

Compositeness of near-threshold states with Coulomb plus short range interaction



Tomona Kinugawa



Tetsuo Hyodo

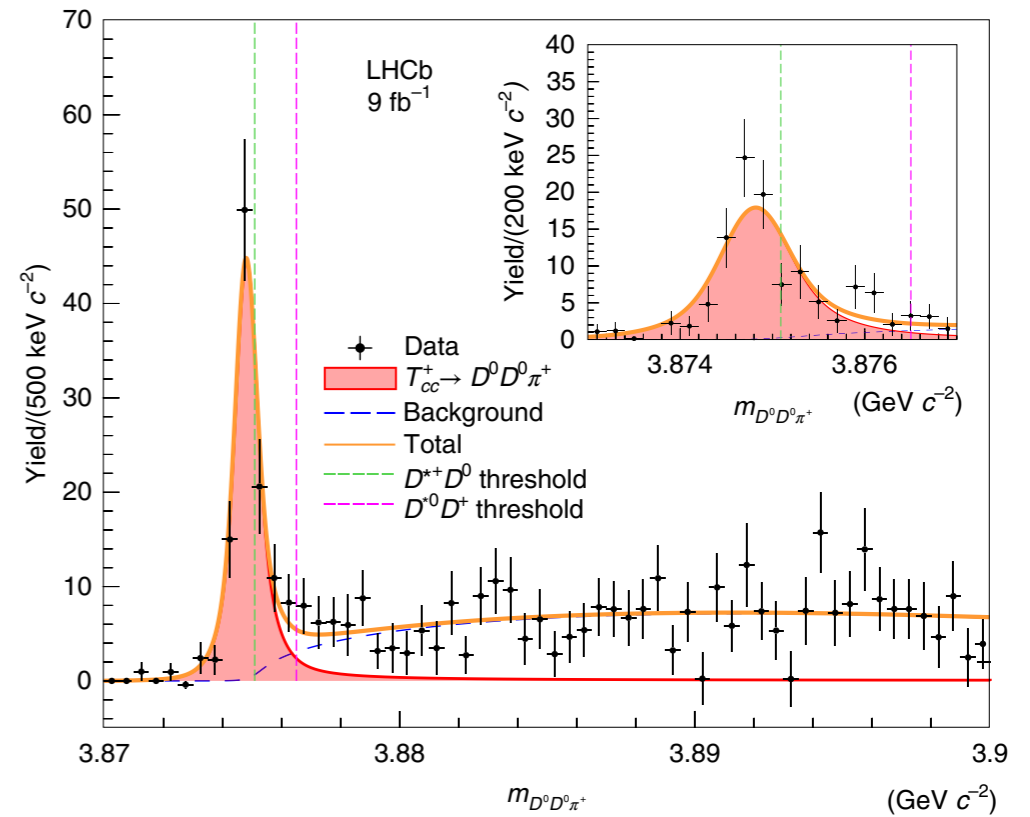
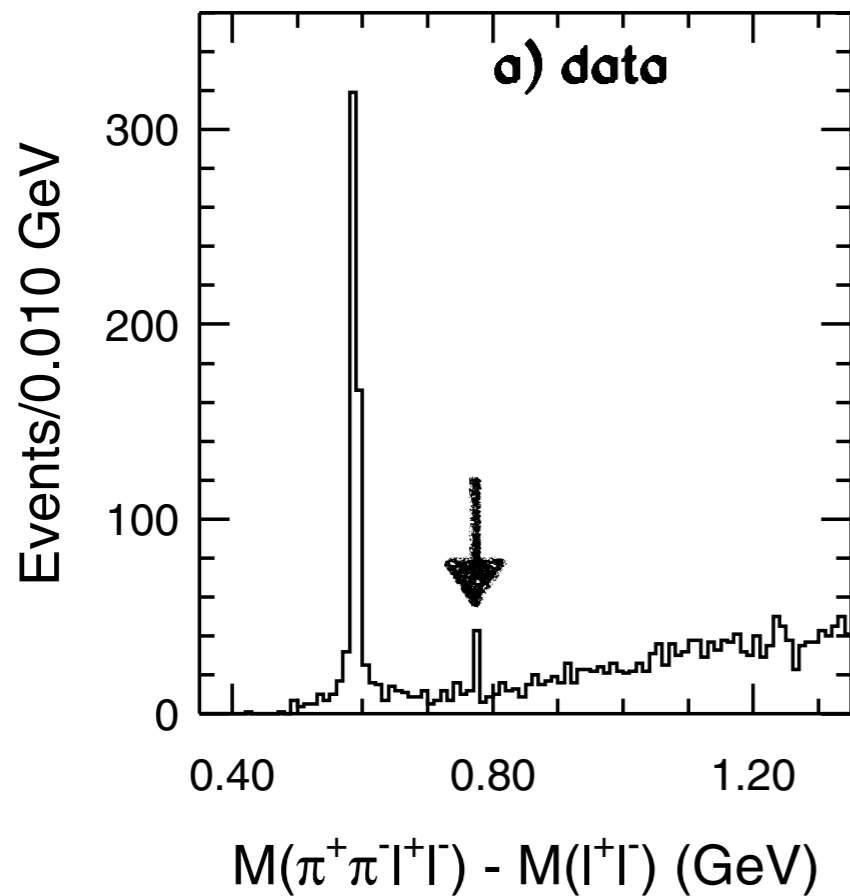
Nishina Center for Accelerator-Based Science, RIKEN

