\right]^{-1}$$ ← Renormalization

$$\to \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - ik\right]^{-1} \quad (\Lambda \to \infty)$$ ← Zero range limit

Effective range model

Properties of the effective range model:

-Single channel: $|\text{hadronic molecule}\rangle$ only $\quad \Leftrightarrow X = 1$

-Zero range limit: $\Lambda \to \infty \Leftrightarrow R_{\text{typ}} = 1/\Lambda \to 0$

$$\Rightarrow a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\} \to R \;\; ?$$

Renormalized scattering amplitude ($\Lambda \to \infty$):

$$1/f(k = i/R) = 0$$

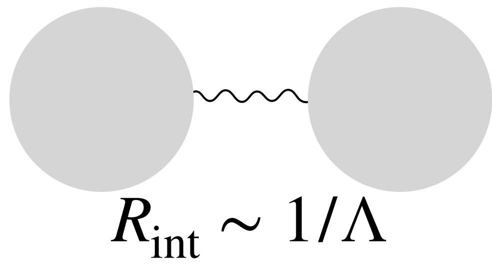
$$a_0 = R\frac{2r_e/R}{1-(r_e/R-1)^2} = R\left[1 + \mathcal{O}\left(\left|\frac{r_e}{R}\right|\right)\right] \Rightarrow a_0 \neq R$$

→ range correction in the weak-binding relation form r_e

Improved weak-binding relation

Weak-binding relation $a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$

interaction range: $R_{\rm typ}$ → $R_{\rm int} \sim 1/\Lambda$



Redefinition of $R_{\rm typ}$:

$$R_{\rm typ} = \max\left\{R_{\rm int}, R_{\rm eff}\right\},$$

$$R_{\rm eff} = \max\left\{|r_e|, \frac{|P_s|}{R^2}, \cdots\right\}.$$

$$f(k) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - \frac{P_s}{4}k^4 + \cdots - ik\right]^{-1}$$

Length scale in the effective range expansion except for a_0

Numerical calculation

$$a_0 = R\left\{ \frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\mathrm{typ}}}{R}\right) \right\}$$

When dose the weak-binding relation work?

Estimation with correction terms ($\xi \equiv R_{\mathrm{typ}}/R$): Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

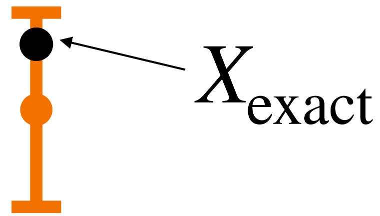
Central value: $X_c = \dfrac{a_0/R}{2 - a_0/R}$

$$X_{\mathrm{upper}}(\xi) = \frac{a_0/R}{2 - a_0/R} + \xi,\ X_{\mathrm{lower}}(\xi) = \frac{a_0/R}{2 - a_0/R} - \xi.$$

Weak-binding relation works when…

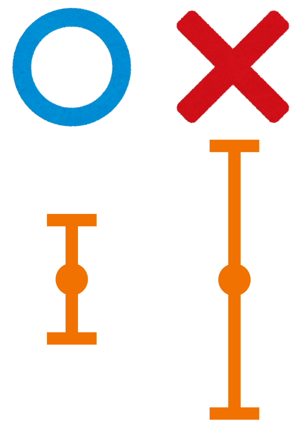
$X_{\mathrm{lower}} < X_{\mathrm{exact}} < X_{\mathrm{upper}}$

→ Validity condition



$(X_{\mathrm{upper}} - X_c)/X_c < 0.1$ and $(X_c - X_{\mathrm{lower}})/X_c < 0.1$

→ Precision condition



Numerical calculation

Effective range model ($\Lambda < \infty$)

$$f(k;\lambda_0,\rho_0,\Lambda) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 + \mathcal{O}\left(R_{\text{int}}\right) - ik\right]^{-1}$$ (two length scales r_e and R_{int})

$$1/f(k = i/R) = 0$$

- $r_e \neq 0$ (range correction): $\xi_{r_e} = |r_e/R|$ → Uncertainty from r_e

  $r_e < 0$ (effective range model)

