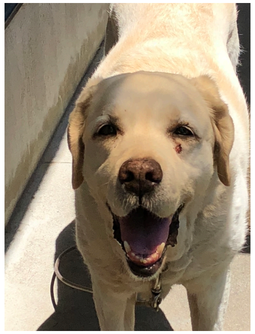
Application of the weak-binding relation with range correction



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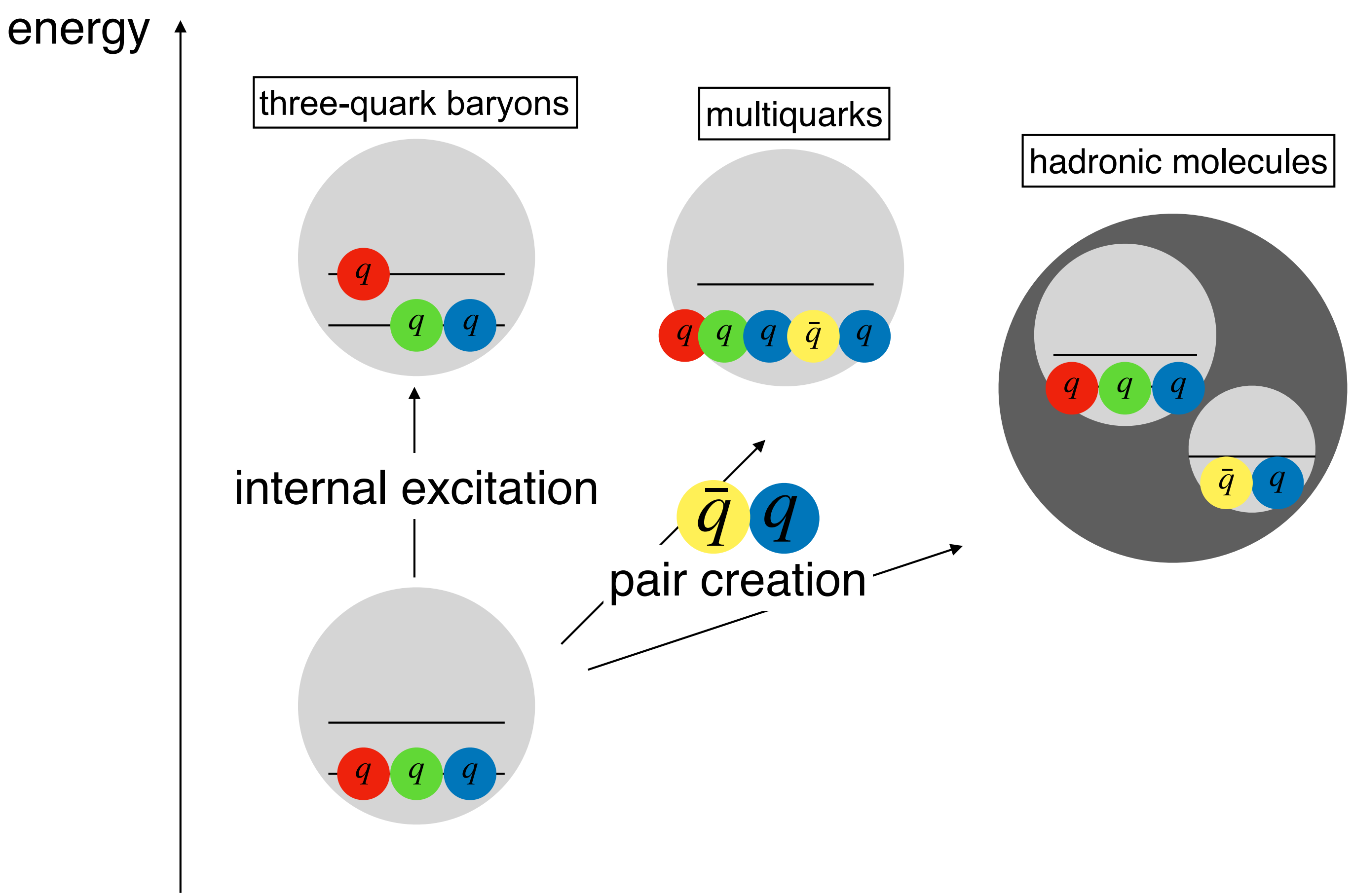
March 24th J-PARCハドロン研究会2022