April 2nd, HIN 2025

Near-threshold exotic hadrons

$$X(3872) \rightarrow \pi^+ \pi^- J/\psi$$

$$T_{cc}(3875)^+ \rightarrow D^0 D^0 \pi^+ \quad (cc\bar{u}\bar{d})$$



LHCb Collaboration, Nature Phys. **18** (2022) no.7, 751-754;

S. K. Choi *et al.* (Belle), Phys. Rev. Lett. **91**, 262001 (2003).

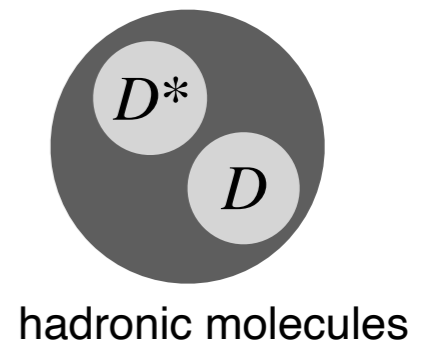
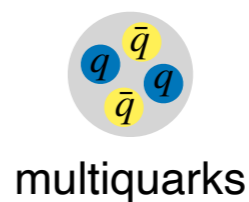
LHCb Collaboration, Nat. Commun. **13** 3351 (2022).

- exotic hadron

$$\neq qqq \text{ or } q\bar{q}$$