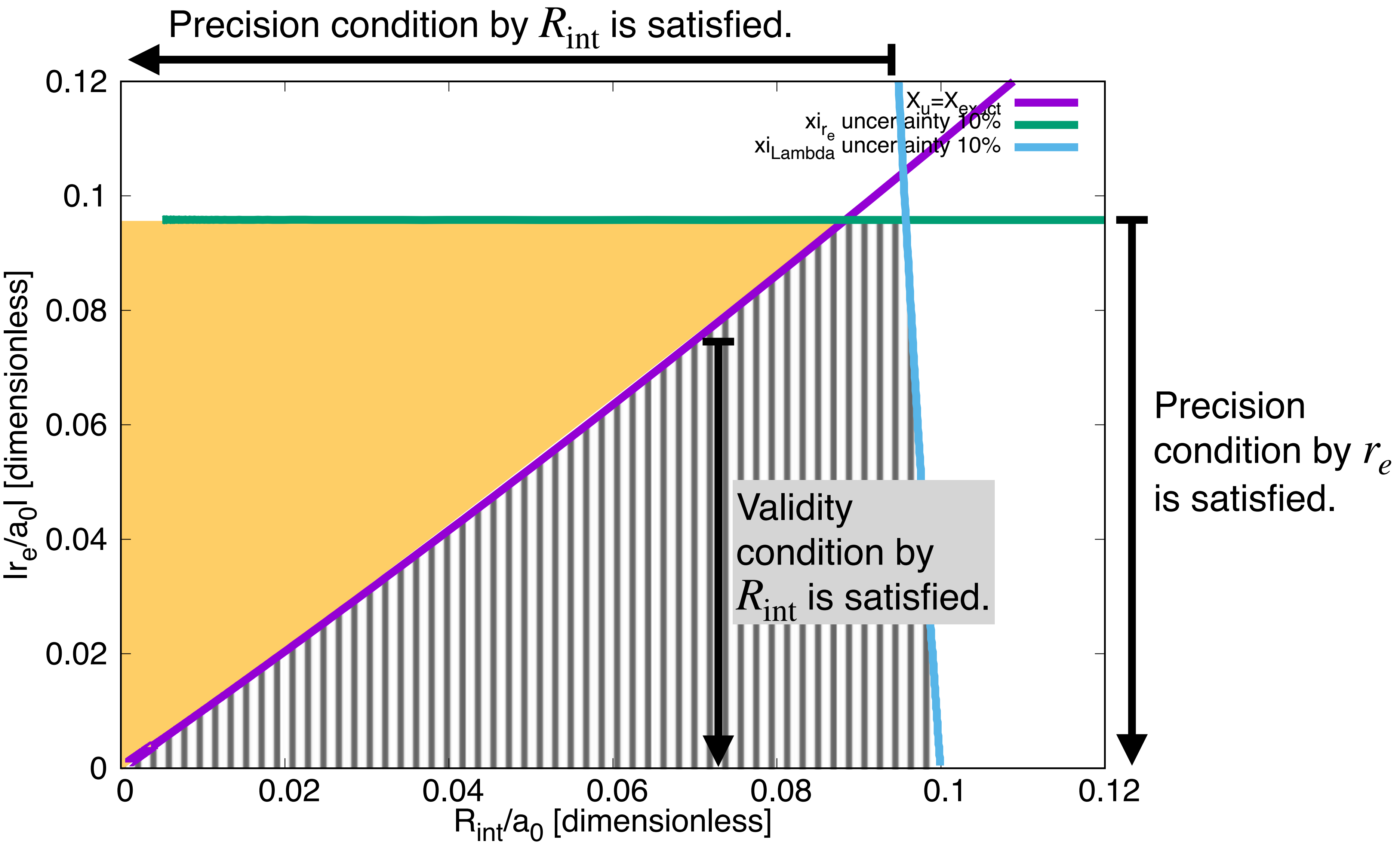
- $R_{\text{int}} = 1/\Lambda \neq 0$: $\xi_{\text{int}} = R_{\text{int}}/R$. → Uncertainty from R_{int}

- $X_{\text{exact}} = 1$

→ We search for the regions of r_e and R_{int} in which validity and precision conditions are satisfied.

