# Compositeness of near-threshold states with Coulomb plus short range interaction

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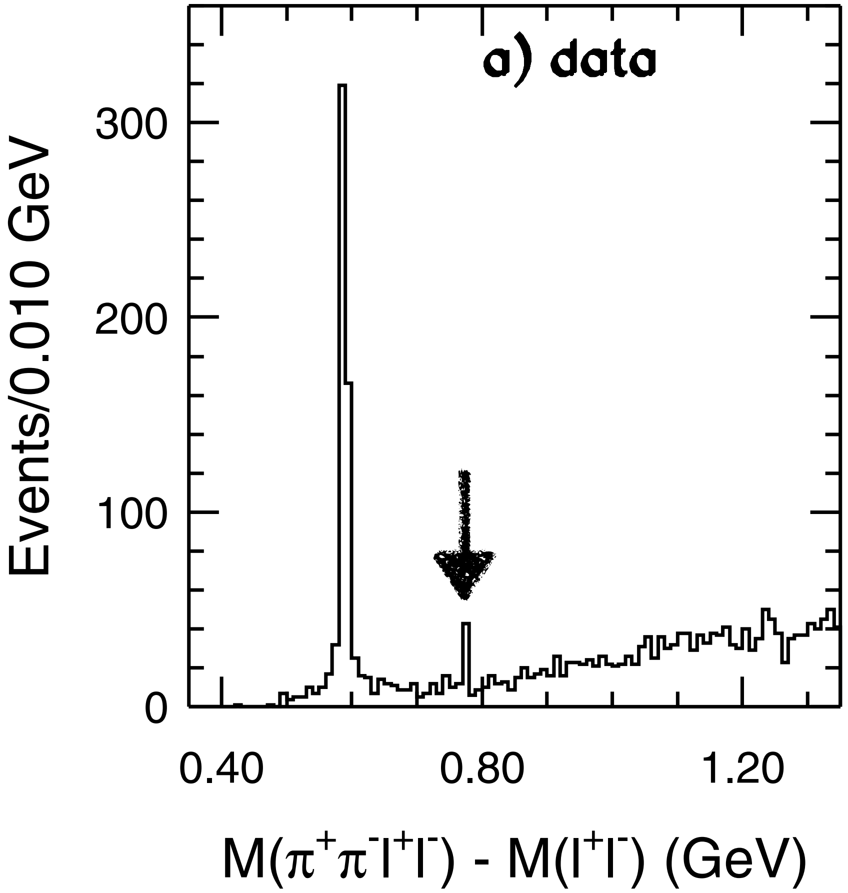
March 27th, HADRON 2025

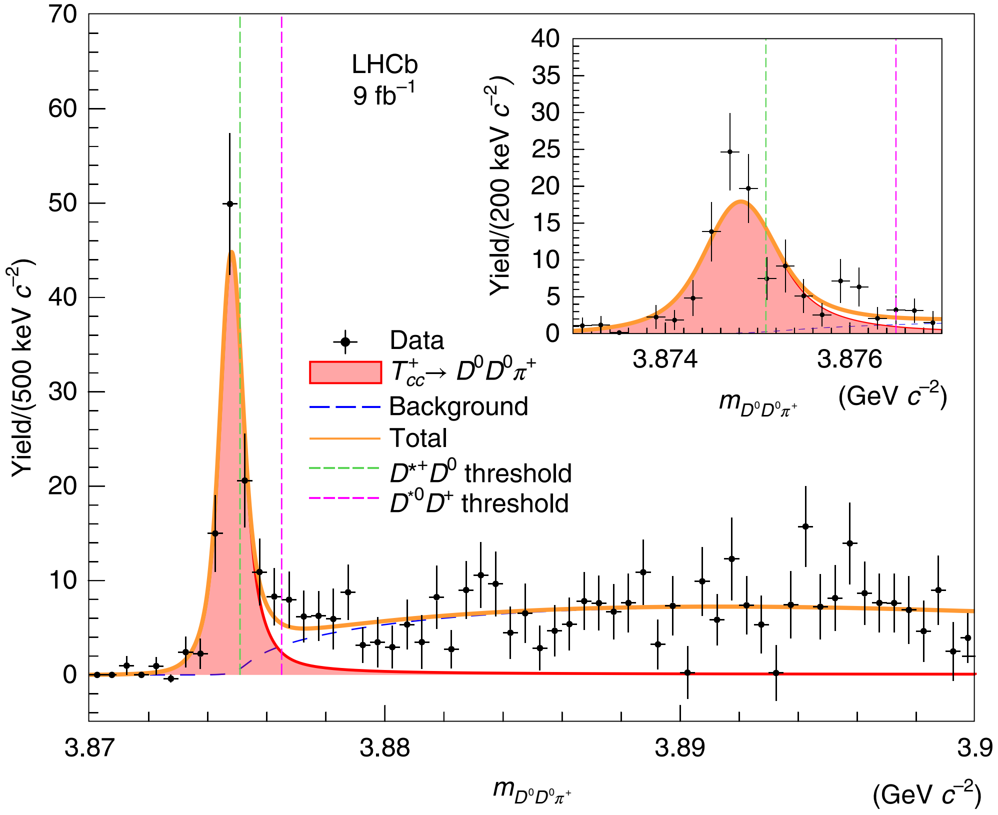
# Near-threshold exotic hadrons

$X(3872) \to \pi^+\pi^- J/\psi$



S. K. Choi *et al*. (Belle), Phys. Rev. Lett. **91**, 262001 (2003).

$T_{cc}(3875)^+ \to D^0D^0\pi^+ \ (cc\bar{u}\bar{d})$

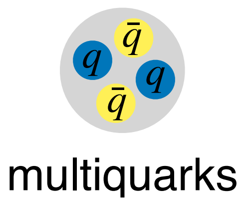


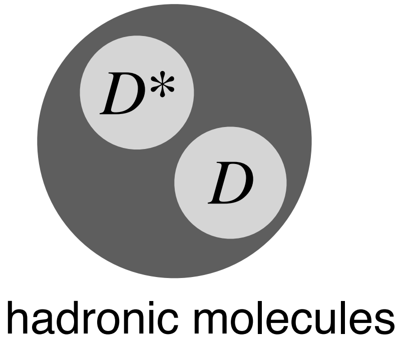
LHCb Collaboration, Nature Phys. **18** (2022) no.7, 751-754;

LHCb Collaboration, Nat. Commun. **13** 3351 (2022).

- exotic hadron

$\neq$ $qqq$ or $q\bar{q}$ → multiquarks
hadronic molecules





- internal structure of near-threshold states?

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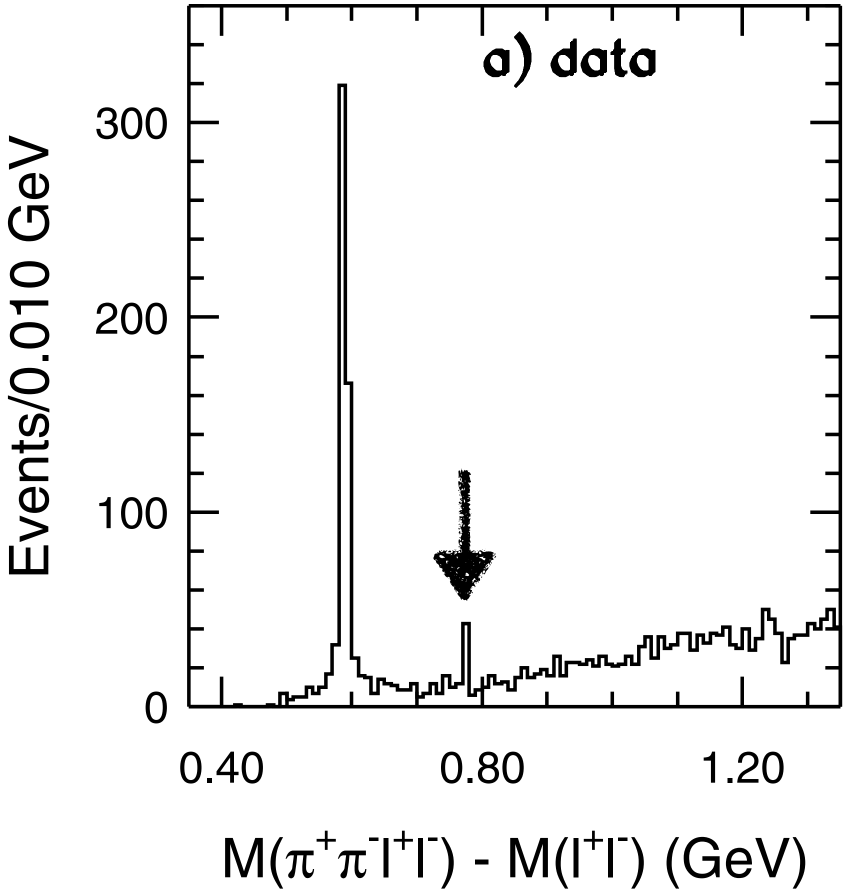

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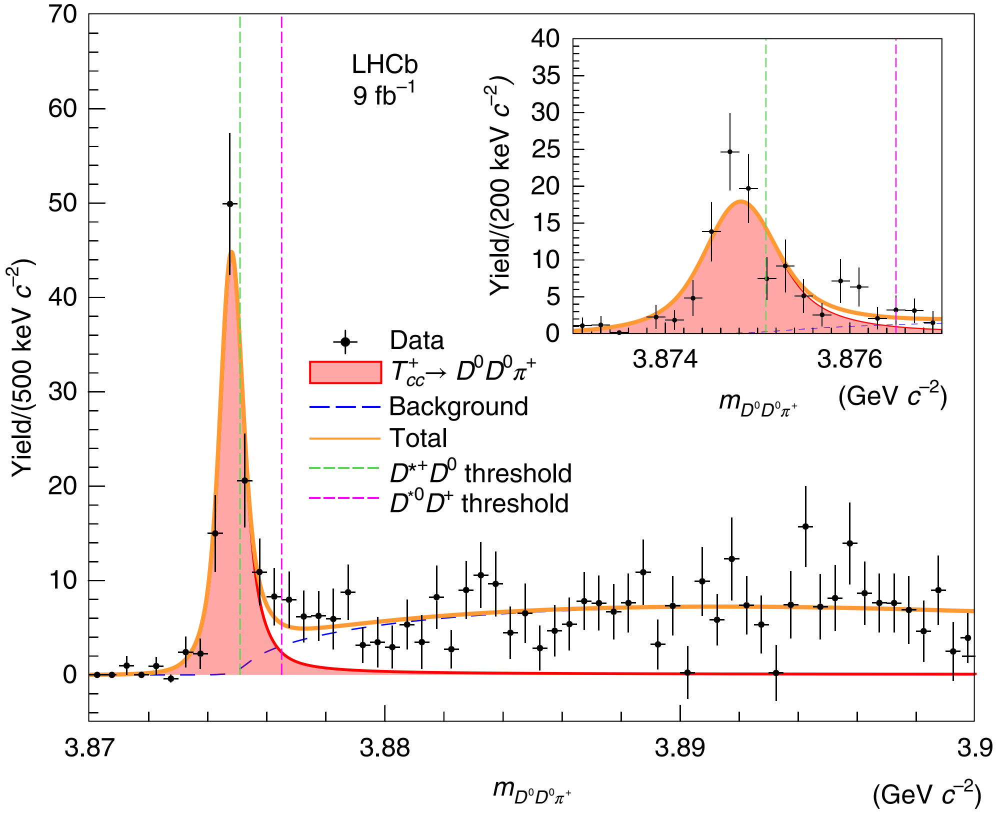

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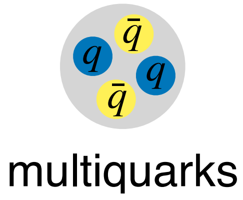

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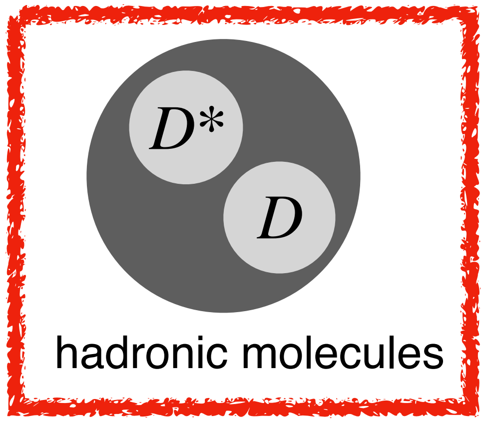
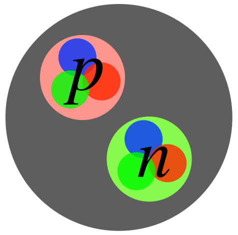


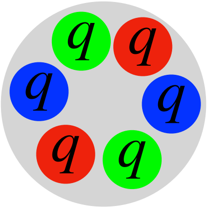
- internal structure of near-threshold states?

# Compositeness

S. Weinberg, Phys. Rev. 137, 672–678 (1965);
T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013).
T. Kinugawa and T. Hyodo, arXiv: 2411.12285 [hep-ph], accepted in EPJ.

## ◉ definition





wavefunction

$$|\Psi\rangle = \sqrt{\underline{X}}\,|\text{composite}\rangle + \sqrt{\underline{1-X}}\,|\text{non composite}\rangle$$

**compositeness** **elementarity**

※ $0 \le X \le 1 \longrightarrow X > 0.5 \Leftrightarrow$ **composite dominant**

$X + Z = 1$   $X < 0.5 \Leftrightarrow$ **elementary dominant**

- **quantitative** analysis of internal structure

deuteron S. Weinberg, Phys. Rev. 137, 672–678 (1965) etc.

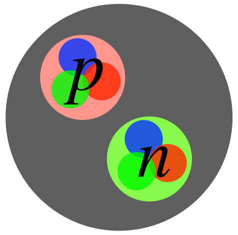
$f_0(980)$, $a_0(980)$ V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett. B 586, 53–61 (2004);
Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016) etc.

$T_{cc}$, $X(3872)$ T. Kinugawa and T. Hyodo, Phys. Rev. C 109 , 045205 (2024);
L. R. Dai, L. M. Abreu, A. Feijoo, and E. Oset, Eur. Phys. J. C 83, 983 (2023) etc.

exotic nuclei, atomic systems T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022) etc.

# Near-threshold states

## ◉ near-threshold states with short range interaction



- at threshold ($E = 0$)

  completely composite ($X = 1$)

  $\because$ low-energy universality $|a_{\mathrm{s.r.}}| \to \infty$

  T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

- near-threshold bound states

  ($E \neq 0$, but small **negative**)

  composite dominant ($X \sim 1$)

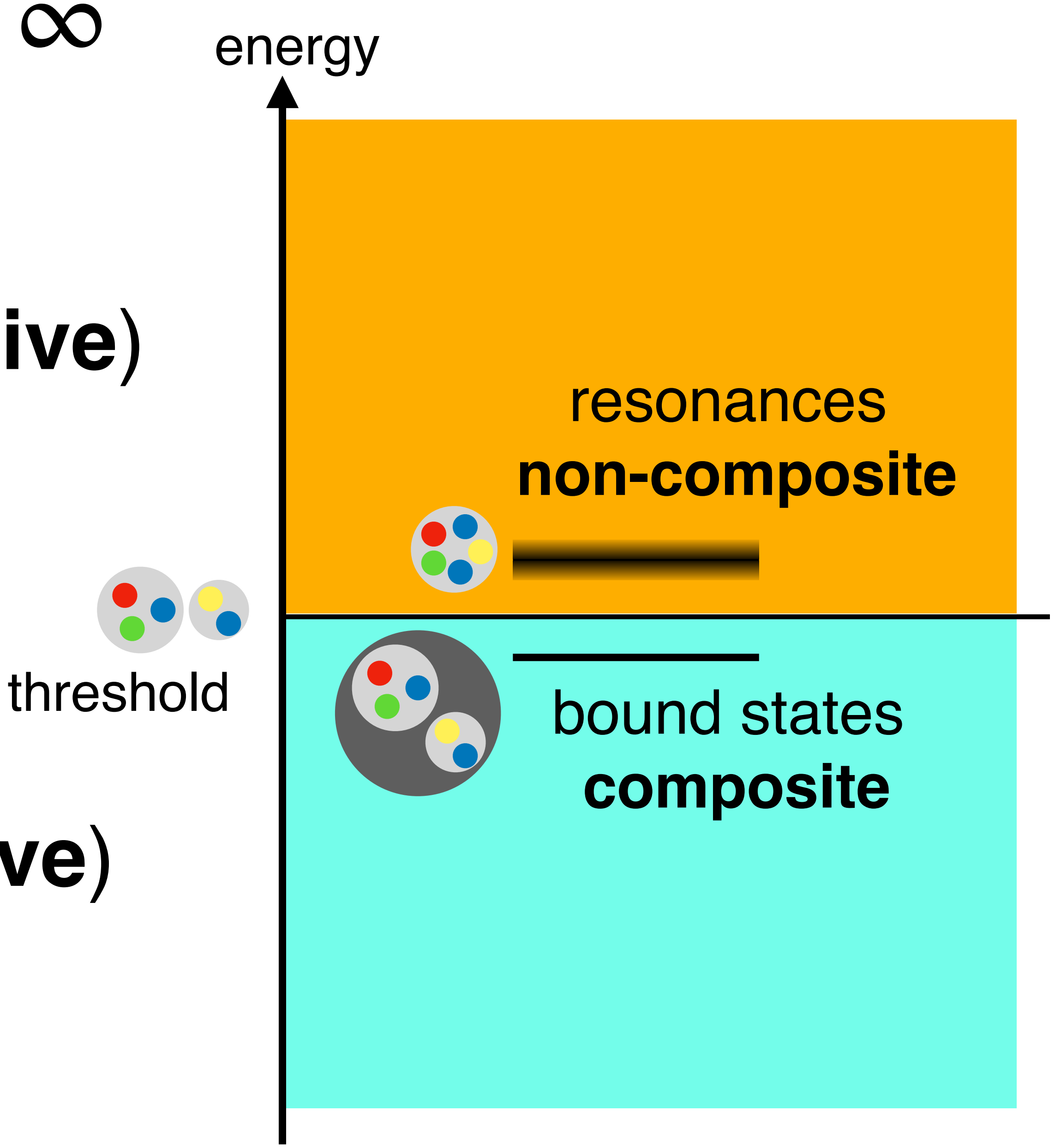
  C. Hanhart, J. R. Pelaez, and G. Rios, Phys. Lett. B **739**, 375 (2014);
  T. Kinugawa and T. Hyodo Phys. Rev. C 109 , 045205 (2024).