multiquarks
hadronic molecules

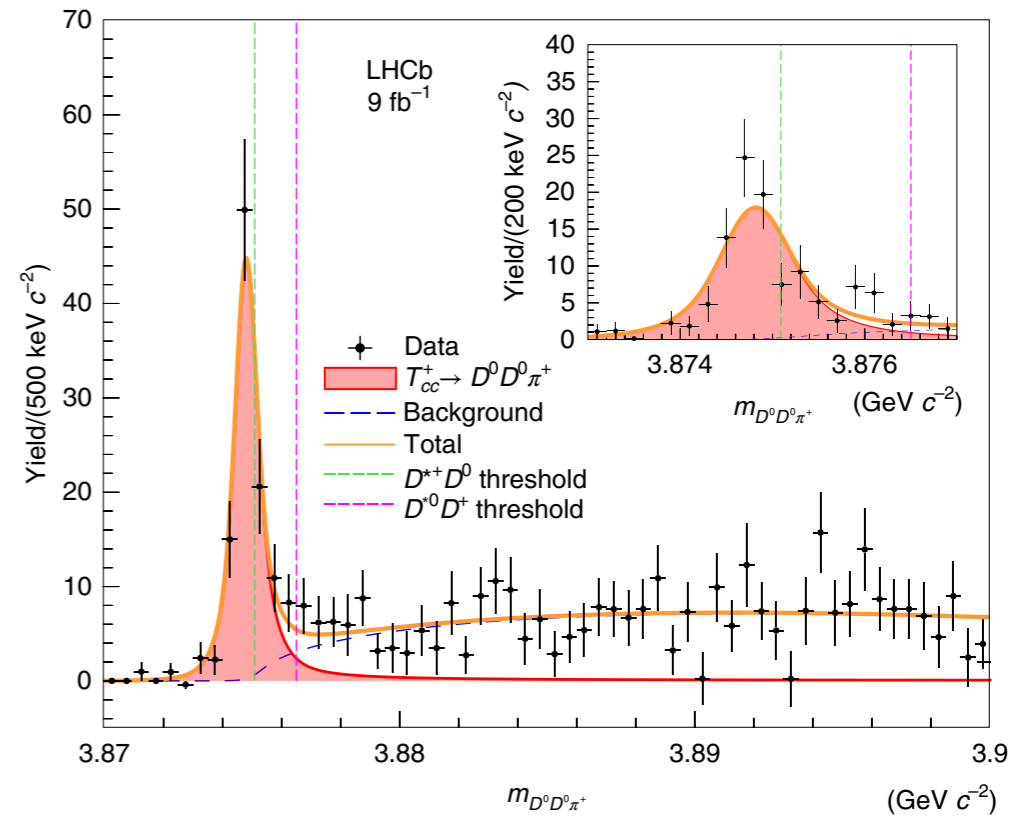
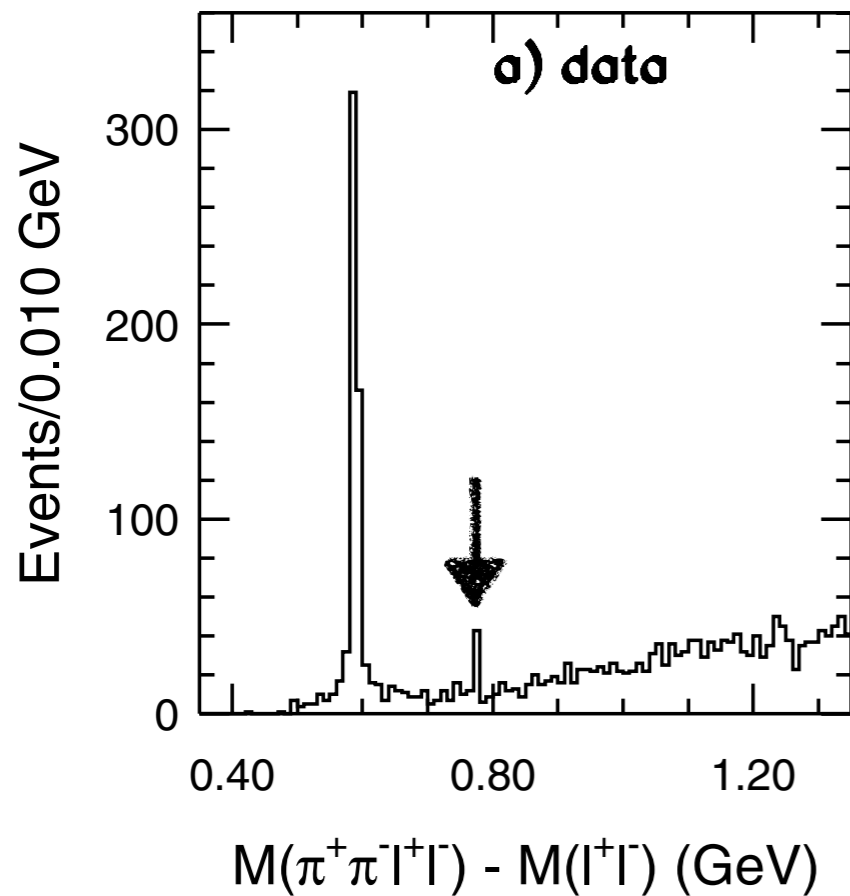


- internal structure of near-threshold states?

Near-threshold exotic hadrons

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LHCb Collaboration, Nature Phys. **18** (2022) no.7, 751-754;

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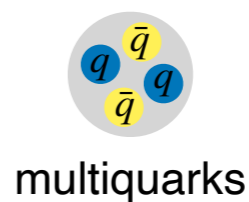
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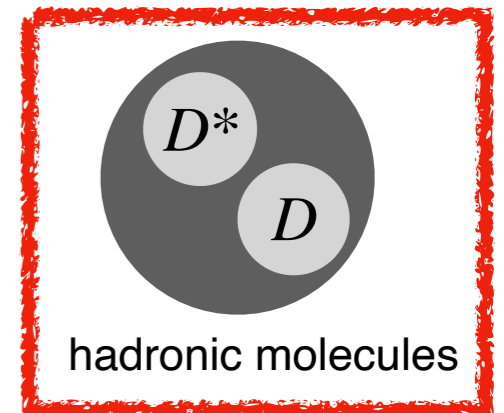