Background

candidates for exotic hadrons → multiquarks, hadronic molecules

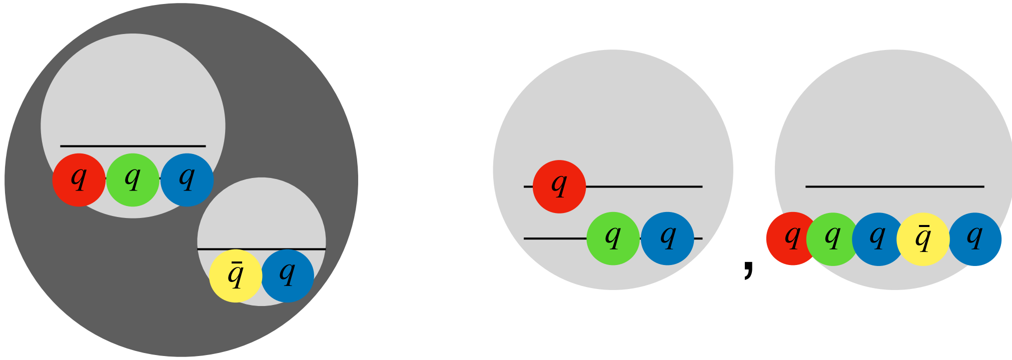
$\Lambda(1405)$, XYZ meson etc…



structure of hadrons ← model independent ← observable

Previous work

Hadron wave function

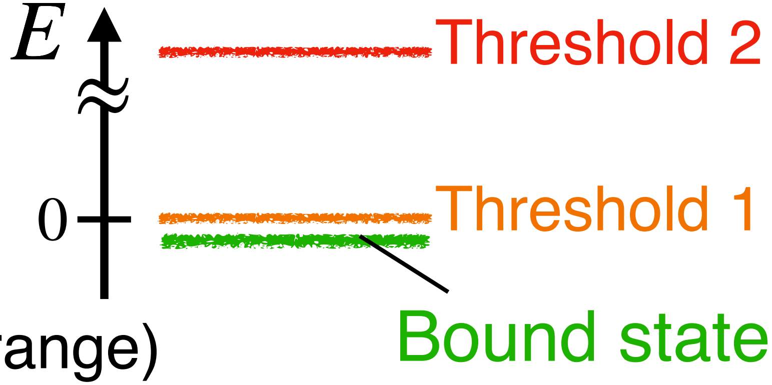


$$|\Psi\rangle = \sqrt{X}\,|\text{hadronic molecule}\rangle + \sqrt{1-X}\,|\text{others}\rangle$$

Compositeness (weight of hadronic molecule)

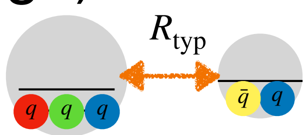
Weak-binding relation for bound state

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\}$$



a_0 (scattering length) R_{typ}(interaction range)

$R \equiv (2\mu B)^{-1/2}$, B (binding energy)



When $R \gg R_{\text{typ}}$: observable(a_0, B) → compositeness(X)

S. Weinberg, Phys. Rev. 137, B672 (1965); Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

Motivation

Weak-binding relation $a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$

<u>Low-energy universality</u>

Physical quantities are scaled by scattering length

→ $R = a_0$

Deviation from $a_0 = R$?

- Deviation by contributions from other channels ← $X \neq 1$
- Deviation by interaction range ← $R_{\rm typ} \neq 0$
- Other length scales ← $f(k) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - \frac{P_s}{4}k^4 + \cdots - ik\right]^{-1}$?

→ We study the range correction in the weak-binding relation by introducing the effective range r_e.

Effective range model

E. Braaten, M. Kusunoki, and D. Zhang, Annals Phys. 323, 1770 (2008), 0709.0499.

Effective range model in the zero-range limit:

- single channel: $|\Psi\rangle = |\text{hadronic molecule}\rangle \Leftrightarrow X = 1$

- point-like interaction (zero range): $R_{\text{typ}} \to 0$

$$\Rightarrow a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\} \to R$$

Renormalized scattering amplitude ($R_{\text{int}} \to 0$):

$$a_0 = R\frac{2r_e/R}{1-(r_e/R-1)^2} = R\left[1 + \mathcal{O}\left(\left|\frac{r_e}{R}\right|\right)\right] \Rightarrow \quad a_0 \neq R$$

range correction in the weak-binding relation form r_e

Range correction in weak-binding relation

Weak-binding relation $a_0 = R\left\{ \frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right) \right\}$

interaction range: $R_{\rm typ} \rightarrow R_{\rm int}$

Redefinition of $R_{\rm typ}$:

$$R_{\rm typ} = \max\left\{ R_{\rm int}, R_{\rm eff} \right\},$$

$$R_{\rm eff} = \max\left\{ |r_e|, \frac{|P_s|}{R^2}, \cdots \right\}.$$

$$f(k) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - \frac{P_s}{4}k^4 + \cdots - ik \right]^{-1}$$

Length scale in the effective range expansion except for a_0

It reduces to previous one when $R_{\rm typ} = R_{\rm int}$

Numerical calculation

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$$

When dose the weak-binding relation work?