- near-threshold resonances

  ($E \neq 0$, but small **positive**)

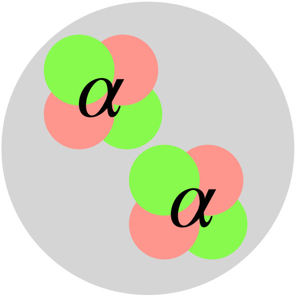
  **non**-composite dominant ($\mathscr{X} \sim 0$)

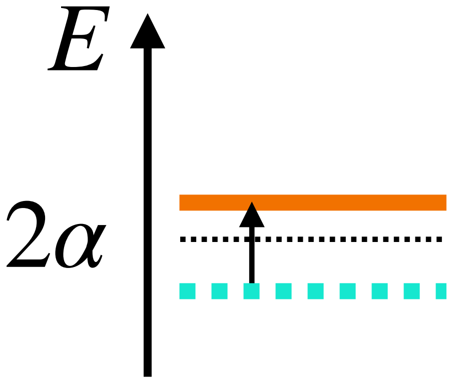
  T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph].



# Coulomb + short range systems

- $^{8}$Be nuclei ($2\alpha^{++}$) J. Hiura, and R. Tamagaki, Prog. Theor. Phys. Suppl. No. 52, 25 (1972).



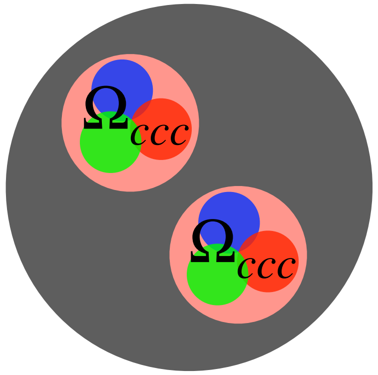


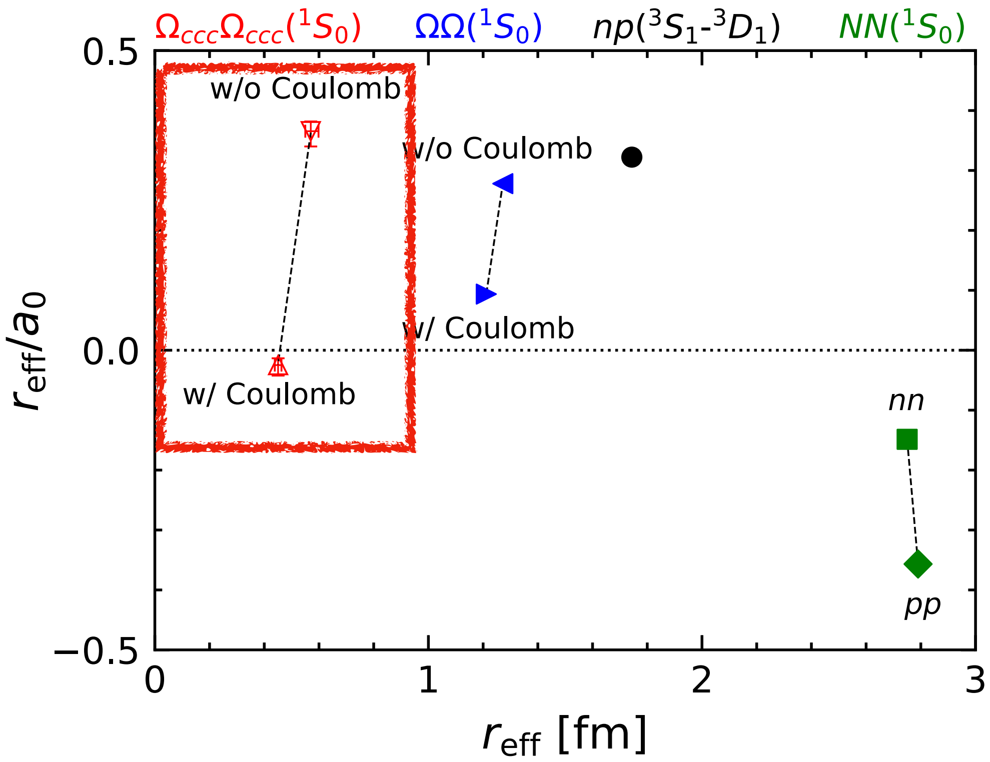
resonance (w/ Coulomb)

↑

bound (w/o Coulomb)

- $\Omega_{ccc}^{++}\,\Omega_{ccc}^{++}$ (HAL QCD) Y. Lyu, H.Tong, *et al*. [HAL QCD Coll.], Phys. Rev. Lett. 127 (2021) 072003.





resonance (w/ Coulomb)

↑

bound (w/o Coulomb)

- $\Xi^-\alpha$ : Coulomb assisted bound state E. Hiyama, M. Isaka, T. Doi, and T. Hatsuda, Phys. Rev. C 106, 064318 (2022).

→ Coulomb is important for near-threshold states!

# Coulomb + short range systems

## ◉ Coulomb + short range interaction

H. A. Bethe, Phys. Rev. 76, 38-50 (1949).

R. Oppenheim Berger and Larry Spruch, Phys. Rev. 138, B1106-B1115 (1965).

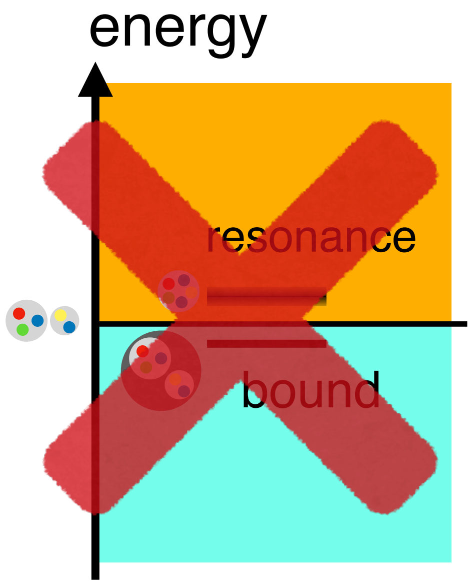
W. Domcke, Atom. Mol. Phys. 16, 359 (1983).

R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809, 171 (2008).

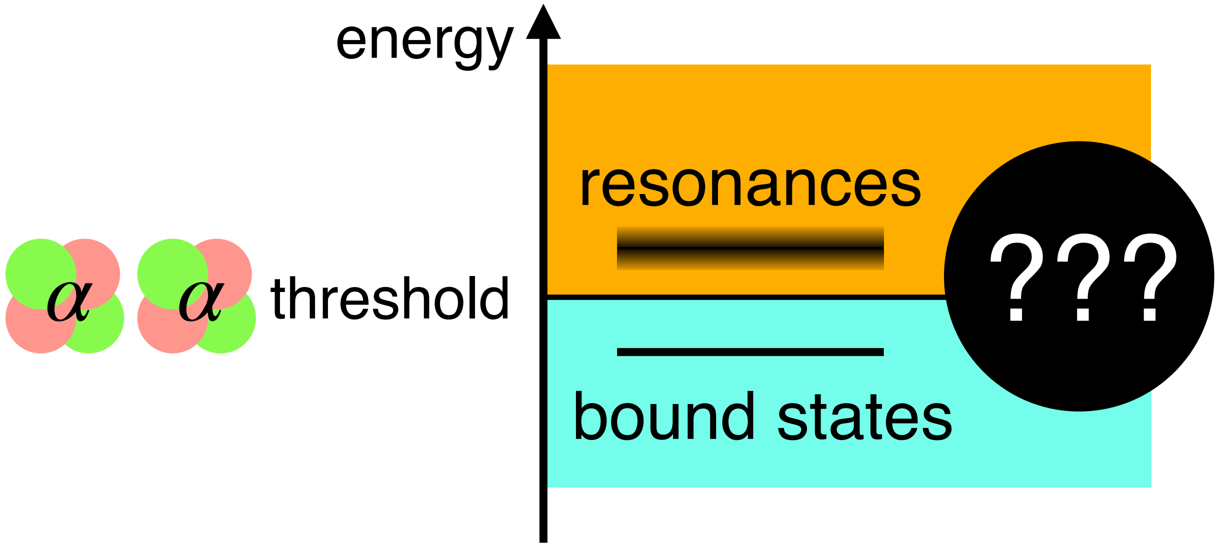
C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798, 135016 (2019).

S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

- low-energy behavior of scattering amplitude is different from that of short range interaction



## ◉ nature of near-threshold state with Coulomb + short range interaction?



- two body
- small $k$ region → $s$-wave
- Coulomb repulsive

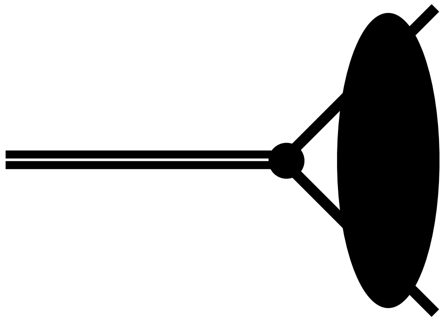
→ pole trajectories?

→ compositeness?

# Coulomb+short range model

## ◉ Hamiltonian

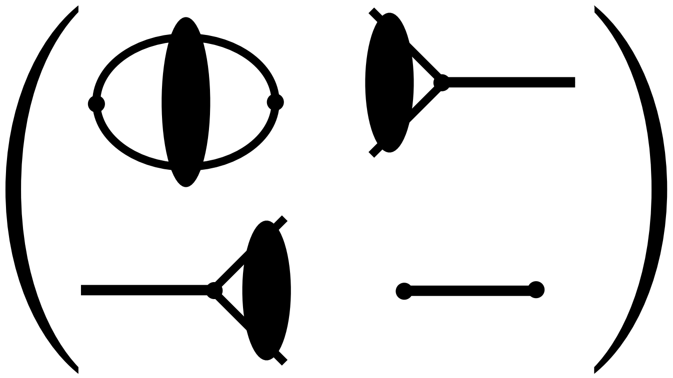
W. Domcke, Atom. Mol. Phys. 16 359 (1983). R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).



Q channel
⇔ bound states
w/ short range

P channel
⇔ scattering w/ Coulomb

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} =$$



H. Feshbach, Annals Phys. 19 287-313 (1962).

## ◉ pole condition

H. A. Bethe, Phys. Rev. 76, 38-50 (1949).
C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798 (2019).

$$-\frac{1}{\boxed{a_s}} + \frac{\boxed{r_e}}{2}k^2 - ik \pm \frac{2}{\boxed{a_B}}\left[\underbrace{\log(-ia_Bk)}_{\text{log cut}} + \psi\left(1 + \frac{i}{a_Bk}\right)\right] = 0$$

## ◉ compositeness $X$

T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

$$X = 1 - \frac{1}{1 - \frac{d}{dE}\boxed{F(E)}}$$

self energy

Bohr radius

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

# Coulomb+short range model

◉ short range limit $a_B \to \infty$

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B}\underline{\left[\log(-ia_Bk) + \psi\left(1 + \frac{i}{a_Bk}\right)\right]} = 0$$

$$\to 0$$

$$\longrightarrow -\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik = 0$$ short range interaction

◉ further low-energy limit $r_e \to 0$

- zero-range theory S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

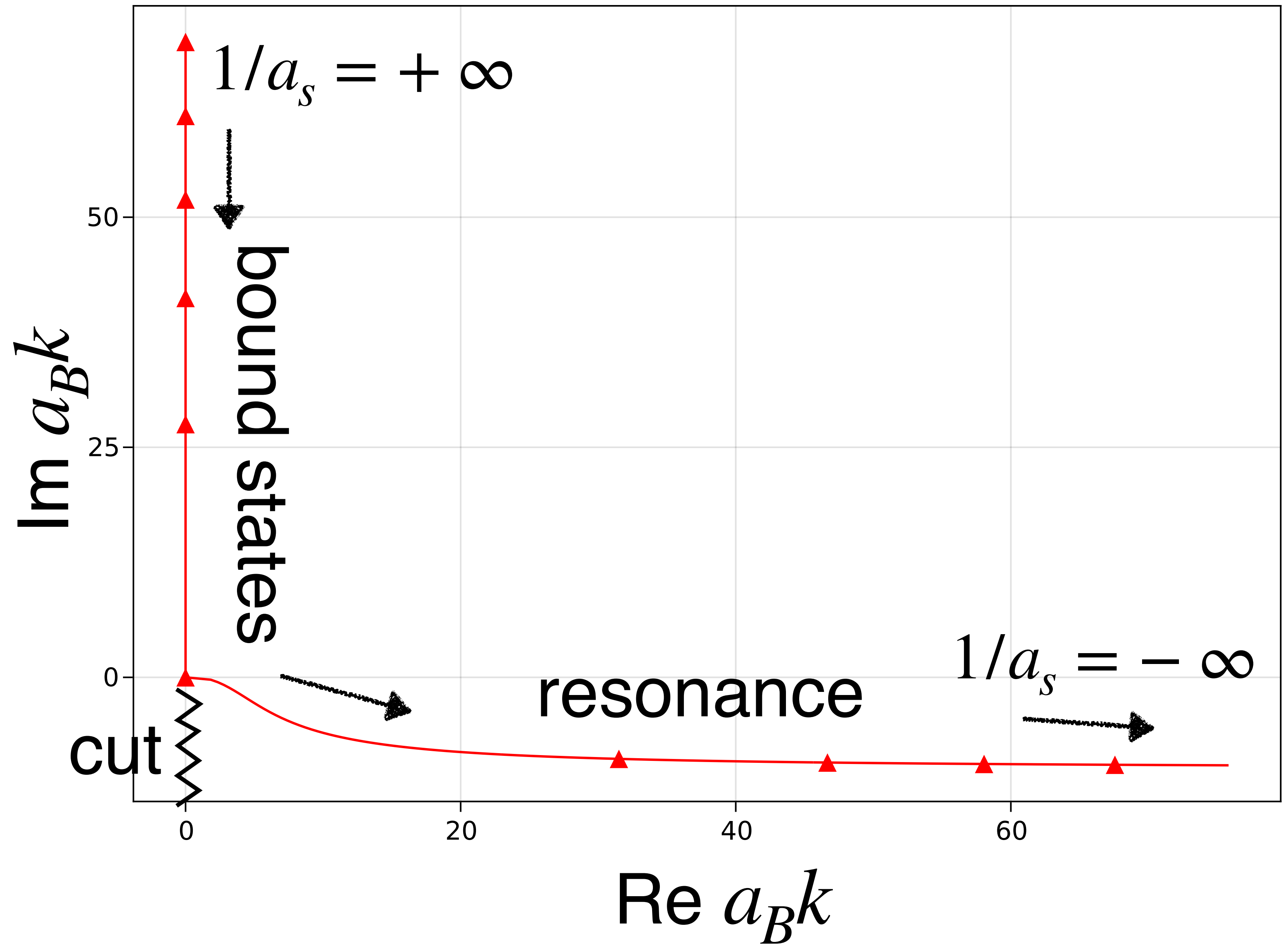
$$\frac{ia_Bk}{2} \mp \log(-ia_Bk) + \psi\left(1 + \frac{i}{a_Bk}\right) + \frac{a_B}{2a_s} = 0$$