multiquarks

hadronic molecules



multiquarks



hadronic molecules

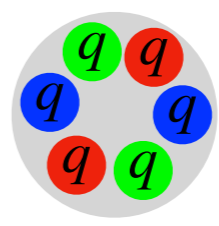
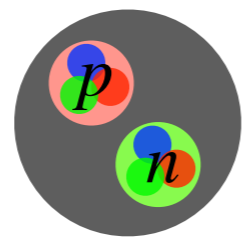
- internal structure of near-threshold states?

Compositeness

S. Weinberg, Phys. Rev. 137, 672–678 (1965);
 T. Hyodo, Int. J. Mod. Phys. A 28, 1330045 (2013).
 T. Kinugawa and T. Hyodo, arXiv: 2411.12285 [hep-ph],
 accepted in EPJ.

● definition

wavefunction



$$|\Psi\rangle = \sqrt{X} |\text{composite}\rangle + \sqrt{1-X} |\text{non composite}\rangle$$

compositeness

elementarity

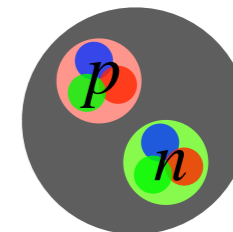
* $0 \leq X \leq 1 \longrightarrow X > 0.5 \Leftrightarrow \text{composite dominant}$
 $X + Z = 1 \quad X < 0.5 \Leftrightarrow \text{elementary dominant}$

- **quantitative** analysis of internal structure

- deuteron** S. Weinberg, Phys. Rev. 137, 672–678 (1965) etc.
 V. Baru, J. Haidenbauer, C. Hanhart, Y. Kalashnikova, A.E. Kudryavtsev,
 $f_0(980), a_0(980)$ Phys. Lett. B 586, 53–61 (2004);
 Y. Kamiya and T. Hyodo, PTEP 2017; Phys. Rev. C 93, 035203 (2016) etc.
- $T_{cc}, X(3872)$ T. Kinugawa and T. Hyodo, Phys. Rev. C 109, 045205 (2024);
 L. R. Dai, L. M. Abreu, A. Feijoo, and E. Oset, Eur. Phys. J. C 83, 983 (2023) etc.
- exotic nuclei, atomic systems** T. Kinugawa, T. Hyodo, Phys. Rev. C 106, 015205 (2022) etc.

Near-threshold states

● near-threshold states with short range interaction



- at threshold ($E = 0$)

completely composite ($X = 1$)

∴ low-energy universality $|a_{\text{s.r.}}| \rightarrow \infty$

T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

- near-threshold **bound states**

($E \neq 0$, but small **negative**)

composite dominant ($X \sim 1$)