Estimation with correction terms ($\xi \equiv R_{\rm typ}/R$): Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

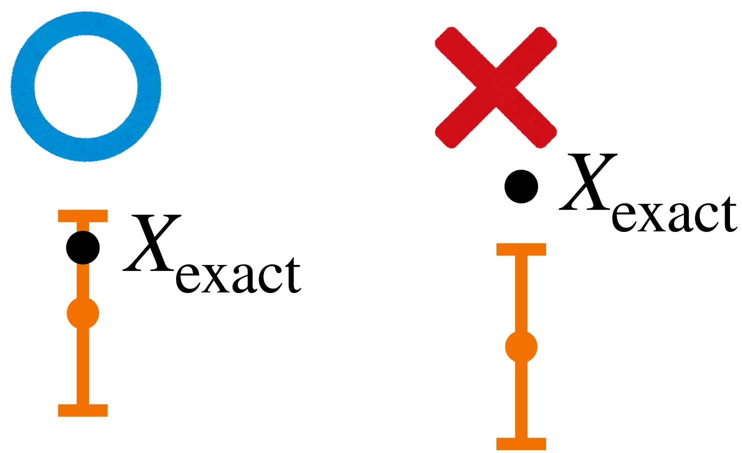
Central value: $X_c = \dfrac{a_0/R}{2 - a_0/R}$

$$X_{\rm upper}(\xi) = \frac{a_0/R + \xi}{2 - a_0/R - \xi}, \quad X_{\rm lower}(\xi) = \frac{a_0/R - \xi}{2 - a_0/R + \xi}.$$

Weak-binding relation works when…

$$X_{\rm lower} < X_{\rm exact} < X_{\rm upper}$$

→ Validity condition



Numerical calculation

Effective range model ($R_{\text{int}} \neq 0$)

$$f(k;\lambda_0,\rho_0,\Lambda) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 + \mathcal{O}\left(\frac{R_{\text{int}}}{R}\right) - ik\right]^{-1}$$ (two length scales r_e and R_{int})

$$1/f(k = i/R) = 0$$

- $r_e \neq 0$ (range correction): $\xi_{r_e} = |r_e/R|$ → Uncertainty from r_e

  $r_e < 0$ (effective range model)

- $R_{\text{int}} \neq 0$: $\xi_{\text{int}} = R_{\text{int}}/R$. → Uncertainty from R_{int}

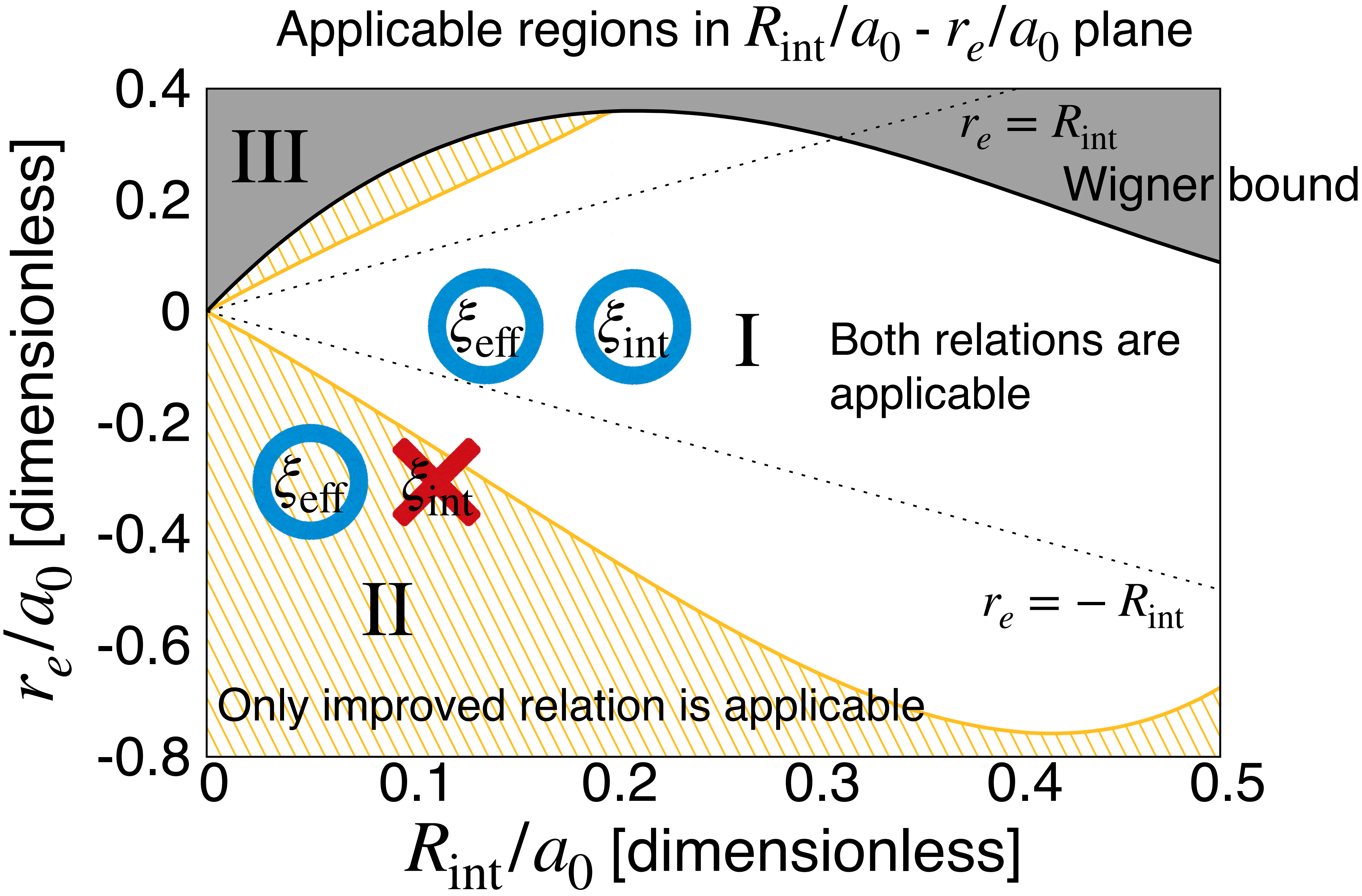
- $X_{\text{exact}} = 1$

→ We search for the region of r_e and R_{int} where validity condition are satisfied.