Numerical calculation

Validity and precision conditions in $R_{\rm int}/a_0$-$|r_e/a_0|$ plane



Only the improved weak-binding relation can be applied.

Extension to unstable states

Candidates for exotic hadrons are unstable states!

Unstable systems → $E, a_0, r_e \in \mathbb{C}$

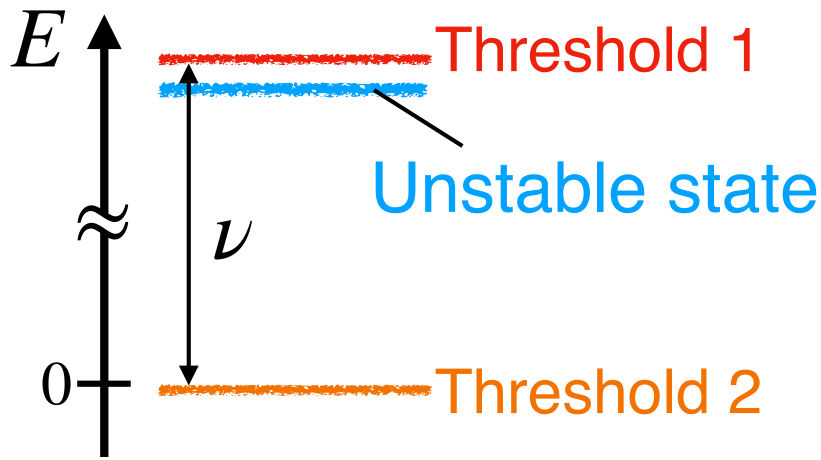
Weak-binding relation for unstable states

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right) + \mathcal{O}\left(\left|\frac{l}{R}\right|^3\right)\right\}$$

$l \equiv (2\mu\nu)^{-1/2}$

ν (deference between
the threshold energies)

$R \equiv (2\mu E)^{-1/2}$, E (eigenenergy)



When $|R| \gg R_{\rm typ}$ and $|R| \gg l$: observables (a_0, E)
→ compositeness(X)

Interpretation of X → $\tilde{X} \equiv \dfrac{1 - |1-X| + |X|}{2} \in \mathbb{R} \quad \because X \in \mathbb{C}$

Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

Extension to unstable states

What are the features of weak-binding relation in the effective range model with zero range limit ($\Lambda \to \infty$)?

$$f(k) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - ik\right]^{-1}$$

One length scale r_e except for a_0

→ $R_{\rm typ} = |r_e|$

- description of unstable states in the **single channel** case

$a_0, r_e \in \mathbb{C}$ → pole of $f(k) \in \mathbb{C}$ → eigenenergy $E \in \mathbb{C}$