C. Hanhart, J. R. Pelaez, and G. Rios, Phys. Lett. B **739**, 375 (2014);

T. Kinugawa and T. Hyodo Phys. Rev. C **109**, 045205 (2024).

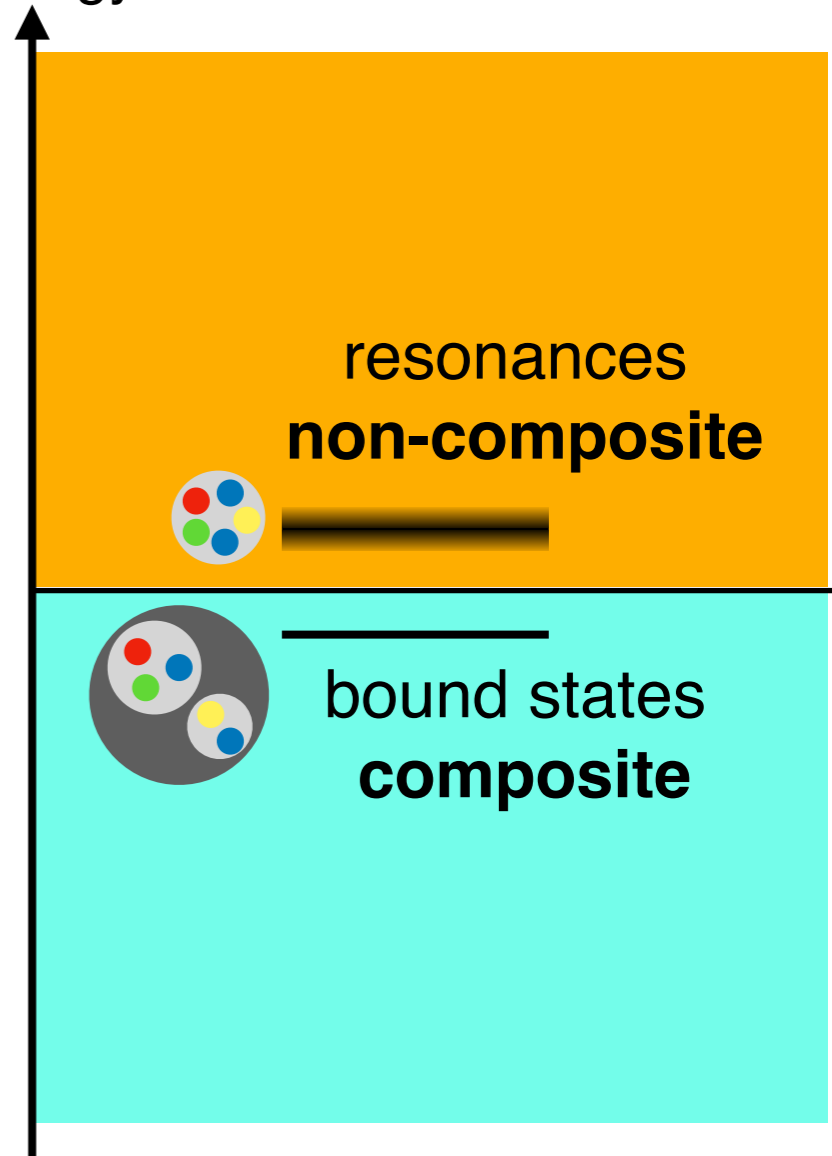
- near-threshold **resonances**

($E \neq 0$, but small **positive**)

non-composite dominant ($X \sim 0$)

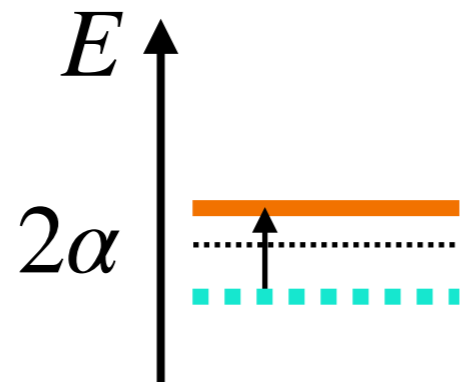
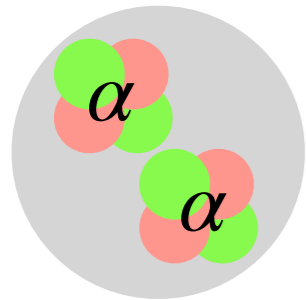
T. Kinugawa and T. Hyodo, arXiv:2403.12635 [hep-ph].