Numerical calculation

$X_{\rm exact}$ ○

T. Kinugawa, T. Hyodo, (2021), arXiv:2111.06619



Applicable region become larger with range correction.

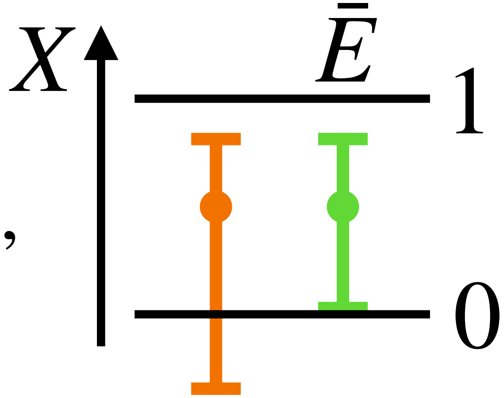
Precision

Precision of X estimated from weak-binding relation ?

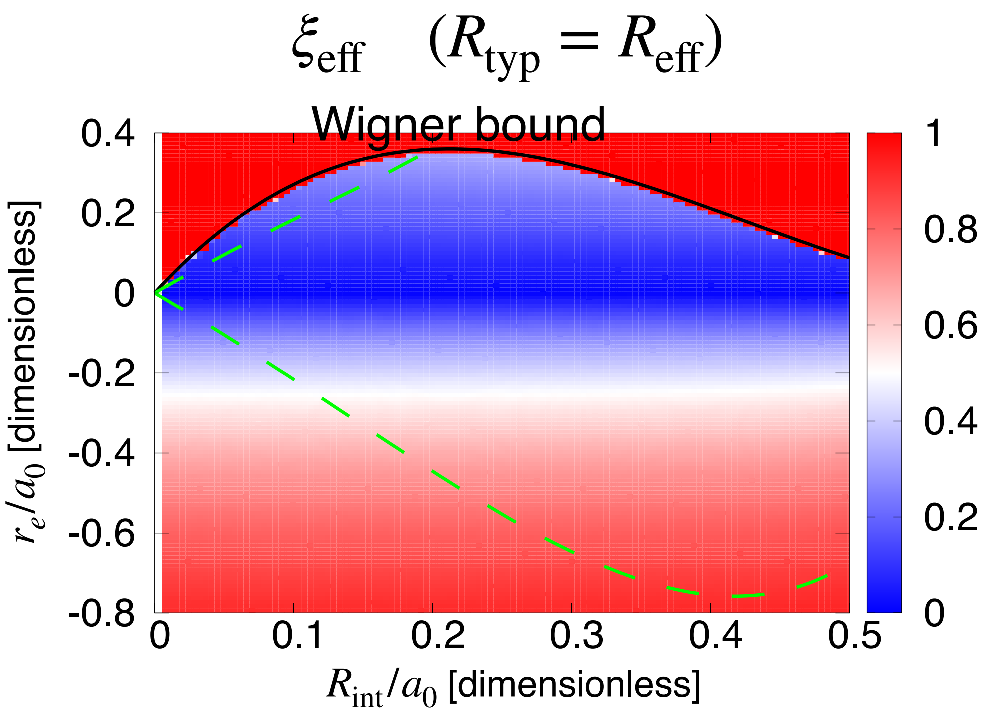
Uncertainty $\bar{E}$

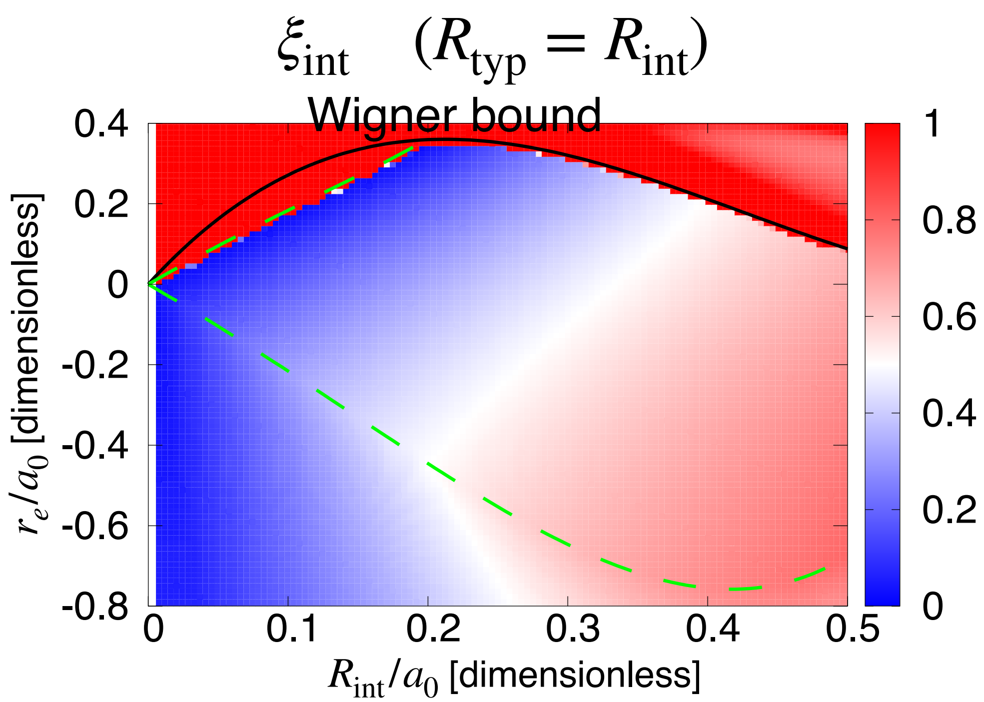
$$\bar{E} = \bar{X}_u - \bar{X}_l, \quad \bar{X}_u = \min\{X_u, 1\}, \quad \bar{X}_l = \max\{X_l, 0\},$$

For the reasonable estimation : $\bar{E} \leq 0.5$



Uncertainty $\bar{E}$ in $R_{\rm int}/a_0$ - r_e/a_0 plane (effective range model)





Application

T. Kinugawa, T. Hyodo, (2022), arXiv:2201.04283

bound state	2-body system		a_0	r_e	$R_{\rm int}$	
d	p	n	5.42 fm	1.75 fm	1.43 fm	[1]