# Pole trajectory (repulsive Coulomb)

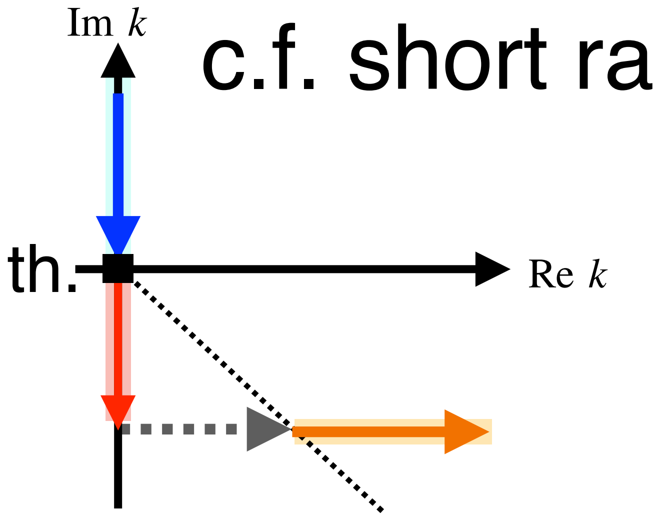
◉ pole trajectory in complex momentum $k$ plane

- varying Coulomb scattering length $a_s$ with fixed $r_e$ and $a_B$

→ pole position (eigenmomentum) moves



- b.s directory goes to resonance



c.f. short range

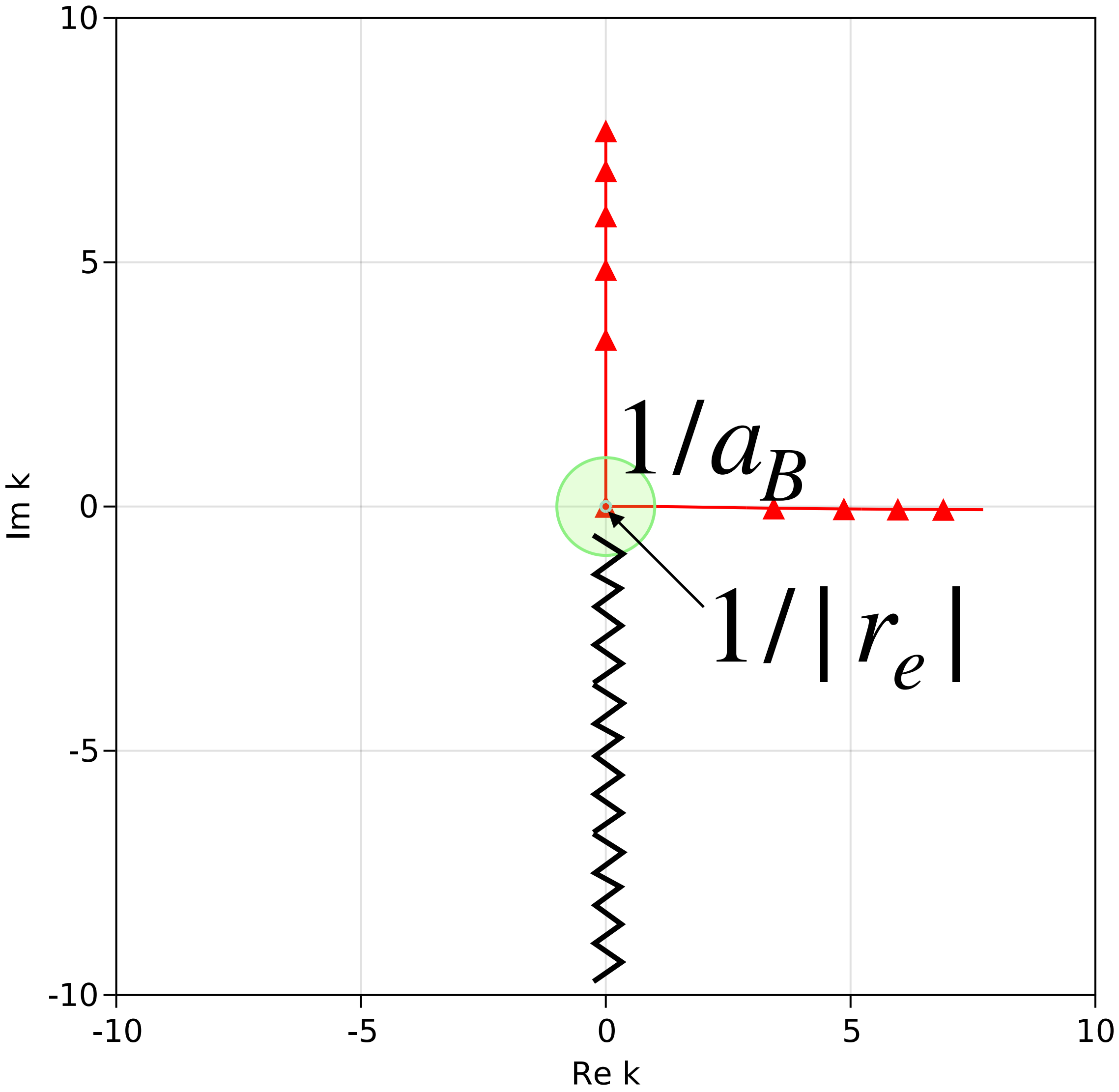
- $a_s \rightarrow \infty$ at threshold
- but no universality

∵ radius of w.f. $< \infty$

S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

# Pole trajectory (repulsive Coulomb)

$a_B = 1, r_e = -10$

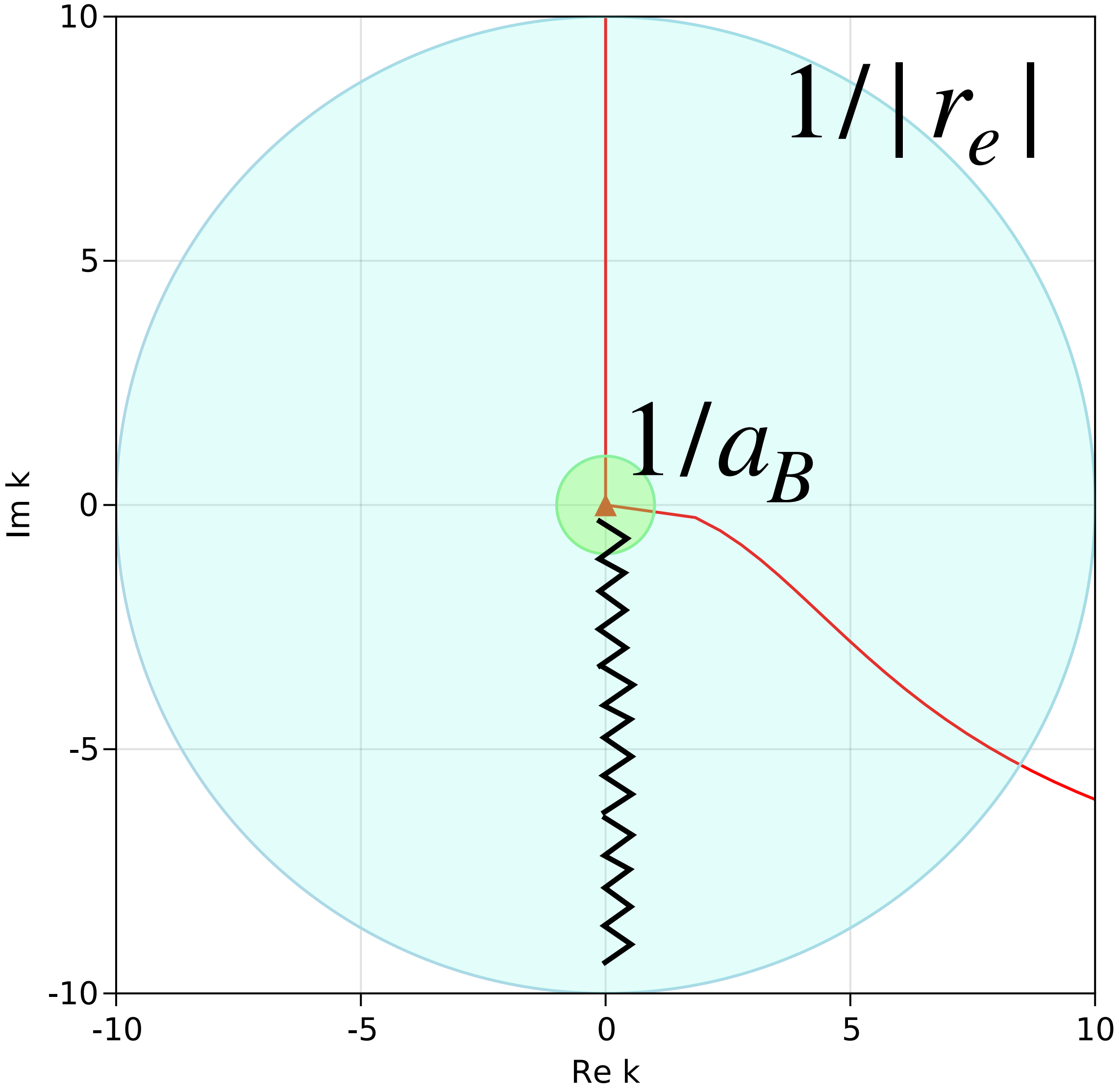


short range universality

$\wedge$

Coulomb dominant region

$a_B = 1, r_e = -0.1$

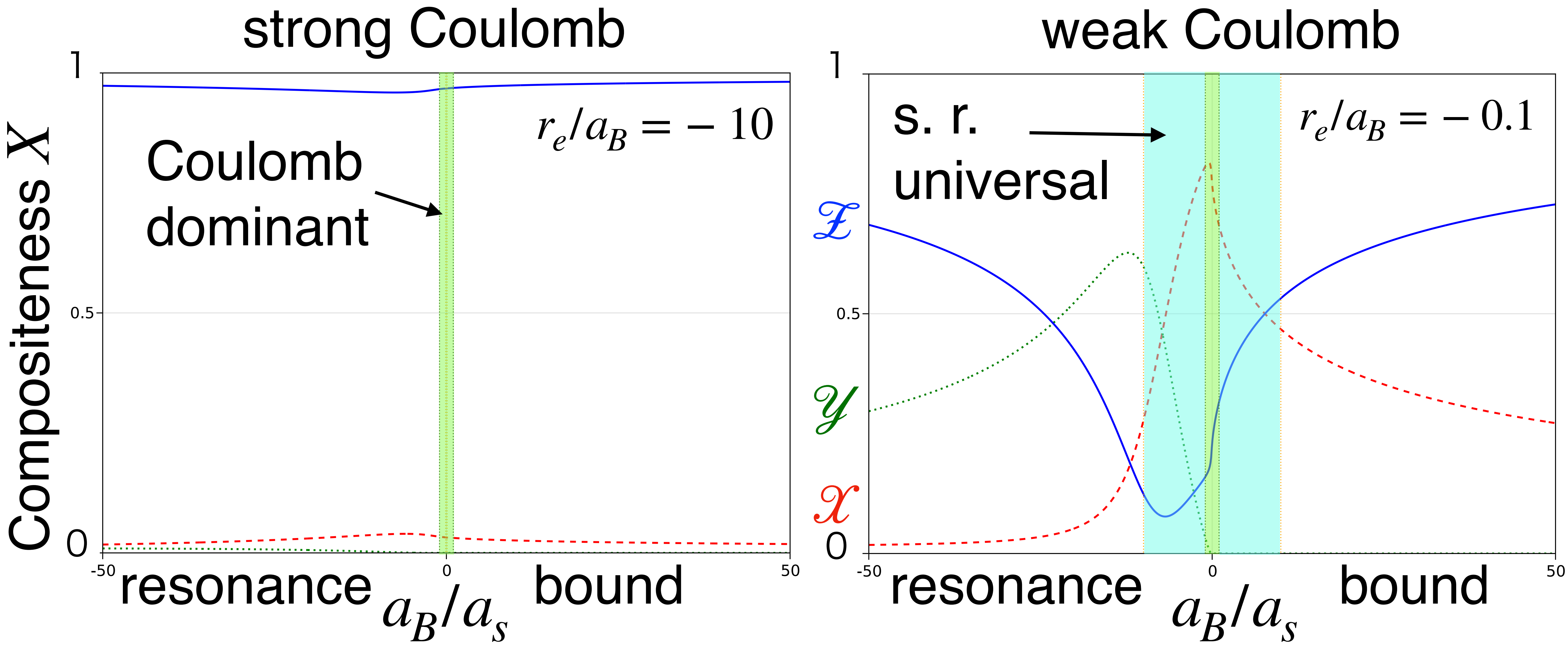


short range universality

$\vee$

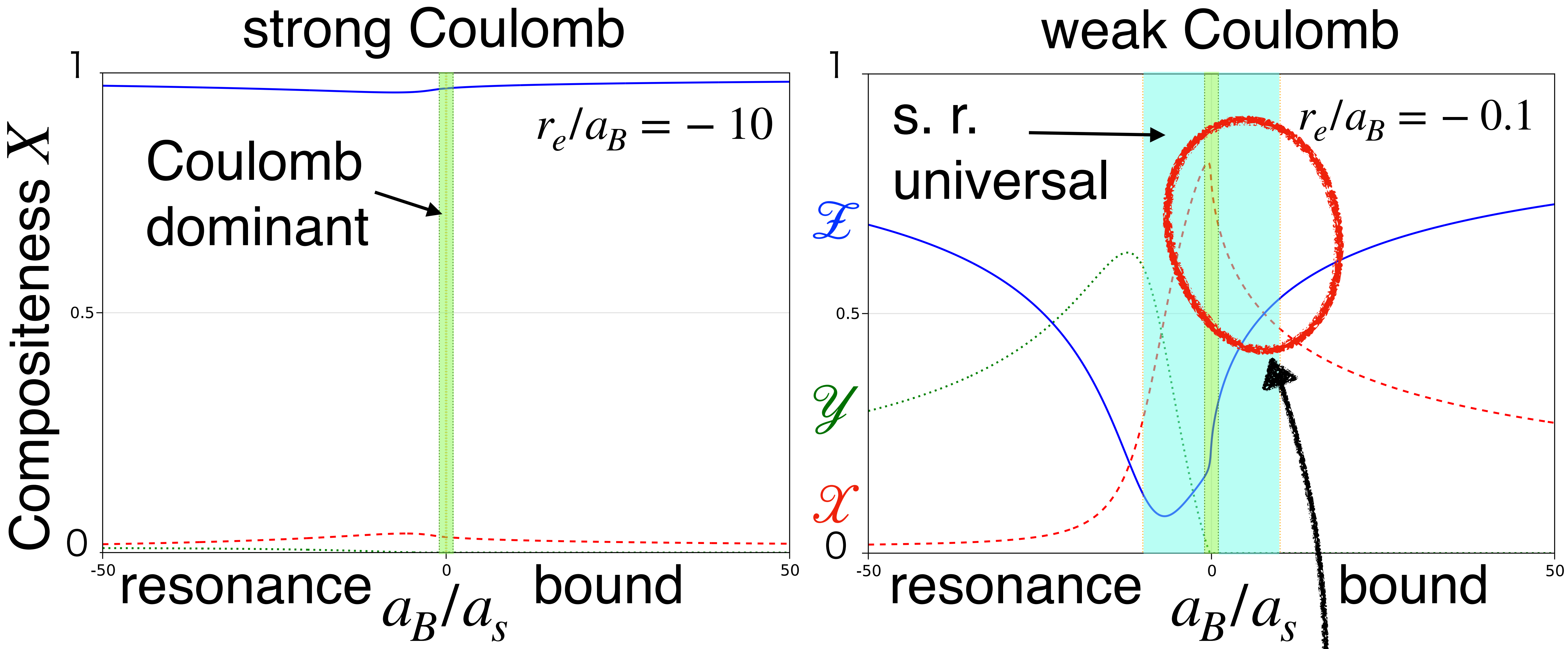
Coulomb dominant region

# Compositeness (repulsive Coulomb)



- complex compositeness ⟵ $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph].
- states with large $|1/a_s|$ are elementary $\mathcal{Z}$ dominant
- structure of bound states ≈ resonances $\because$ continuous $X$

# Compositeness (repulsive Coulomb)



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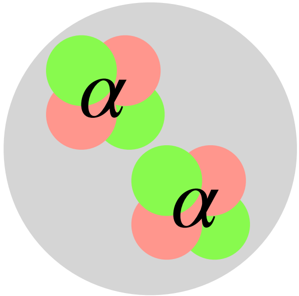
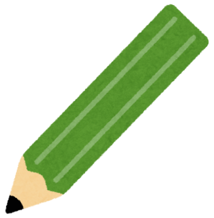
- states with large $|1/a_s|$ are elementary $\mathscr{Z}$ dominant
- structure of bound states ≈ resonances ∵ continuous $X$
- remnant of short range universality in $|r_e| \ll |a_B|$ case
  $X \to 1$ in $B \to 0$ limit in short range

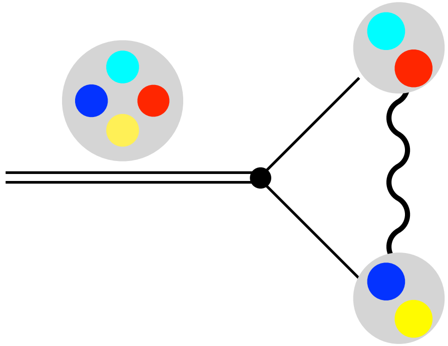
# Summary

near-threshold bound states & resonances
with **Coulomb + short range** interaction





- bare state which couples to Coulomb scattering
- pole condition ← $a_s$, $r_e$, $a_B$



- repulsive Coulomb

bound → resonance (does not become virtual states)

$X$ is not necessary to be unity at threshold

if Coulomb < s.r., remnant of s.r. universality can be seen

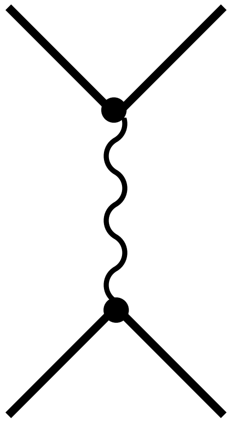
nature of b.s. ≈ nature of resonance

**Back up**

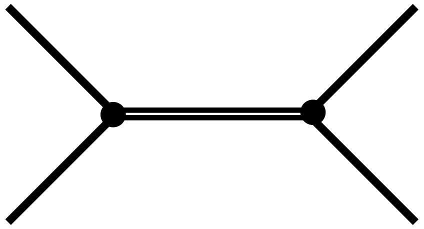
# Coulomb+short range model

◉ model

Coulomb



short range (s.r.)