unstable states

$$a_0 = |a_0| e^{i\theta_{a_0}}, r_e = -|r_e| e^{i\theta_{r_e}}$$

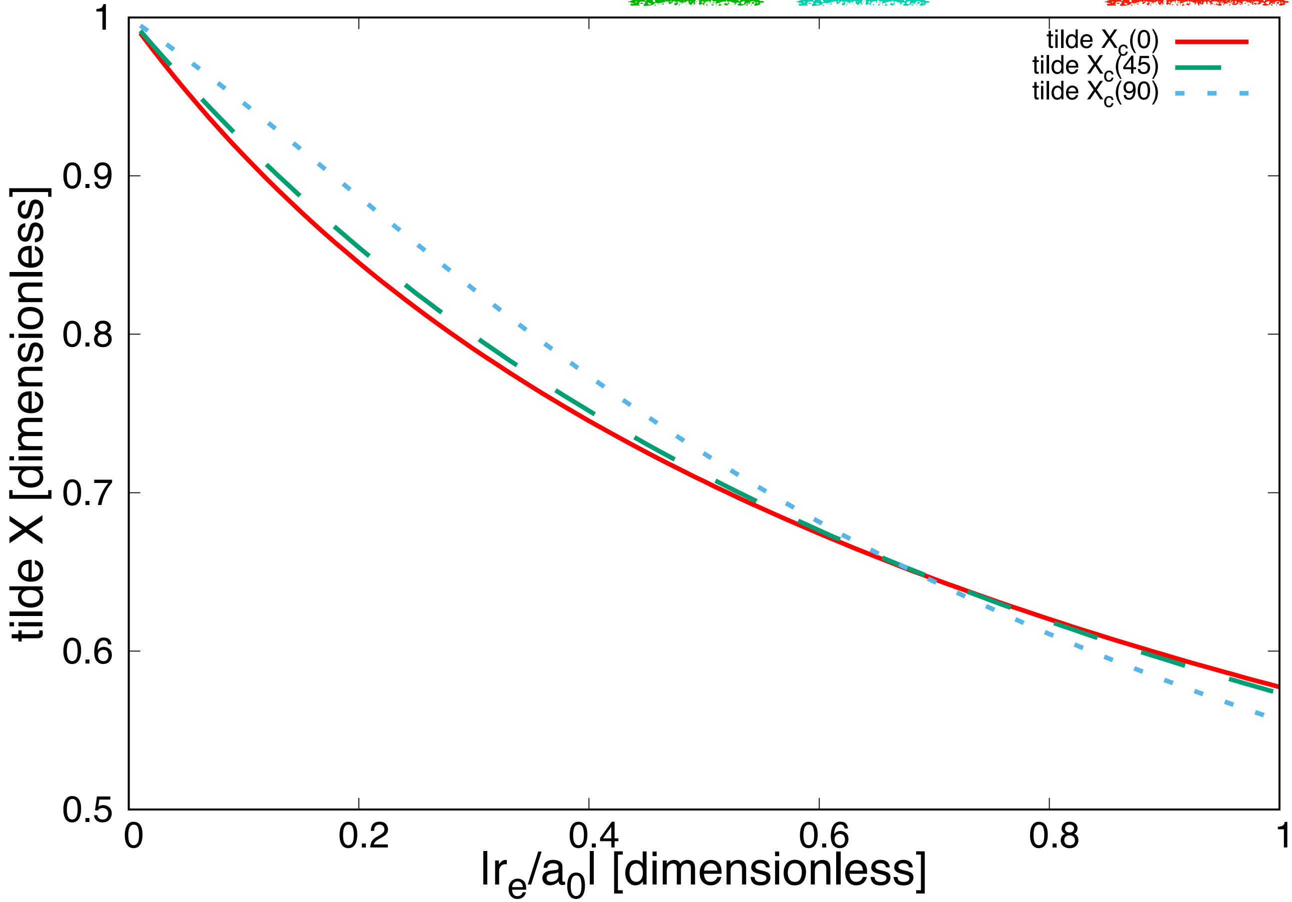
- $X(a_0, r_e) = X(a_0/r_e)$ in the zero range limit

→ X depends only on $\theta \equiv \theta_{a_0} - \theta_{r_e}$.

$$a_0 = |a_0| e^{i\theta},\ r_e = -|r_e|$$

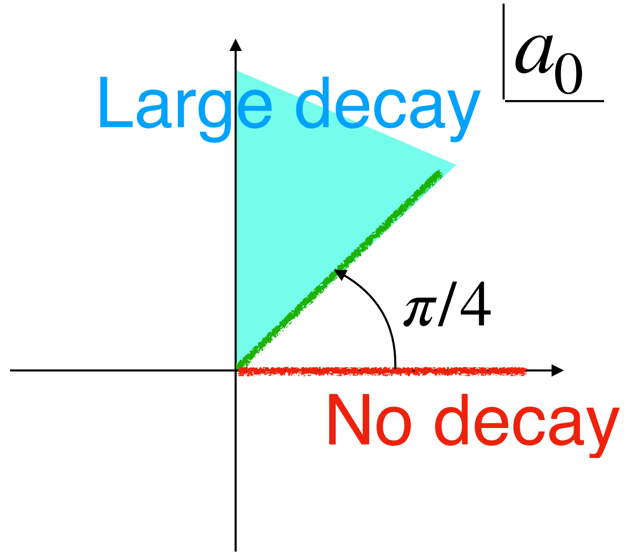
Extension to unstable states

$\tilde{X}$ for unstable ($\theta = \pi/4,\ \pi/2$) and stable ($\theta = 0$) states



$\theta = \pi/4,\ \pi/2$

→ Far from the stable state



There are almost no differences between $\tilde{X}$ for unstable and stable states.