energy



Coulomb + short range systems

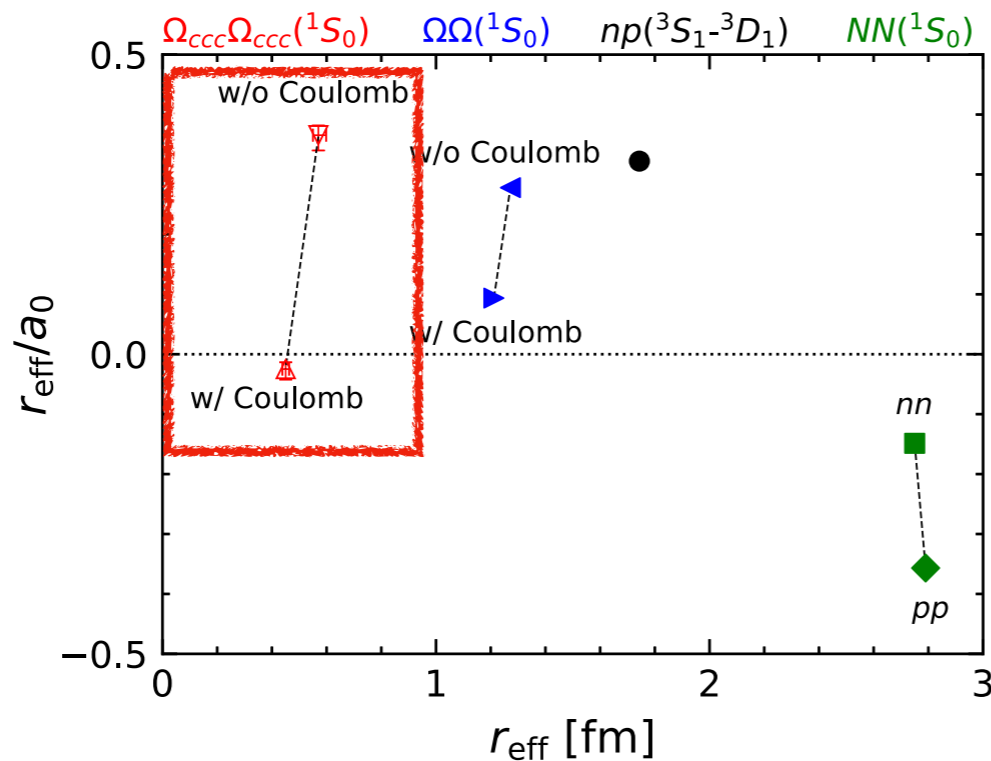
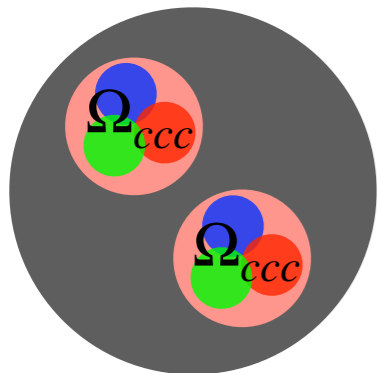
- ${}^8\text{Be}$ nuclei ($2\alpha^{++}$) J. Hiura, and R. Tamagaki, Prog. Theor. Phys. Suppl. No. 52, 25 (1972).



resonance (w/ Coulomb)

bound (w/o Coulomb)

- $\Omega_{ccc}^{+++} \Omega_{ccc}^{+++}$ (HAL QCD) Y. Lyu, H.Tong, *et al.* [HAL QCD Coll.], Phys. Rev. Lett. 127 (2021) 072003.



resonance (w/ Coulomb)

bound (w/o Coulomb)

- $\Xi^{-} \alpha$: Coulomb assisted bound state E. Hiyama, M. Isaka, T. Doi, and T. Hatsuda, Phys. Rev. C 106, 064318 (2022).

→ Coulomb is important for near-threshold states!