Weinberg, S. Phys. Rev. 137, 672–678 (1965);
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B 586, 53–61 (2004);
T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

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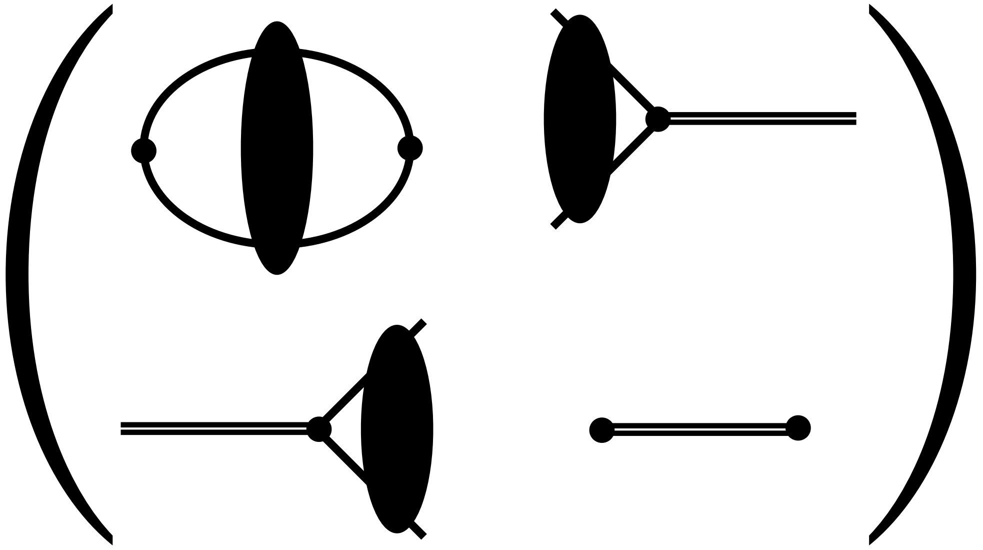
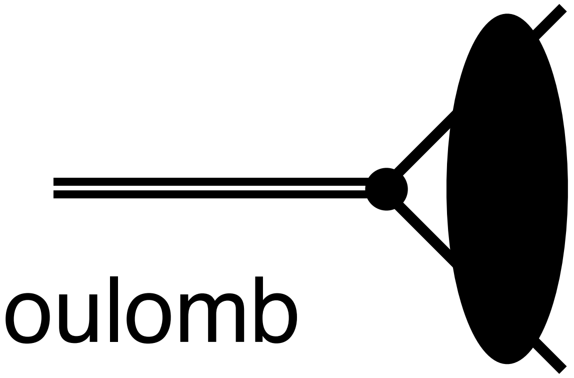


Q channel
⇔ b.s. w/o Coulomb



P channel
⇔ scattering w/ Coulomb

# Coulomb+short range model

◉ Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$

$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

◉ effective Hamiltonian (channel eliminating)

$$\hat{H}_{P\mathrm{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\mathrm{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$

# Coulomb+short range model

◉ Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$

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◉ effective Hamiltonian (channel eliminating)

$$\hat{H}_{P\mathrm{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\mathrm{ch}} = \underbrace{\boxed{\hat{H}_{PP}}}_{=\hat{H}^0+\hat{V}_P} + \underbrace{\boxed{\hat{H}_{PQ}(E-\hat{H}_{QQ})^{-1}\hat{H}_{QP}}}_{=\hat{V}_Q}$$

$$\longrightarrow \hat{H}_{P\mathrm{ch}} = \hat{H}^0 + (\hat{V}_P + \hat{V}_Q)$$

$\hat{H}^0$ : free Hamiltonian

$\hat{V}_P$ : pure Coulomb interaction

$\hat{V}_Q$ : short range interaction

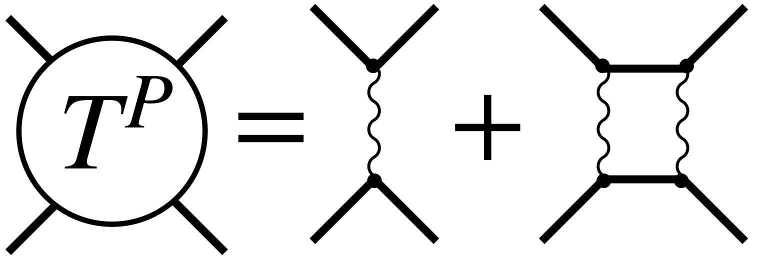
# Coulomb+short range model

◉ *T*-matrix H. Feshbach, Annals Phys. 19 287-313 (1962); W. Domcke, Atom. Mol. Phys. 16 359 (1983); R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).

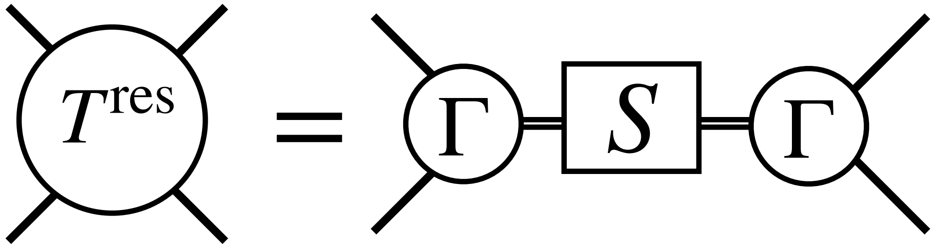
Lippmann-Schwinger eq. : $\hat{T} = [(\hat{V}_P + \hat{V}_Q)^{-1} - \hat{G}^0]^{-1}$

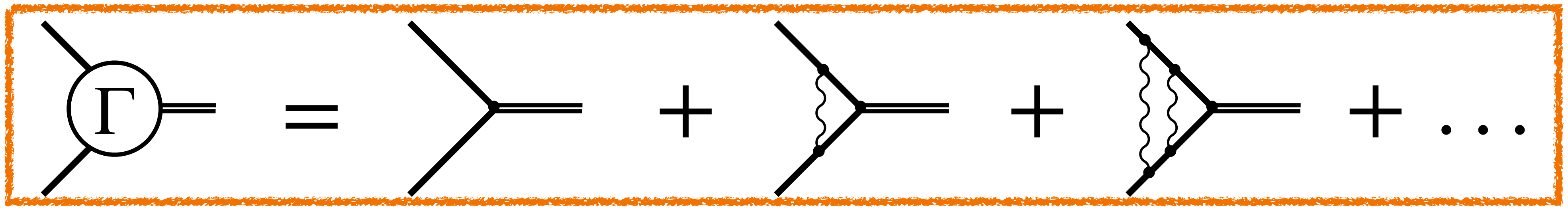
⇔ Feshbach method : $\hat{T} = \hat{T}^P + \hat{T}^{\mathrm{res}}$

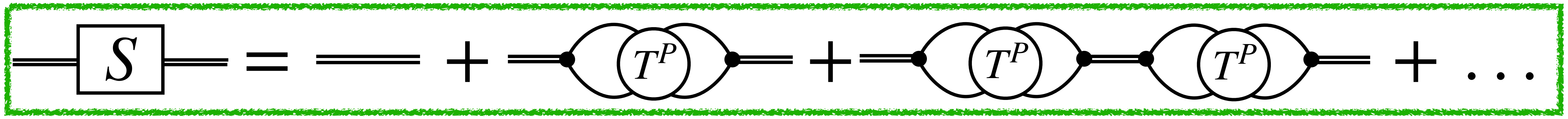
$$\hat{T}^P = [\hat{V}_P^{-1} - \hat{G}^0]^{-1}, \quad \hat{T}^{\mathrm{res}} = \boxed{\hat{T}^P \hat{V}_P^{-1}} \boxed{[\hat{V}_Q^{-1} - \hat{G}_P]^{-1}} \boxed{\hat{V}_P^{-1} \hat{T}^P}$$



pure Coulomb



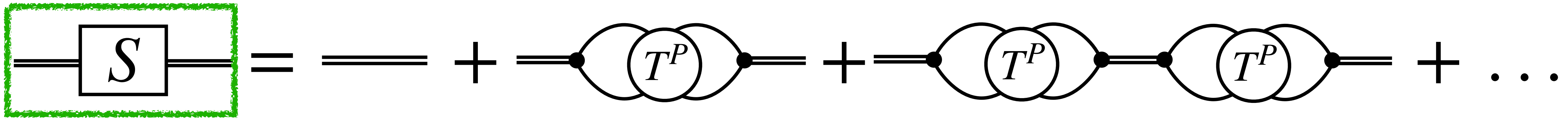


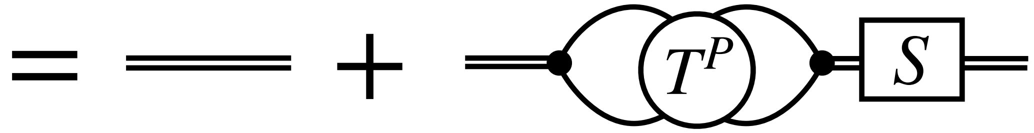


# Coulomb+short range model

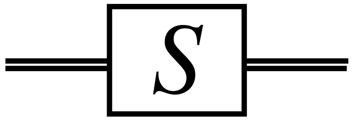
● pole condition $\hat{T} = \hat{T}^P + \hat{T}^{\rm res}$

pole of $T(k,k')$ ⇔ pole of $T^{\rm res}(k,k')$ ⇔ $\boxed{[V_Q^{-1} - G_P]^{-1}} = \infty$

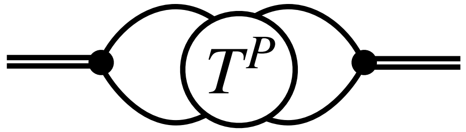




$H_{QP}G_PH_{PQ}$



$= S(E)$, $= (E - \varepsilon_d)^{-1}$,



$= F(E)$

$$\longrightarrow S(E) = (E-\varepsilon_d)^{-1} + (E-\varepsilon_d)^{-1}F(E)S(E)$$
$$= [E - \varepsilon_d - F(E)]^{-1}$$

⟶ pole condition : $E - \boxed{\varepsilon_d} - \boxed{F(E)} = 0$

bare state energy    self energy

# Coulomb+short range model

◉ self energy $F(E)$ **in low-energy limit**

W. Domcke, Atom. Mol. Phys. 16 359 (1983).

- attractive Coulomb

$$F(k) = \frac{A}{2\pi}\left[c - \frac{1}{2}ia_B k + \log(-ia_B k) + \psi\left(1 - \frac{i}{a_B k}\right)\right]$$

- repulsive Coulomb

$$F(k) = -\frac{A}{2\pi}\left[c + \frac{1}{2}ia_B k + \log(-ia_B k) + \psi\left(1 + \frac{i}{a_B k}\right)\right]$$

$A$ : constant with dimension of energy

$c$ : dimensionless constant

$\psi(x) = \frac{d}{dx}\log(\Gamma(x))$ : digamma function

# Coulomb+short range model

◉ pole condition **in low-energy limit**

- Coulomb scattering length $a_s$ and effective range $r_e$

$$(\text{amplitude})^{-1} = -\frac{1}{a_s} + \frac{r_e}{2}k^2 + \mathcal{O}(k^4) - ik + 2\log(-ik) + 2\psi\left(1 + \frac{i}{k}\right) + \ldots,$$

R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).

$$\longrightarrow a_s = -a_B\left[\frac{4\pi}{A}\varepsilon_d \pm 2c\right]^{-1}, \quad r_e = -\frac{4\pi}{Aa_B\mu}$$

- pole condition with $a_s$ and $r_e$ H. A. Bethe, Phys. Rev. 76, 38-50 (1949).

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B}\left[\log(-ia_Bk) + \psi\left(1 + \frac{i}{a_Bk}\right)\right] = 0$$

◉ compositeness $X$ T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

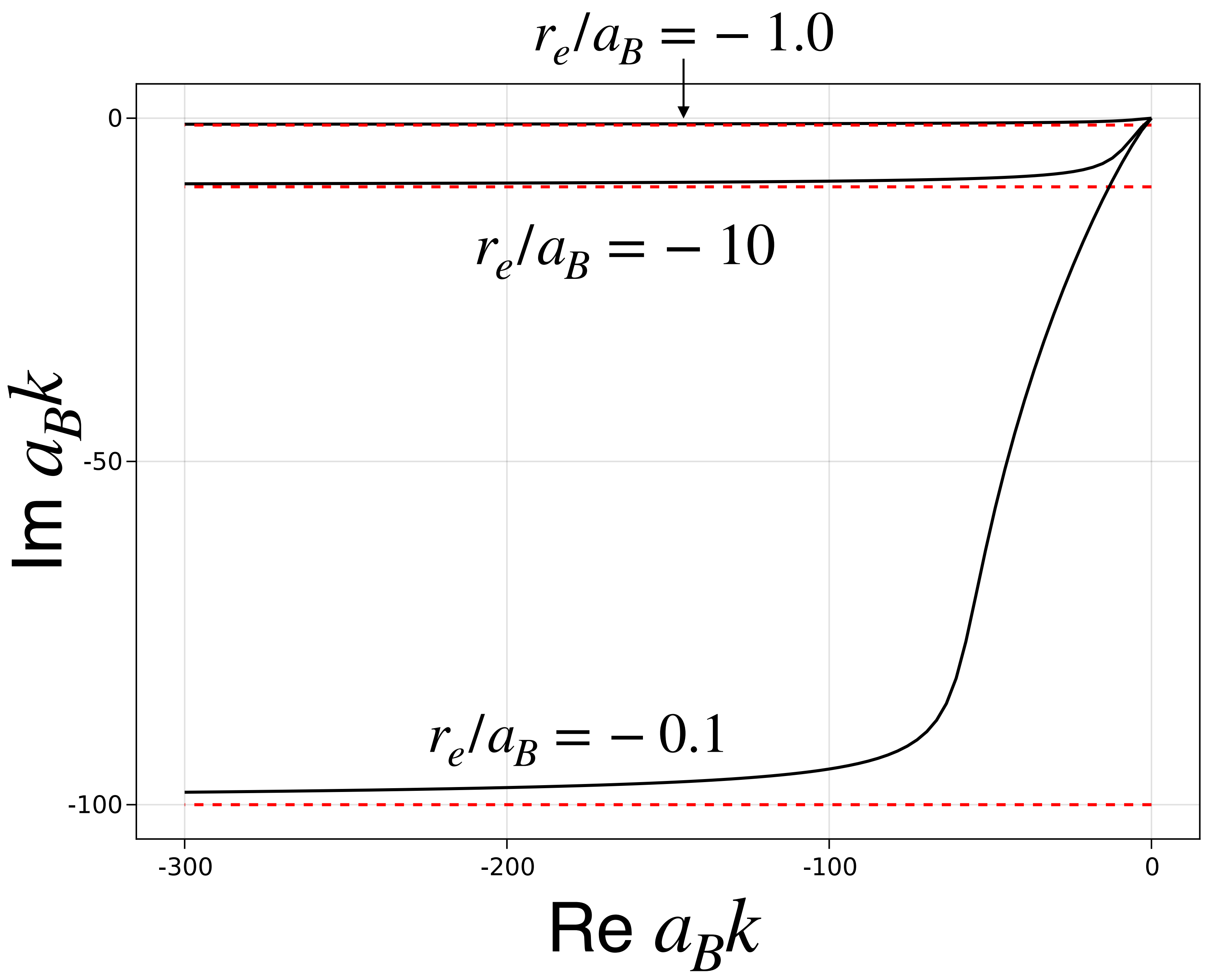
$$X = 1 - \frac{1}{1 - \frac{d}{dE}\boxed{F(E)}}$$