→ Is this a feature of effective range model in the zero range limit?

Extension to unstable states

Estimation of uncertainty of interpretation (U):

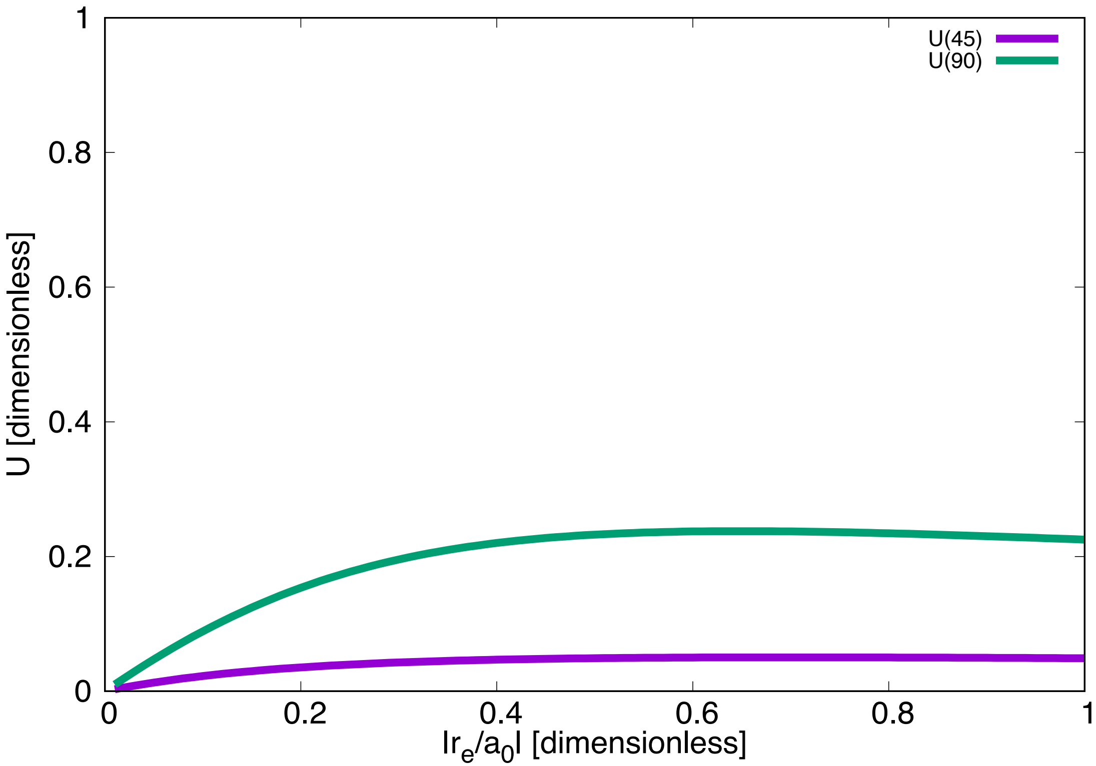
$$U \equiv |1 - X| + |X| - 1.$$

Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

-When U is small, we can interpret $\tilde{X}$ as compositeness and the system is similar to the stable state.

-When $X \in \mathbb{R}$, $U = 0$.

U ($\theta = \pi/2$ and $\pi/4$)



$U \lesssim 0.25$ in $\theta = \pi/2$ case

Far from the stable state

$\because a_0 = |a_0| e^{i\theta}$ is pure imaginary

This system has similar features to the stable state.

$\because X$ depends only on a_0/r_e.

Conclusion and future prospect

- Weak-binding relation : observable → compositeness (X)

$$a_0 = R\left\{ \frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right) \right\}$$

- We study the range correction in weak-binding relation from r_e.

- Improved weak-binding relation by redefinition of $R_{\rm typ}$:

$$R_{\rm typ} = \max\left\{ R_{\rm int}, |r_e|, \cdots \right\}$$

- We find the region where only the improved weak-binding relation can be applied.

- In effective range model in the zero range limit, the unstable state is similar to the stable state.

- Future prospect: Apply the improved relation to hadron systems.