$X(3872)$	D^0	$\bar{D}^{*0}$	28.5 fm	-5.34 fm	1.43 fm	[2]
$N\Omega$ dibaryon	N	Ω	5.30 fm	1.26 fm	0.676 fm	[3]
$\Omega\Omega$ dibaryon	Ω	Ω	4.6 fm	1.27 fm	0.949 fm	[4]
${}^3_\Lambda$H	d	Λ	16.8 fm	2.3 fm	4.31 fm	[5]
^{4}He dimer	^{4}He	^{4}He	189 B.R.	13.8 B.R.	10.2 B.R.	[6]

Low energy universality $\because a_0 > R_{\rm int}$

States other than ${}^3_\Lambda$H : $|r_e| > R_{\rm int}$ → Range correction is important?

$R_{\rm typ} = R_{\rm eff}$

[1] R. Machleidt, Phys. Rev. C **63**, 024001 (2001), nucl-th/0006014.

[2] A. Esposito, L. Maiani, A. Pilloni, A. D. Polosa and V. Riquer (2021), 2108.11413.

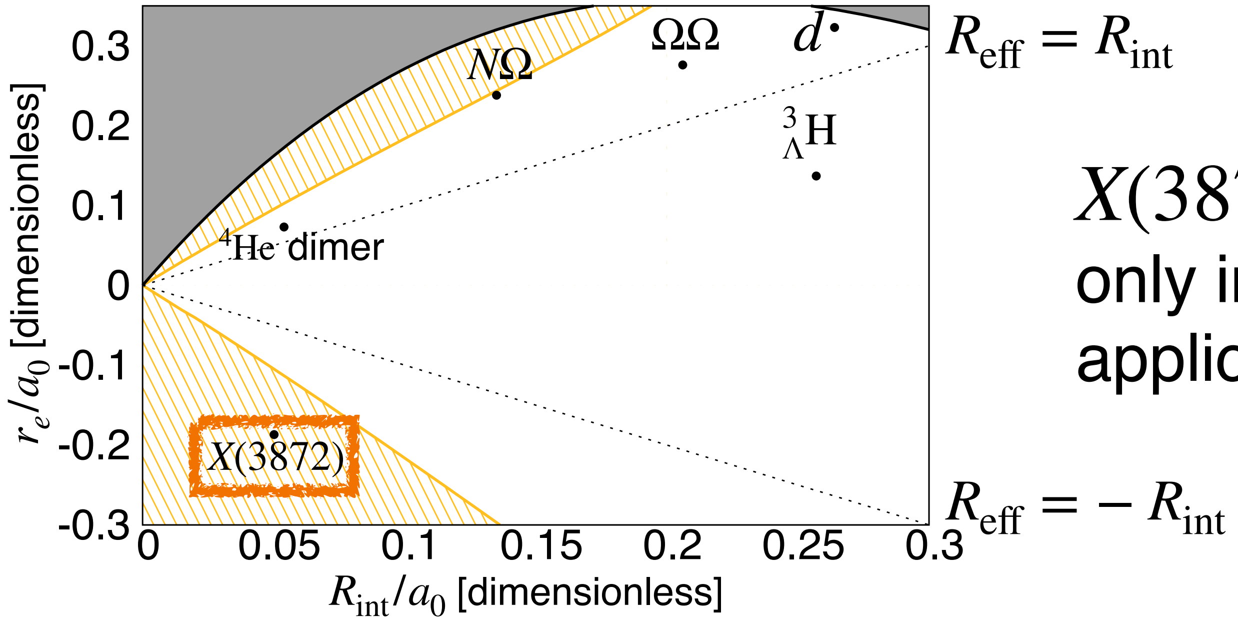
[3] HAL QCD, T. Iritani *et al.*, Phys. Lett. B **792**, 284 (2019), 1810.03416.