self energy

# far from threshold (repulsive Coulomb)

## ◉ imaginary part of eigenenergy in complex momentum $k$ plane

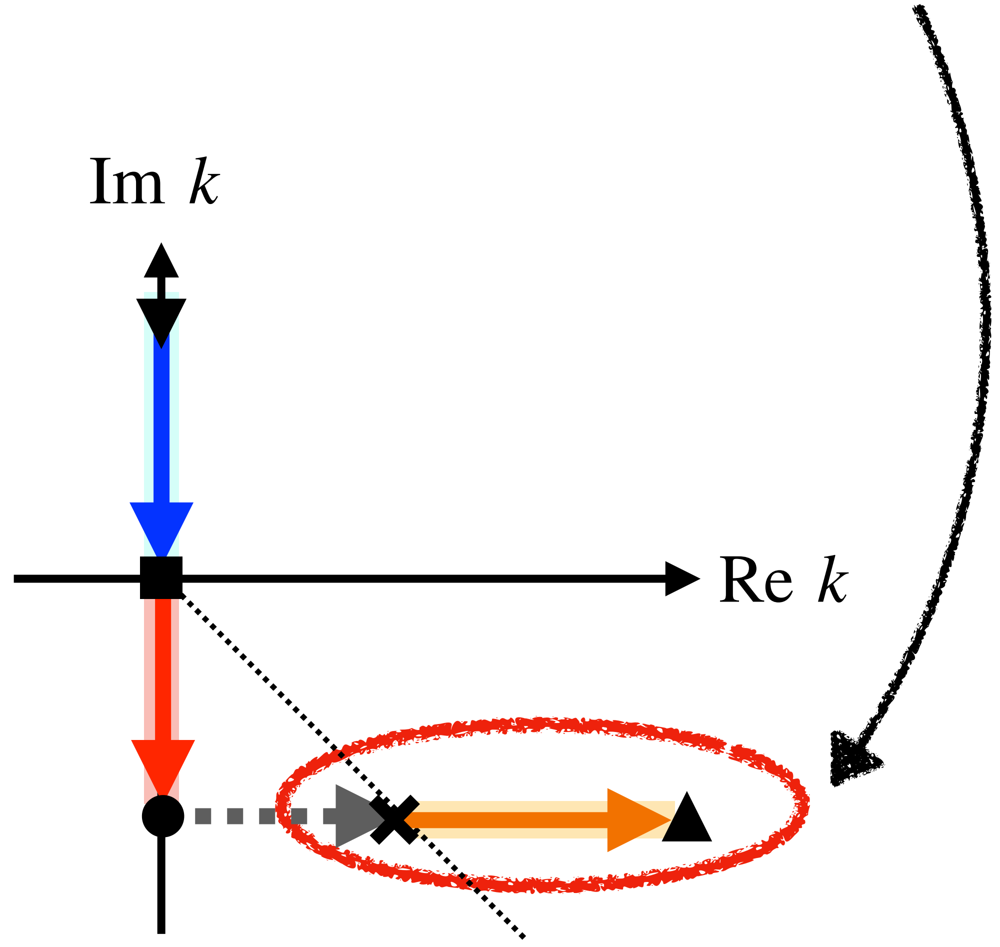
- far from threshold in $1/a_s \to -\infty$ limit



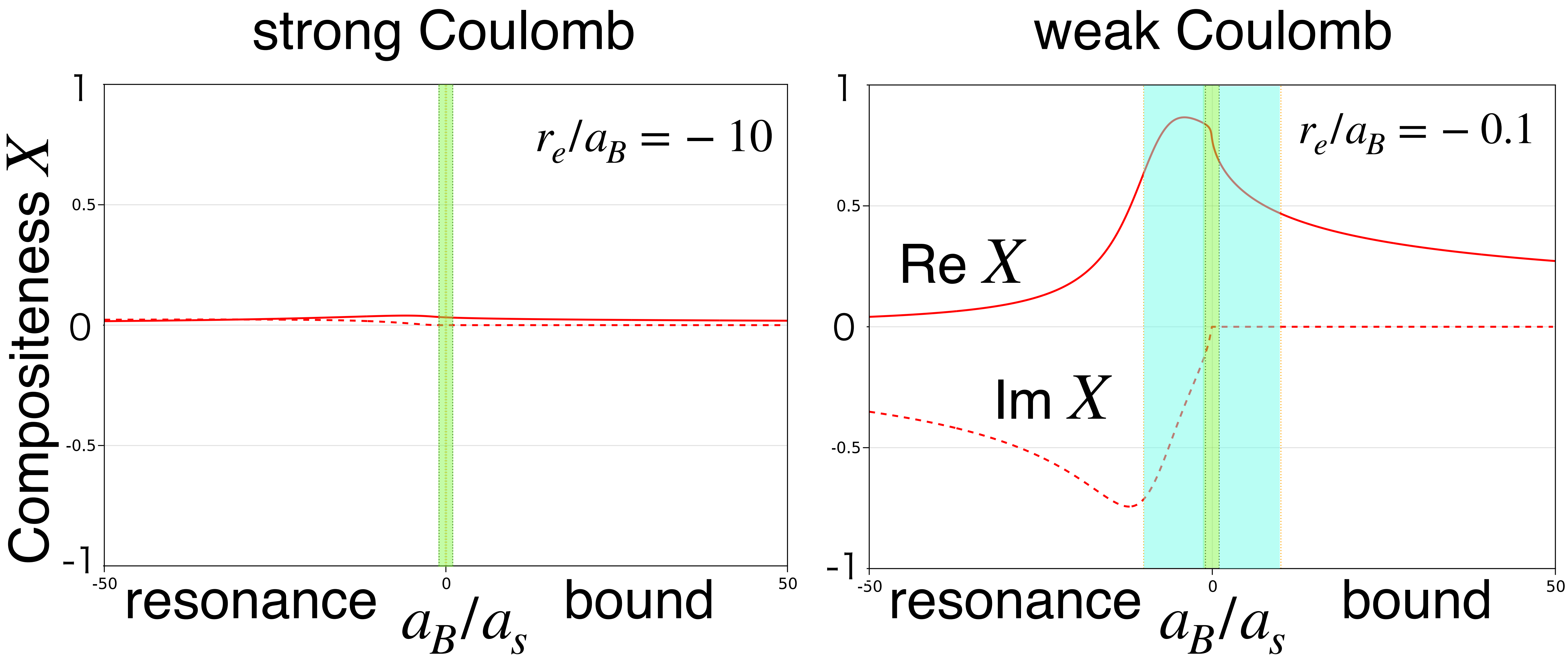
dotted lines : $a_B/r_e$

- Im $k \to 1/r_e$

→ trajectory close to that of resonance in ERE

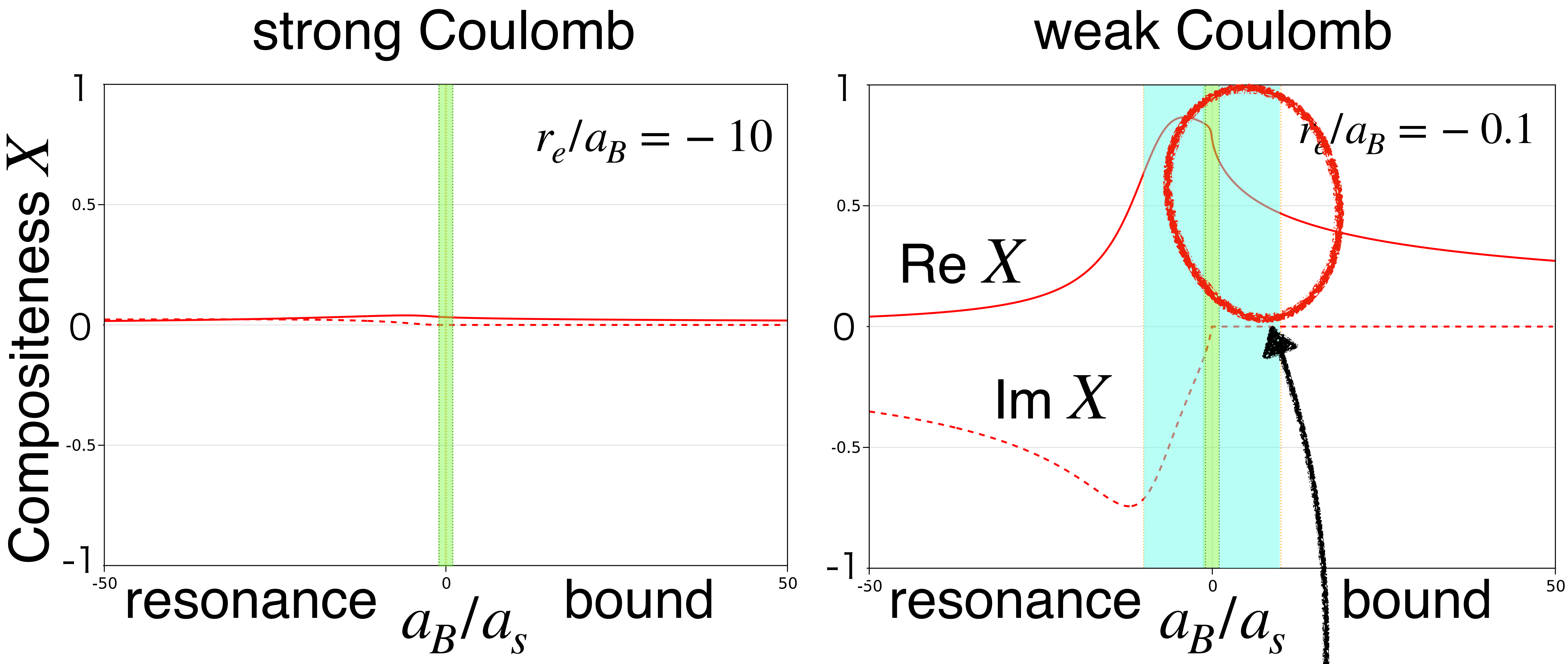


# Compositeness (repulsive Coulomb)



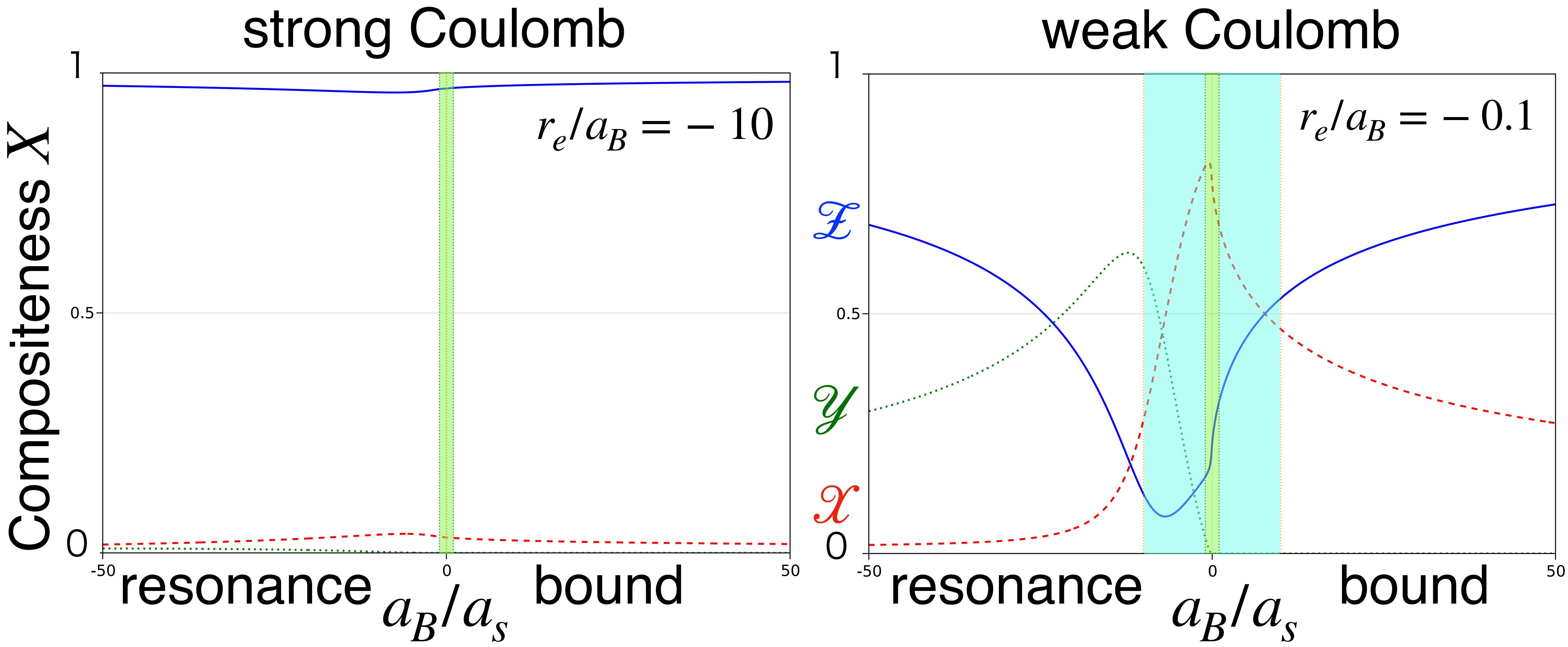
- $\pm 1/a_B$ : Coulomb force dominant region
- $\pm 1/|r_e|$ : short range universal region

# Compositeness (repulsive Coulomb)



- $\pm 1/a_B$ : Coulomb force dominant region
- $\pm 1/|r_e|$ : short range universal region
- remnant of short range universality in $|r_e| \ll |a_B|$ case
  $X \to 1$ in $B \to 0$ limit in short range