[4] S. Gongyo *et al.*, Phys. Rev. Lett. **120**, 212001 (2018), 1709.00654.

[5] H. W. Hammer, Nucl. Phys. A **705**, 173 (2002), nucl-th/0110031.

[6] A. Kievsky and M. Gattobigio, Phys. Rev. A **87**, 052719 (2013), 1212.3457.

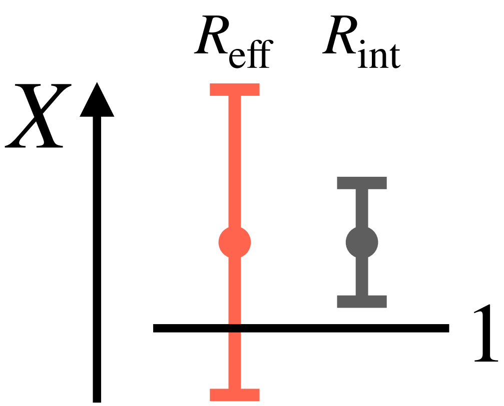
Application



$X(3872)$ is in region where only improved relation is applicable

lower bound of X (X_l)

束縛状態	R_{eff}	R_{int}
d	0.738	0.857
$X(3872)$	0.530	0.681
$N\Omega$	0.787	1.04
$\Omega\Omega$	0.775	0.934
${}^3_\Lambda$H	0.745	0.745
^{4}He dimer	0.929	0.967



X of $N\Omega$ dibaryon from previous relation

$\longrightarrow X_l > 1$

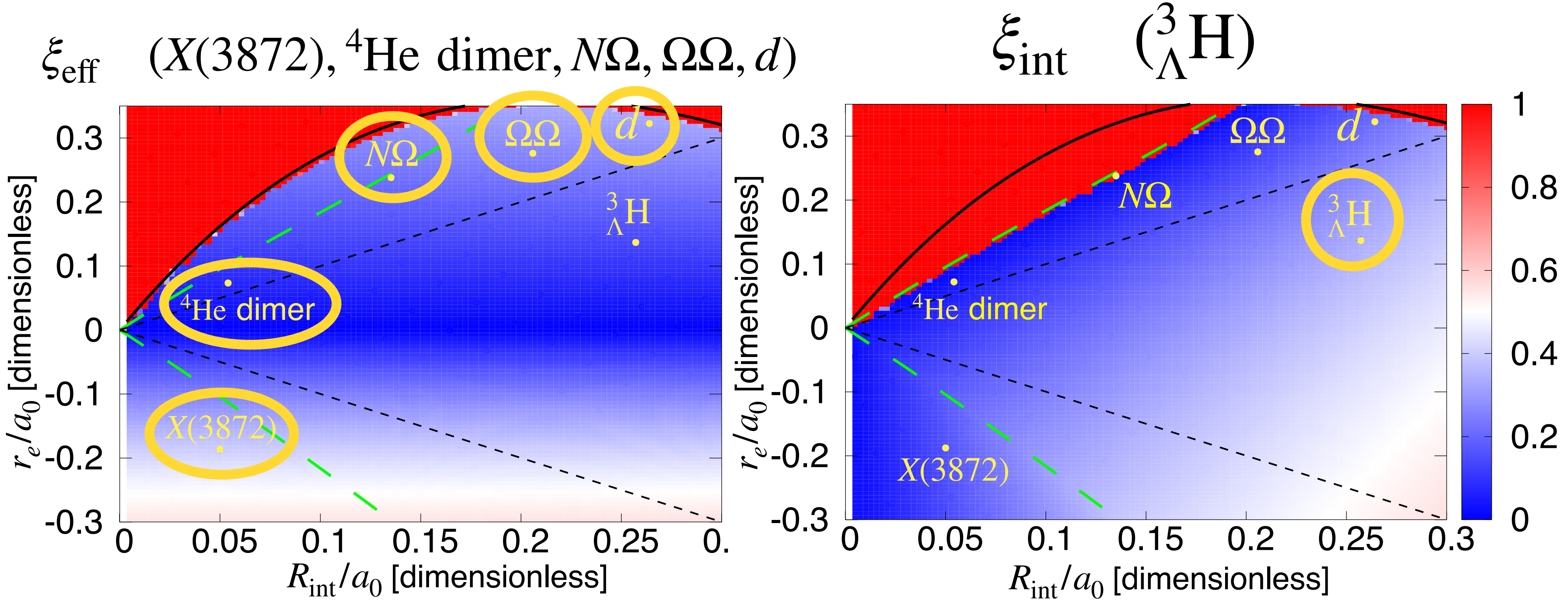
c.f. definition of X

$$0 \leq X \leq 1$$

We need range correction to estimate X of $X(3872)$ and $N\Omega$.

Application

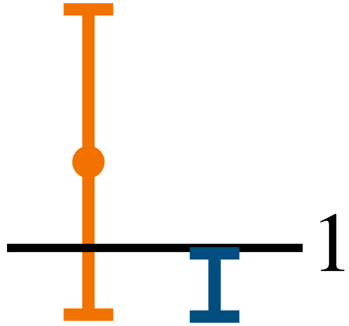
Uncertainty $\bar{E}$ with systems in $R_{\rm int}/a_0$ - r_e/a_0 plane



We can estimate X of all states with small uncertainty $\bar{E} \lesssim 0.5$.

Application

T. Kinugawa, T. Hyodo, (2022), arXiv:2201.04283



Estimated X

(X from relation) $\cap\ 0 \leq X \leq 1$

bound state	X
d	$0.738 \leq X \leq 1$
$X(3872)$	$0.530 \leq X \leq 1$
$N\Omega$	$0.801 \leq X \leq 1$
$\Omega\Omega$	$0.791 \leq X \leq 1$
${}^{3}_{\Lambda}\mathrm{H}$	$0.745 \leq X \leq 1$
${}^{4}\mathrm{He}$ dimer	$0.92 \leq X \leq 1$

- All states are dominated by the composite components; $0.5 < X$

- model calculation of $X(3872)$

M. Takizawa and S. Takeuchi, PTEP 2013, 093D01 (2013), arXiv:1206.4877.

$$|X(3872)\rangle = c_1 |c\bar{c}\rangle + c_2 |D^0\bar{D}^{*0}\rangle + c_3 |D^+D^{*-}\rangle$$

$$|c_2|^2 = X$$

model calculation $0.759 \leq X \leq 0.897$

This work (model-independent) $0.530 \leq X \leq 1$

Conclusion and future prospect

- Weak-binding relation : observable → compositeness (X)

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$$

- We study the range correction in weak-binding relation from r_e.

- Improved weak-binding relation by redefinition of $R_{\rm typ}$:

$$R_{\rm typ} = \max\left\{R_{\rm int}, |r_e|, \cdots\right\}$$

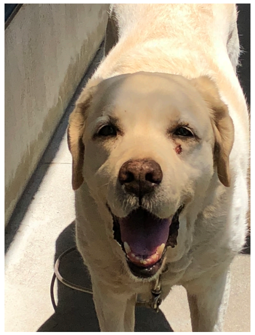
- We find the region where only the improved weak-binding relation can be applied.

- We apply the relations to the actual systems.

→ find that the range correction is important for some states.

- Future prospect: extend the improved relation to the unstable states

レンジ補正を考慮した弱束縛関係式の応用



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applicable region

Resonance model ($X_{\mathrm{exact}} \leq 1$)

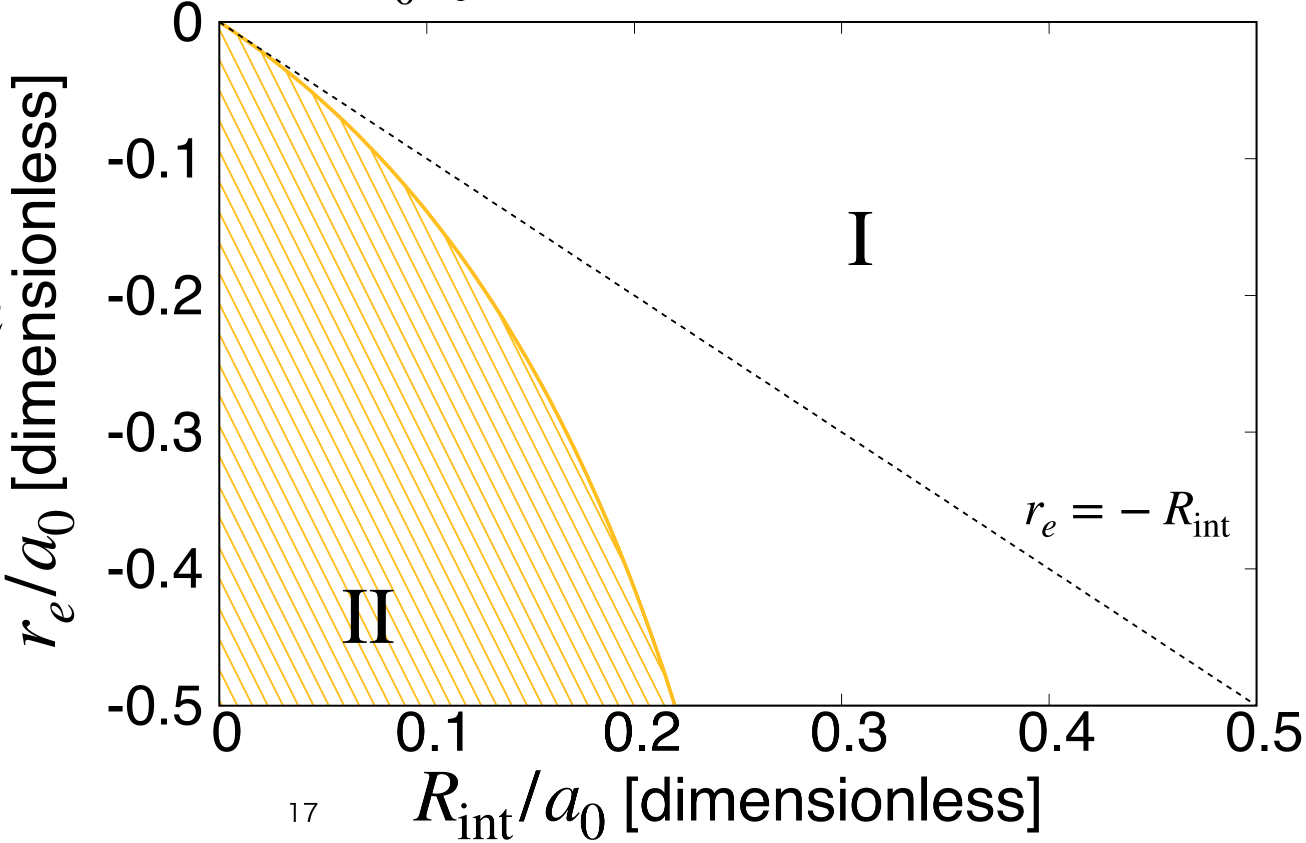
$$X_{\mathrm{exact}}^{-1} = 1 + 16\pi\kappa \frac{g_0^2}{\left\{ (-\kappa^2 - \nu_0)\left(\frac{8\pi}{1 - \frac{2}{\pi}\Lambda} + \frac{g_0^2}{\nu_0} \right) - g_0^2 \right\}^2},$$