# Compositeness (repulsive Coulomb)

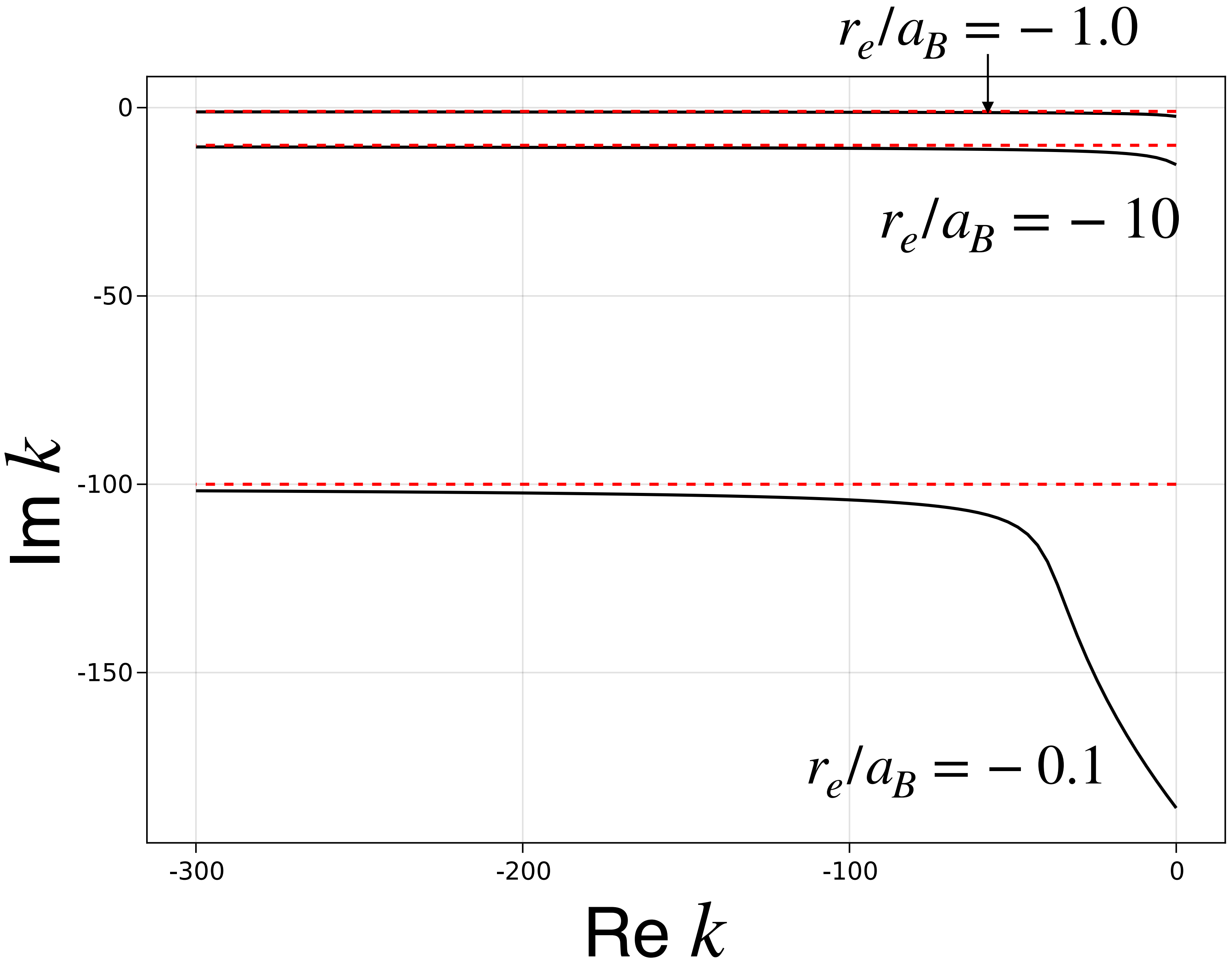


- compositeness of resonances ← $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph].
- all states are interpretable $\because$ no virtual states
- states with large $|1/a_s|$ are elementary $\mathscr{Z}$ dominant
- nature of bound states = nature of resonances
  $\because$ $X$ is continuous across threshold

# far from threshold (attractive Coulomb)

## ◉ imaginary part of eigenenergy in complex momentum $k$ plane

- far from threshold in $1/a_s \to -\infty$ limit



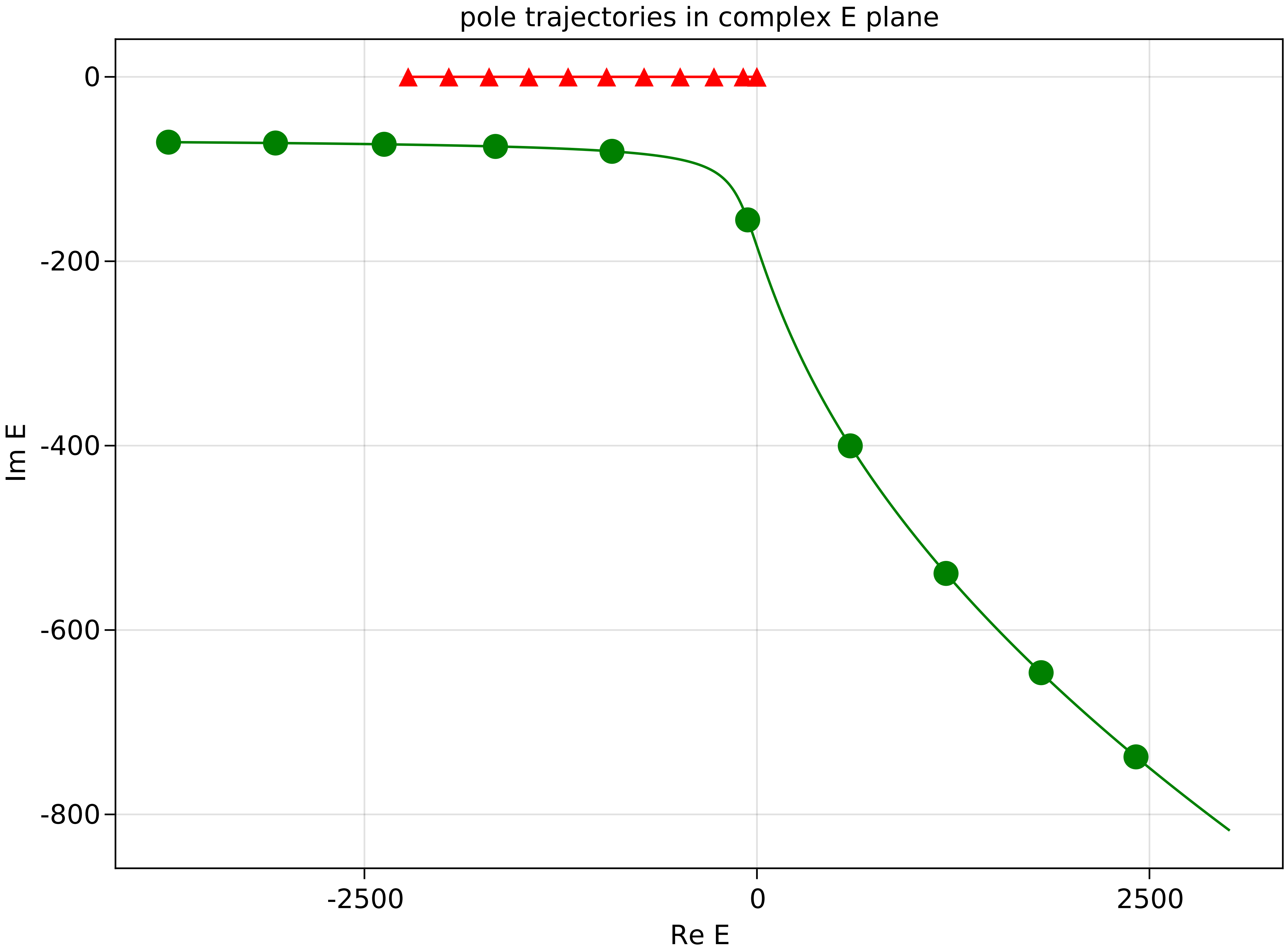
dotted lines :$1/r_e$

- Im $k$ of resonance pole
- Im $k \to 1/r_e$

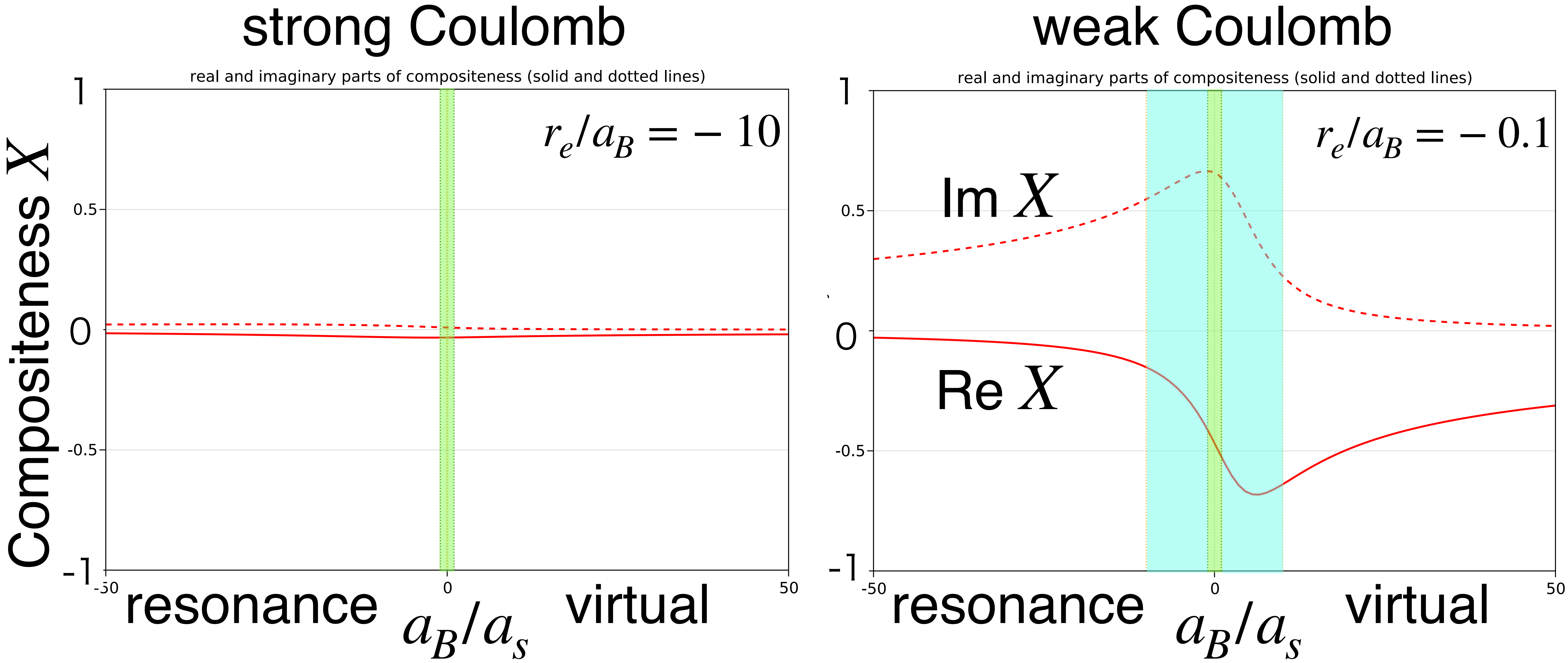
→ trajectory close to that of resonance in ERE

- same as repulsive case

# **Attractive**

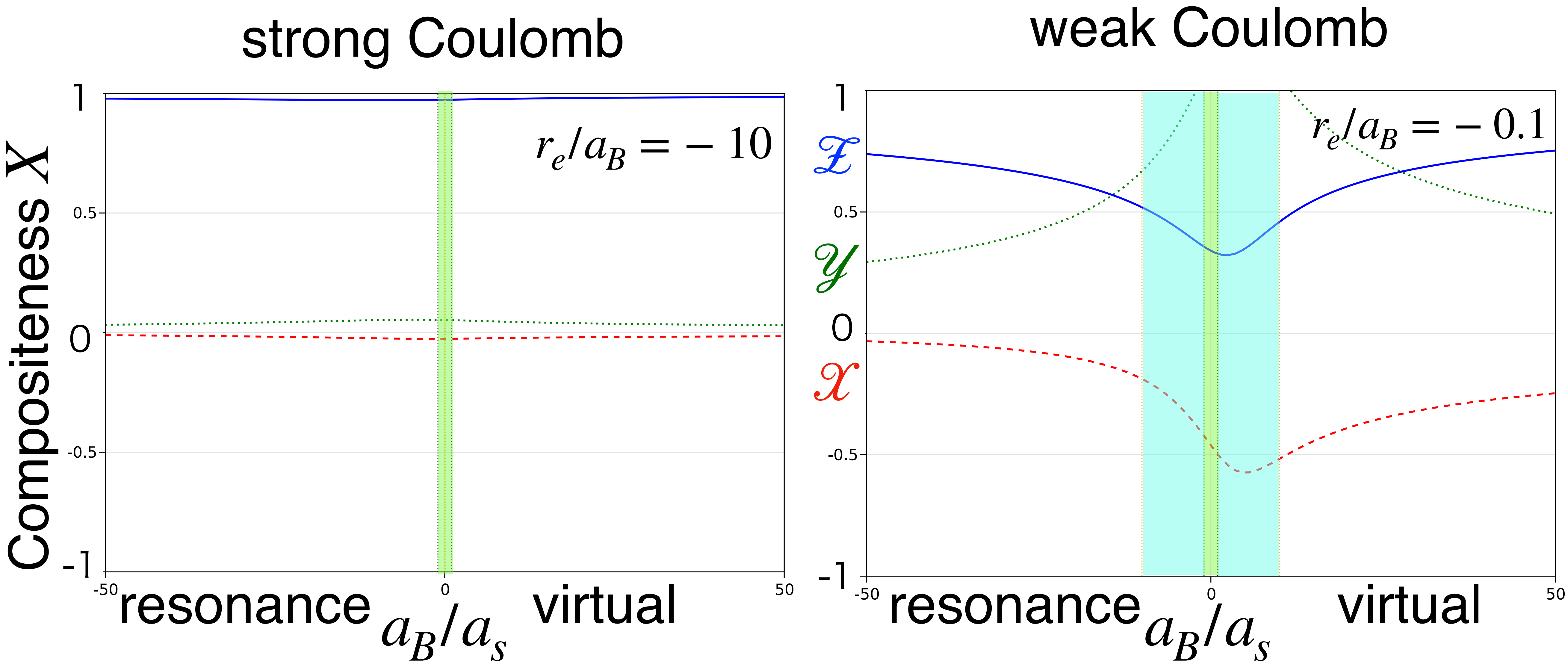

pole trajectories in complex E plane
Im E
0
-200
-400
-600
-800
-2500
0
2500
Re E

# Compositeness (att. Coulomb resonance)



- $\pm 1/a_B$ : Coulomb force dominant region
- $\pm 1/|r_e|$ : short range universal region
- compositeness of unstable resonances are complex $X \in \mathbb{C}$

# Compositeness (att. Coulomb resonance)



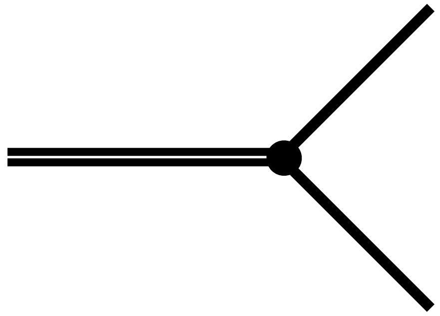
- $\mathcal{X} < 0$ → non-interpretable in this region
- but $\mathcal{X} \geq 0$ in far-threshold region with large $|1/a_s|$

  → states are $\mathcal{Z}$ dominant with large bare state contribution

# Pole trajectory (only w/ s.r.)

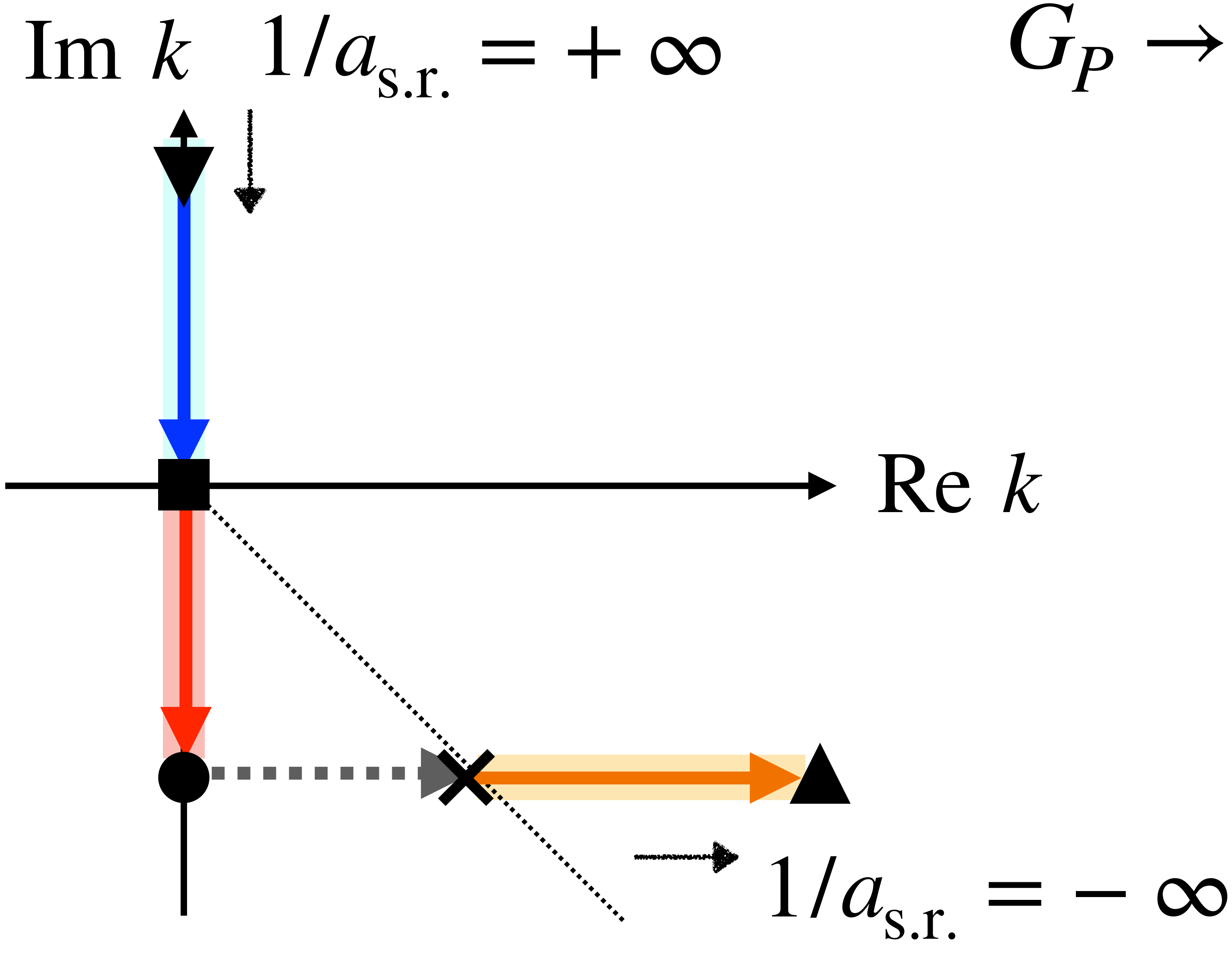
## ◉ pole trajectory in complex momentum $k$ plane (No Coulomb)