$$f(k)^{-1} = -\frac{8\pi}{m}\left(\lambda_0 + \frac{g_0^2}{E - \nu_0} \right)^{-1} - \frac{2}{\pi}\Lambda - ik,$$

$$r_e = -\frac{16\pi g_0^2}{m^2\nu_0^2}\left(-\frac{g_0^2}{\nu_0} + \lambda_0 \right)^{-2}$$

< 0 (Wigner bound)

Applicable region in R_{int}/a_0 - r_e/a_0 plane ($ma_0^2\nu_0 = -0.5$)



Discussion of the range correction

T. Kinugawa, T. Hyodo, (2021), arXiv:2112.00249

What is the range correction for the central value ?

From effective range model ($X = 1$)

$$a_0 = R\frac{2r_e/R}{1-(r_e/R-1)^2} = R\left[1 + \mathcal{O}\left(\left|\frac{r_e}{R}\right|\right)\right] \quad \Rightarrow a_0 \neq R$$

→ $$\bar{R} \equiv R\frac{2r_e/R}{1-(r_e/R-1)^2}$$

→ $a_0 = \bar{R}$ and $X = 1$

→ $$a_0 = \bar{R}\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm int}}{\bar{R}}\right)\right\} \; ?$$

↑ Range correction for the central value ?

Discussion of the range correction

What are the origins of the effective range?

$$\mathscr{H}_{\text{int}} = \frac{1}{4}\lambda_0(\psi^\dagger\psi)^2 + \frac{1}{4}\rho_0\nabla(\psi^\dagger\psi)\cdot\nabla(\psi^\dagger\psi) + \frac{1}{2}g_0(\phi^\dagger\psi^2 + \psi^{\dagger 2}\phi)$$

derivative couplings — $X = 1$

Bare field ϕ — $X \neq 1$

One of the origins of the deviation of a_0 form R

$$\rightarrow r_e = \frac{16\pi}{m}\frac{[1+(m/12\pi^2)\rho_0\Lambda^3]^2\{2\rho_0[1+(m/24\pi^2)\rho_0\Lambda^3] - g_0{}^2/(mv_0{}^2)\}}{\left[\lambda_0 - (m/20\pi^2)\rho_0^2\Lambda^5 - \frac{g_0^2}{v_0}\right]^2}$$

v_0 : bare mass Λ: cut off

Both derivative coupling ($X = 1$, ρ_0 term) and contribution from bare field ($X \neq 1$, g_0 term) can be the origins of r_e

Discussion of the range correction

If these terms were separable…

$$r_e = \text{Contribution of derivative couplings} + \text{Contribution of bare field } \phi$$

we would consider the range correction for X_c

$$\rightarrow \quad a_0 = \bar{R}\left\{ \frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{\bar{R}}\right) \right\}$$

Contribution of derivative couplings ($\bar{R}$)

Contribution of bare field ϕ ($\frac{2X}{1+X}$)

However

→ we can not consider the range correction for the central value because r_e is not separable.

$$r_e = \frac{16\pi}{m} \frac{[1 + (m/12\pi^2)\rho_0\Lambda^3]^2\{2\rho_0[1 + (m/24\pi^2)\rho_0\Lambda^3] - g_0{}^2/(m v_0{}^2)\}}{\left[\lambda_0 - (m/20\pi^2)\rho_0^2\Lambda^5 - \frac{g_0^2}{v_0}\right]^2}$$