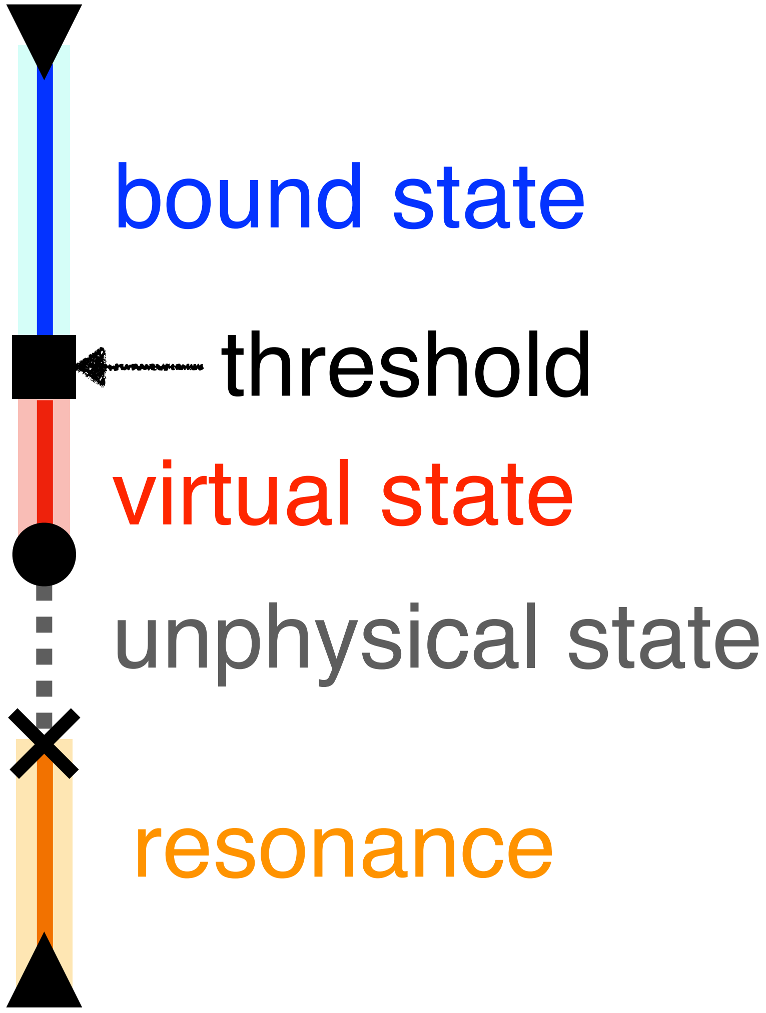
Q channel
⇔ bare state



P channel
⇔ free scattering

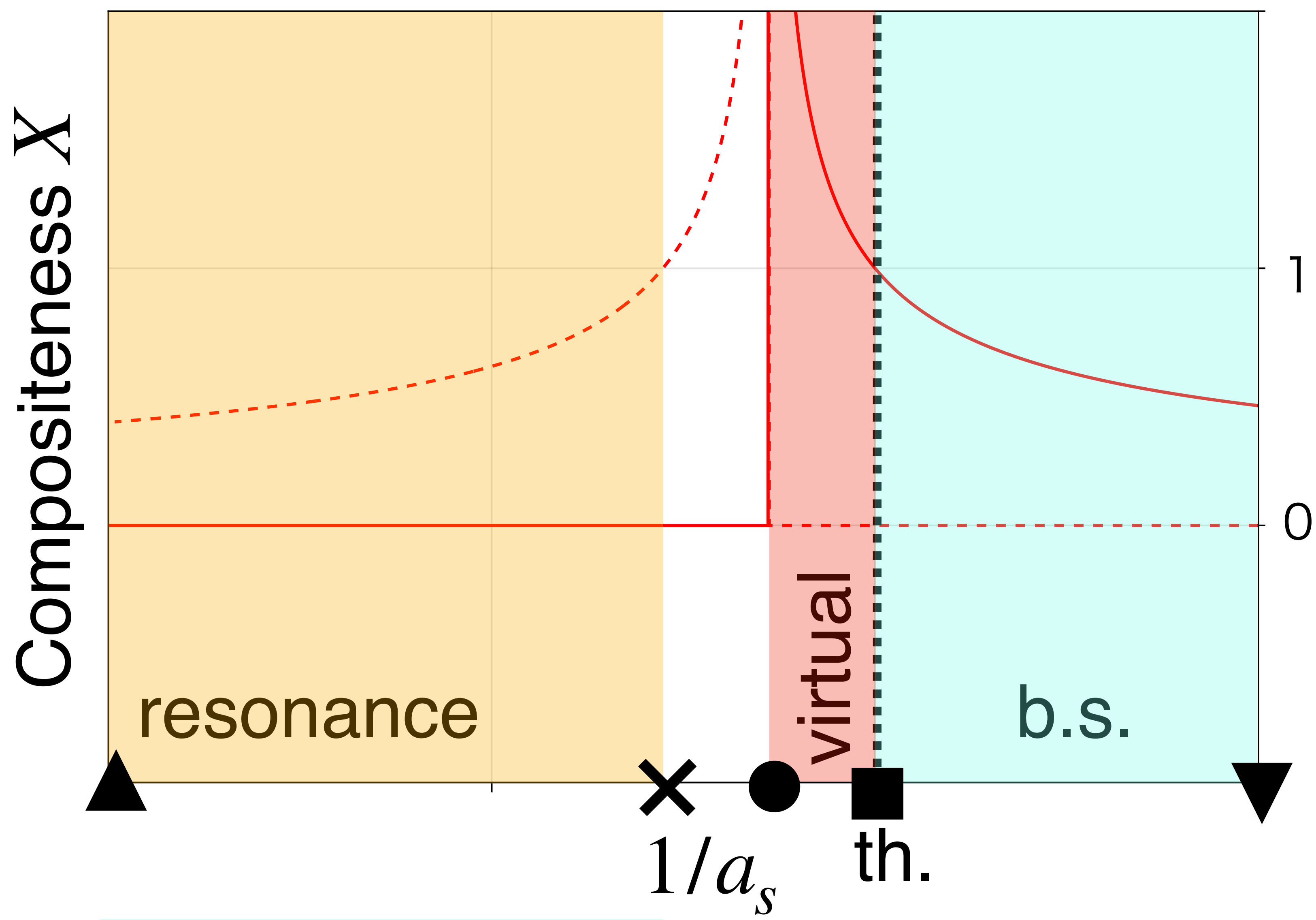
$G_P \to G^0$ free Green's function

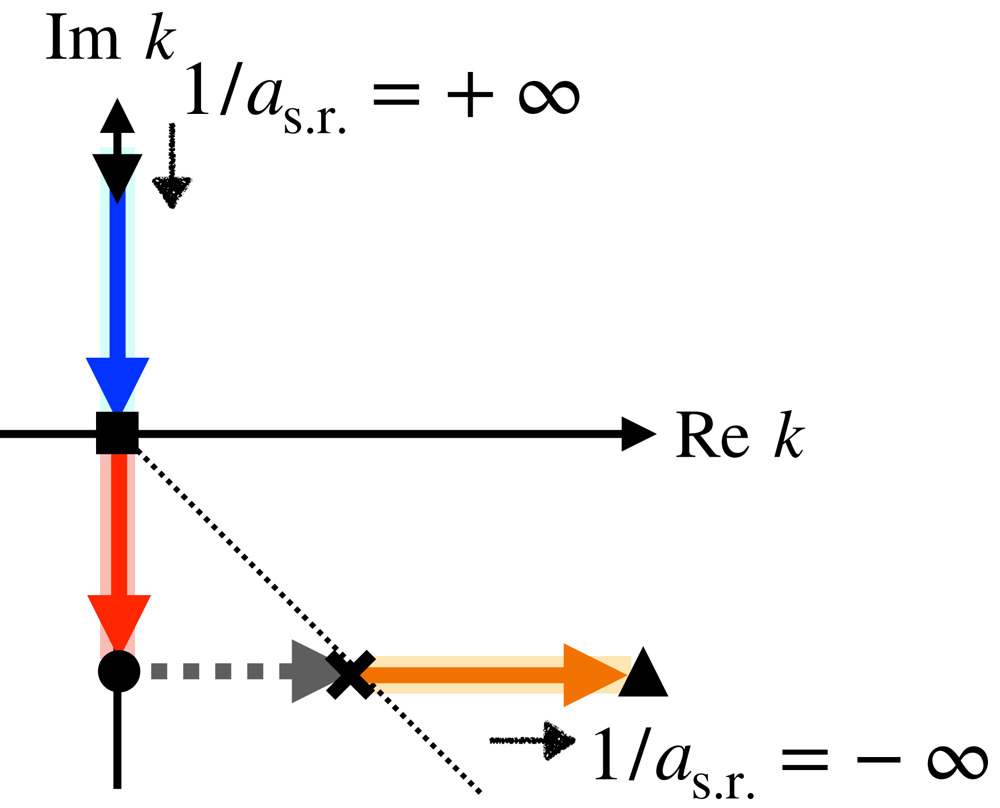




T. Hyodo, Phys. Rev. Lett. **111**, 132002 (2013) .

# Compositeness (only w/ s.r.)



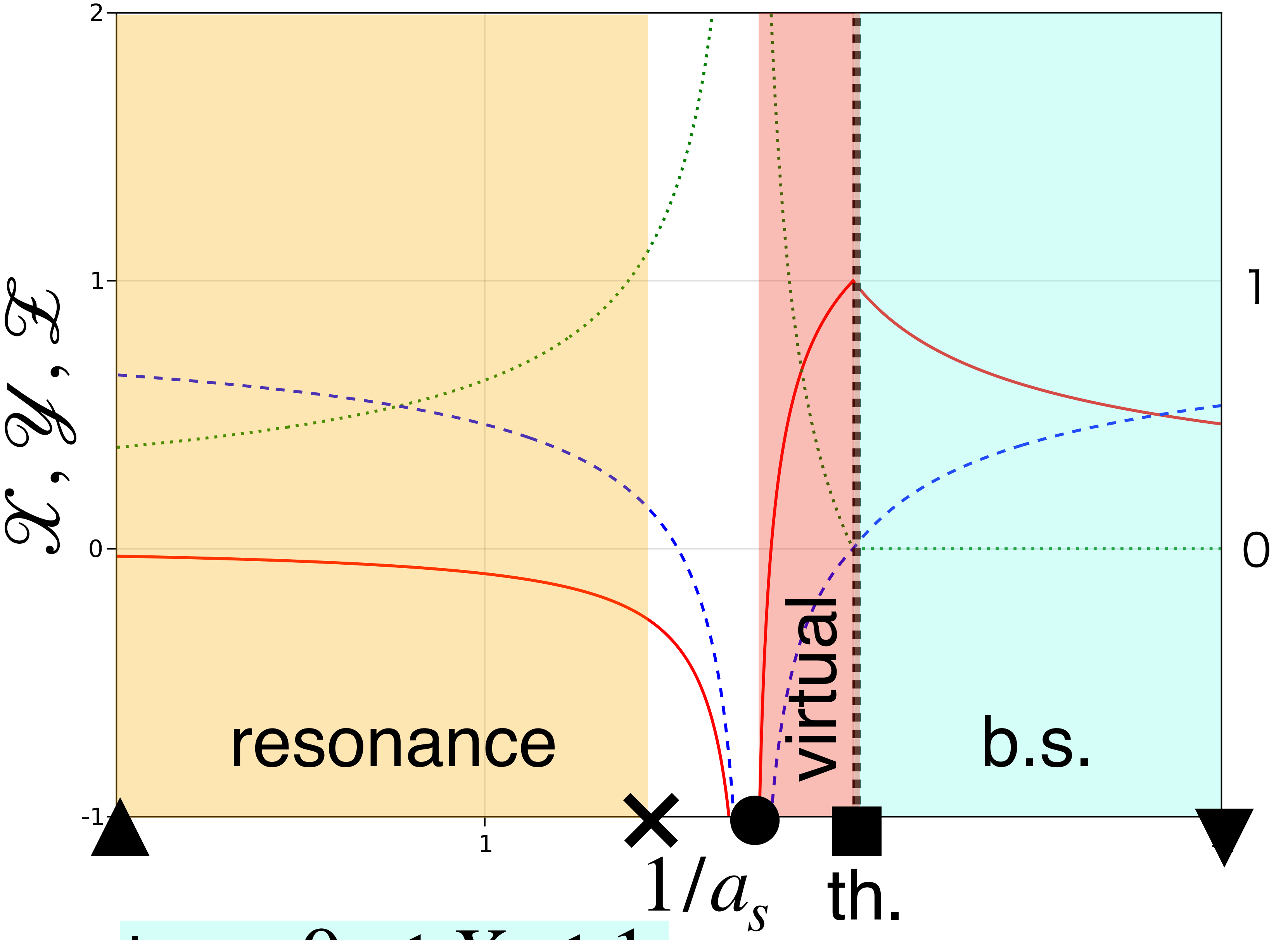


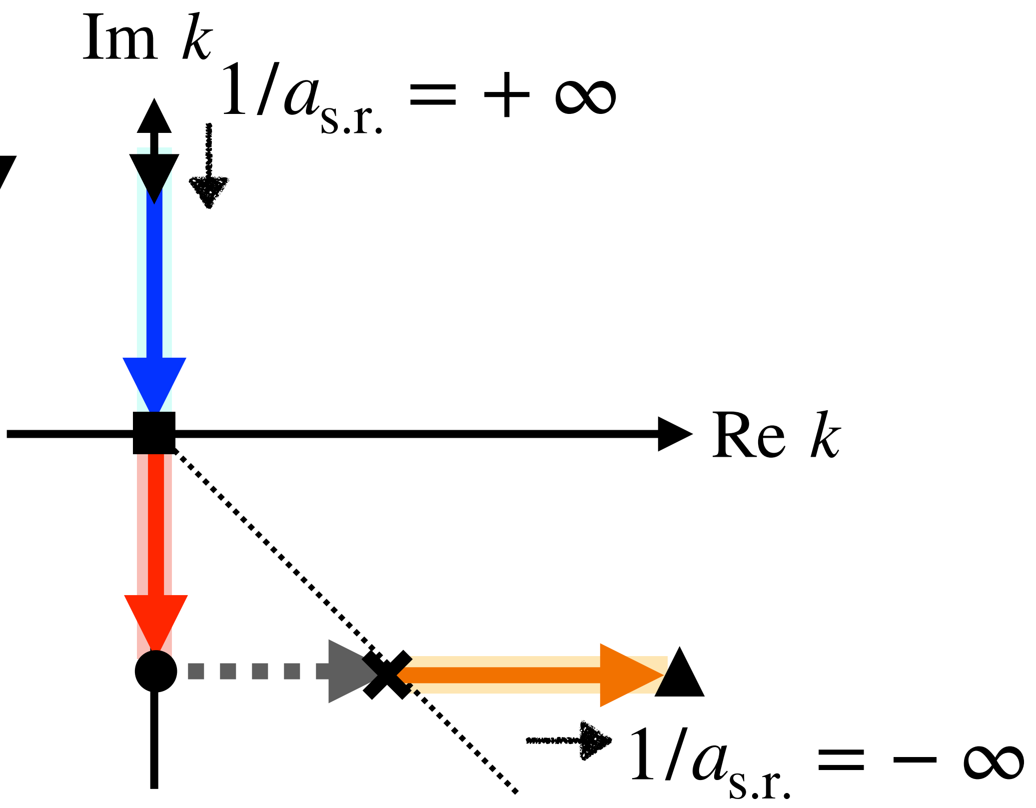
- b.s. : $0 \leq X \leq 1$
- virtual : $1 < X$

● divergence $X \to \infty$

- resonance : $\mathrm{Im}\ X \leq 1$

# $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ (only w/ s.r.)





- b.s. : $0 \leq X \leq 1$
- virtual : $\mathcal{Z} < 0$ (non-interpretable)

● divergence $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \to \infty$

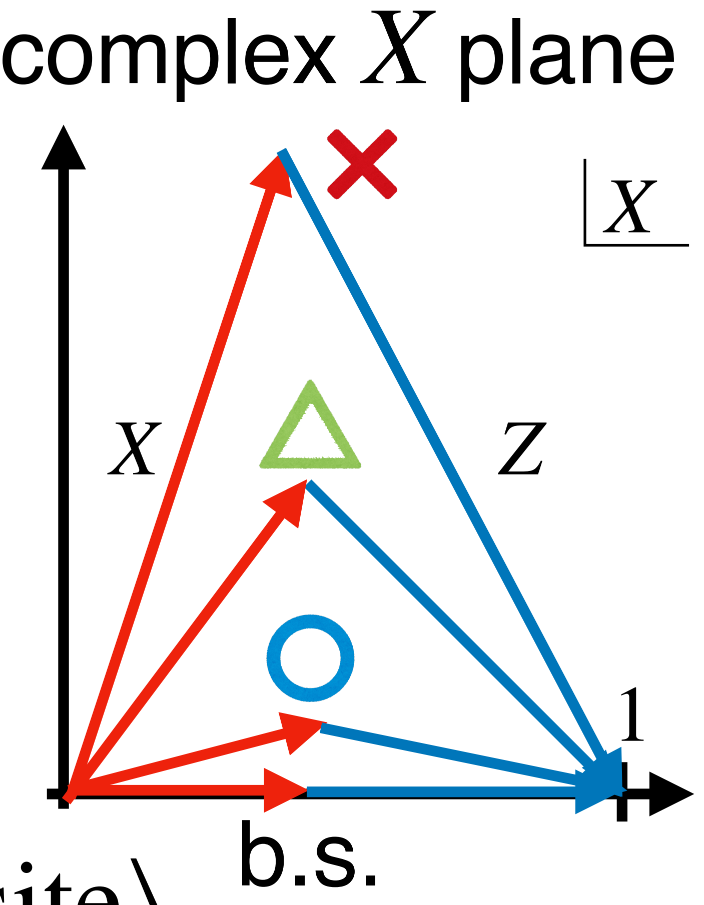
- resonance : $\mathcal{Z}$ dominant

# Complex compositeness

- probabilistic interpretation?

$X \in \mathbb{C}$ and $X+Z = 1$

- If Im $X$ is large, it seems that reasonable interpretation is impossible ✗ △

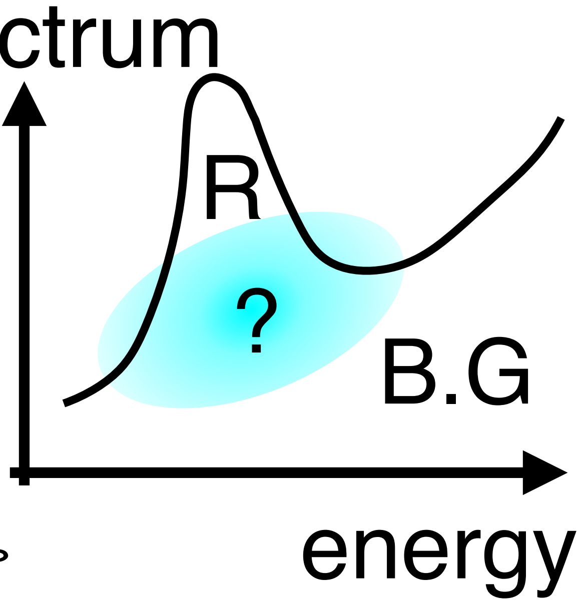


- our proposal

i) $\mathscr{X}$: probability of certainly finding $|\mathrm{composite}\rangle$
ii) $\mathscr{Z}$: probability of certainly finding $|\mathrm{elementary}\rangle$
iii) $\mathscr{Y}$: probability of uncertain identification

uncertain appears from T. Berggren, Phys. Lett. B 33, 547 (1970).

- finite lifetime (uncertainty in energy)
- separation from B.G.



complex compositeness $X \in \mathbb{C} \longrightarrow \mathscr{X}, \mathscr{Y}, \mathscr{Z}$

# Definition

T. Kinugawa and T. Hyodo
arXiv:2403.12635 [hep-ph].

## ◉ conditions for sensible interpretation

- normalization : $\mathscr{X}+\mathscr{Y}+\mathscr{Z}=1$ for probabilistic interpretation
- in bound state limit : $\mathscr{X}\to X$, $\mathscr{Z}\to Z$ and $\mathscr{Y}\to 0$

$\mathscr{Y}$ characterizes uncertainty of resonance

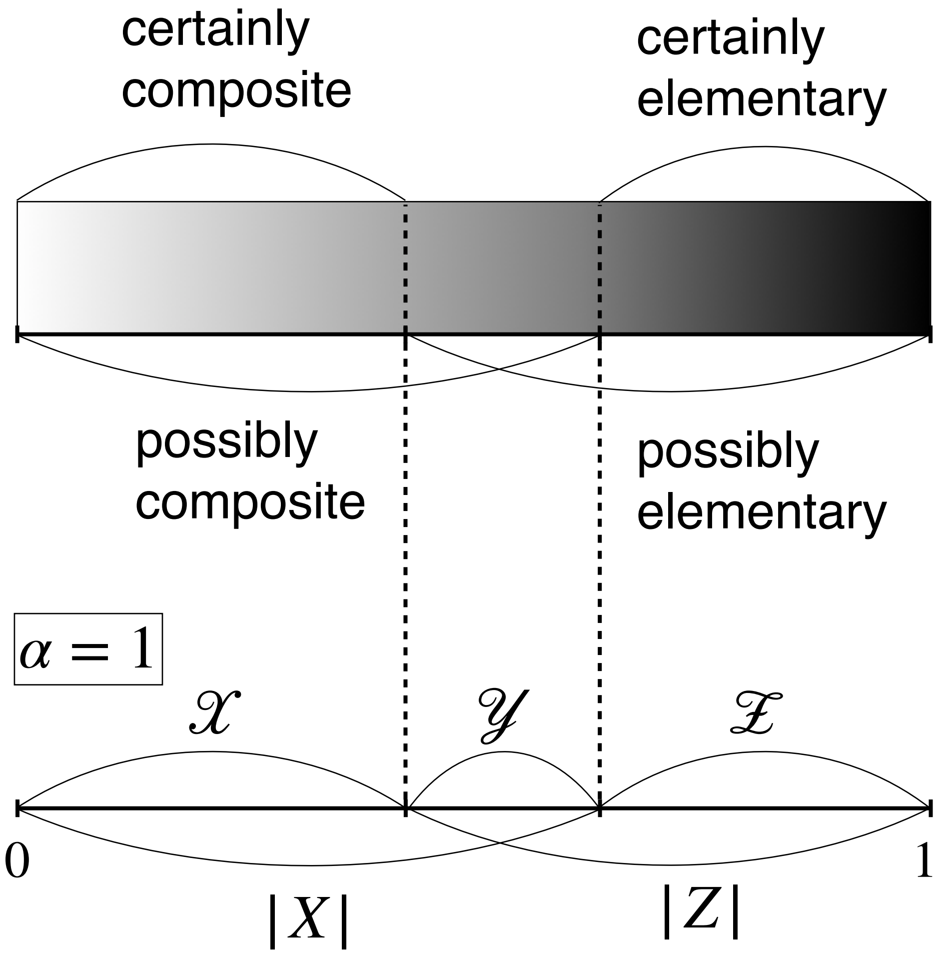
## ◉ new interpretation

$$\mathscr{X}+\alpha\mathscr{Y}=|X|,\ \ \mathscr{Z}+\alpha\mathscr{Y}=|Z|$$

$$\mathscr{X}=\frac{(\alpha-1)|X|-\alpha|Z|+\alpha}{2\alpha-1}$$

$$\mathscr{Z}=\frac{(\alpha-1)|Z|-\alpha|X|+\alpha}{2\alpha-1}$$

$$\mathscr{Y}=\frac{|X|+|Z|-1}{2\alpha-1}$$

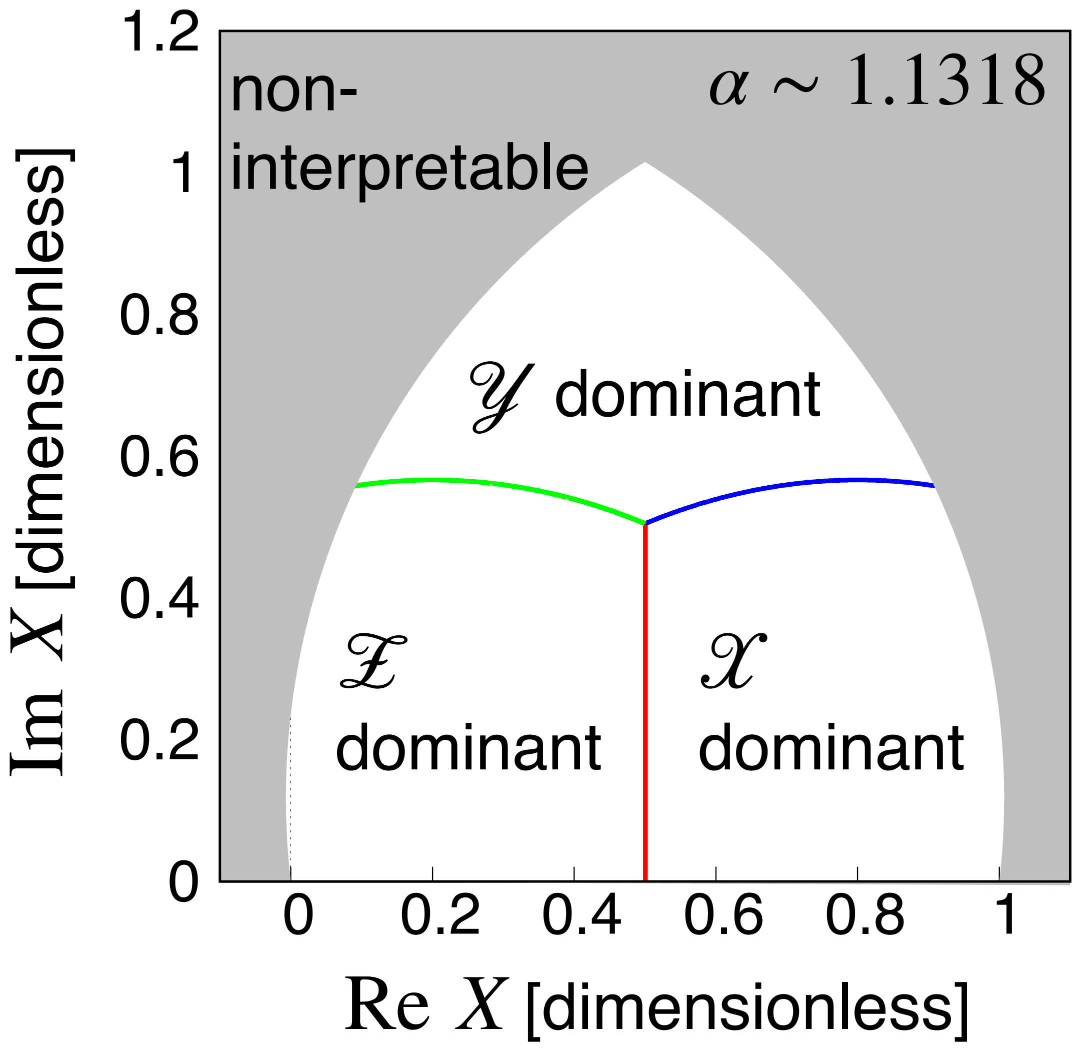


$\alpha$ reflects uncertain nature of resonances

# Definition

- if $\alpha > 1/2$, $\mathscr{Y}$ is always positive but $\mathscr{X}$, $\mathscr{Z}$ can be negative

| | | |
|---|---|---|
| $\mathcal{X} \geq 0$ and $\mathcal{Z} \geq 0$ | $\mathcal{X} > \mathcal{Y}, \mathcal{Z}$ | composite dominant |
| | $\mathcal{Z} > \mathcal{Y}, \mathcal{X}$ | elementary dominant |
| | $\mathcal{Y} > \mathcal{X}, \mathcal{Z}$ | uncertain |
| $\mathcal{X} < 0$ or $\mathcal{Z} < 0$ | | non-interpretable |



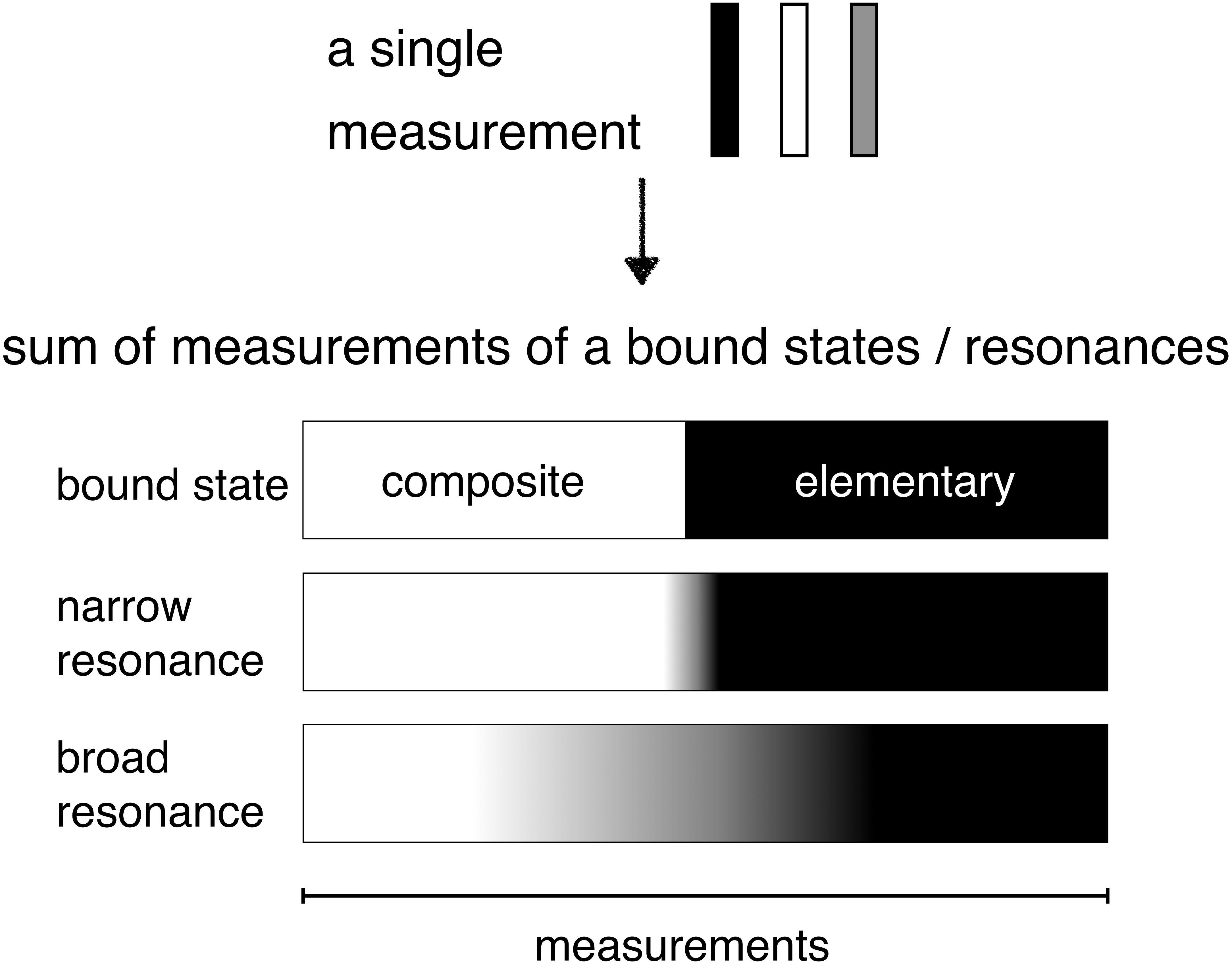
our criterion for physical "state"

$$\Gamma \leq \mathrm{Re}\ E$$

- exclude poles which we cannot regard as physical "state" from probabilistic interpretation

$\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ dominant regions
and
non-interpretable region

# uncertainty in resonances



a single measurement

sum of measurements of a bound states / resonances

bound state

composite

elementary

narrow resonance

broad resonance

measurements