Coulomb + short range systems

● Coulomb + short range interaction

H. A. Bethe, Phys. Rev. 76, 38-50 (1949).

R. Oppenheim Berger and Larry Spruch, Phys. Rev. 138, B1106-B1115 (1965).

W. Domcke, Atom. Mol. Phys. 16, 359 (1983).

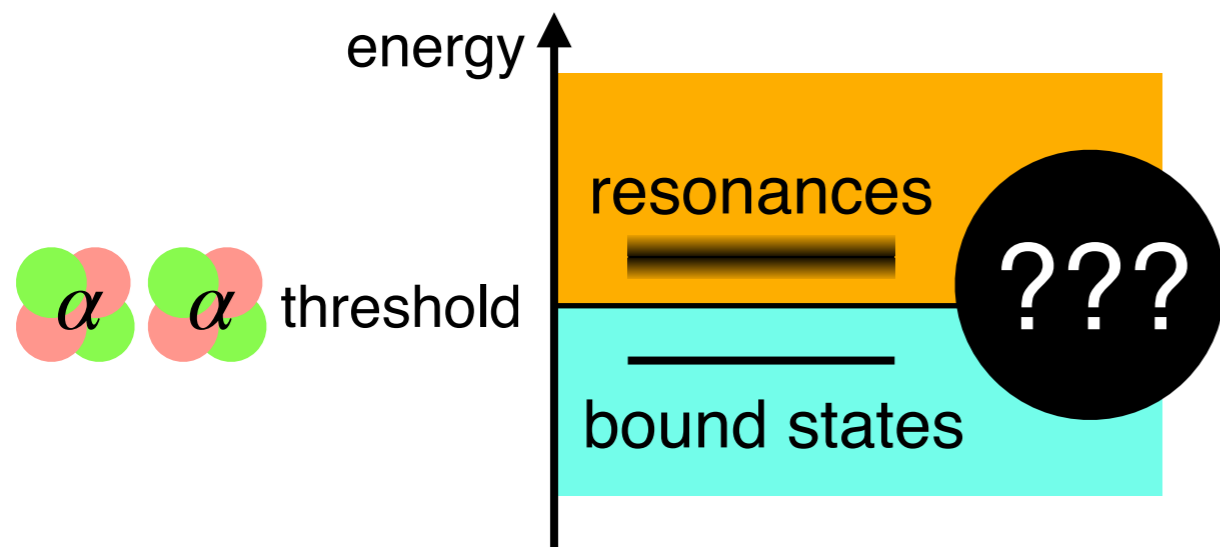
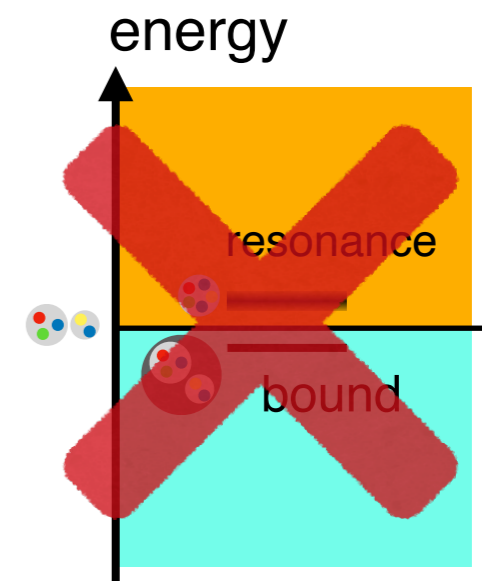
R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809, 171 (2008).

C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798, 135016 (2019).

S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

- low-energy behavior of scattering amplitude is different from that of short range interaction

● nature of near-threshold state with Coulomb + short range interaction?

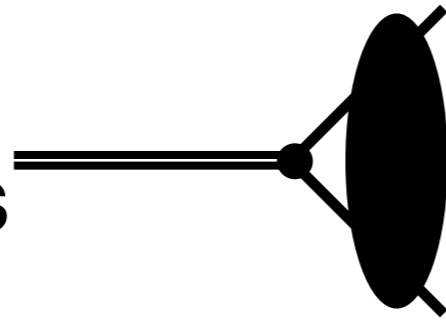


- two body
- small k region \rightarrow s -wave
- Coulomb repulsive
- \rightarrow pole trajectories?
- \rightarrow compositeness?

Coulomb+short range model

● Hamiltonian W. Domcke, Atom. Mol. Phys. 16 359 (1983). R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).

Q channel
 \Leftrightarrow bound states
 w/ short range



P channel
 \Leftrightarrow scattering w/ Coulomb

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} = \begin{pmatrix} \text{Coulomb} & \text{Short Range} \\ \text{Short Range} & \text{Coulomb} \end{pmatrix}$$

H. Feshbach, Annals Phys. 19 287-313 (1962).

● pole condition H. A. Bethe, Phys. Rev. 76, 38-50 (1949).
 C. H. Schmickler, H.-W. Hammer, and A.G. Volosniev, Physics Letters B 798 (2019).

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B} \left[\underbrace{\log(-ia_Bk)}_{\text{log cut}} + \psi \left(1 + \frac{i}{a_Bk} \right) \right] = 0$$

● compositeness X T. Hyodo, Phys. Rev. C 90, 055208 (2014).

Bohr radius

$$X = 1 - \frac{1}{1 - \frac{d}{dE} F(E)} \text{ self energy}$$

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

Coulomb+short range model

8

● short range limit $a_B \rightarrow \infty$

$$a_B = \frac{\hbar c}{\mu c^2 Z_1 Z_2}$$

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B} \left[\log(-ia_B k) + \psi \left(1 + \frac{i}{a_B k} \right) \right] = 0$$

$\rightarrow 0$

→ $-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik = 0$ short range interaction

● further low-energy limit $r_e \rightarrow 0$

- zero-range theory S. Mochizuki, and Y. Nishida, Phys. Rev. C 110 , 064001 (2024).

$$\frac{ia_B k}{2} \mp \log(-ia_B k) + \psi \left(1 + \frac{i}{a_B k} \right) + \frac{a_B}{2a_s} = 0$$

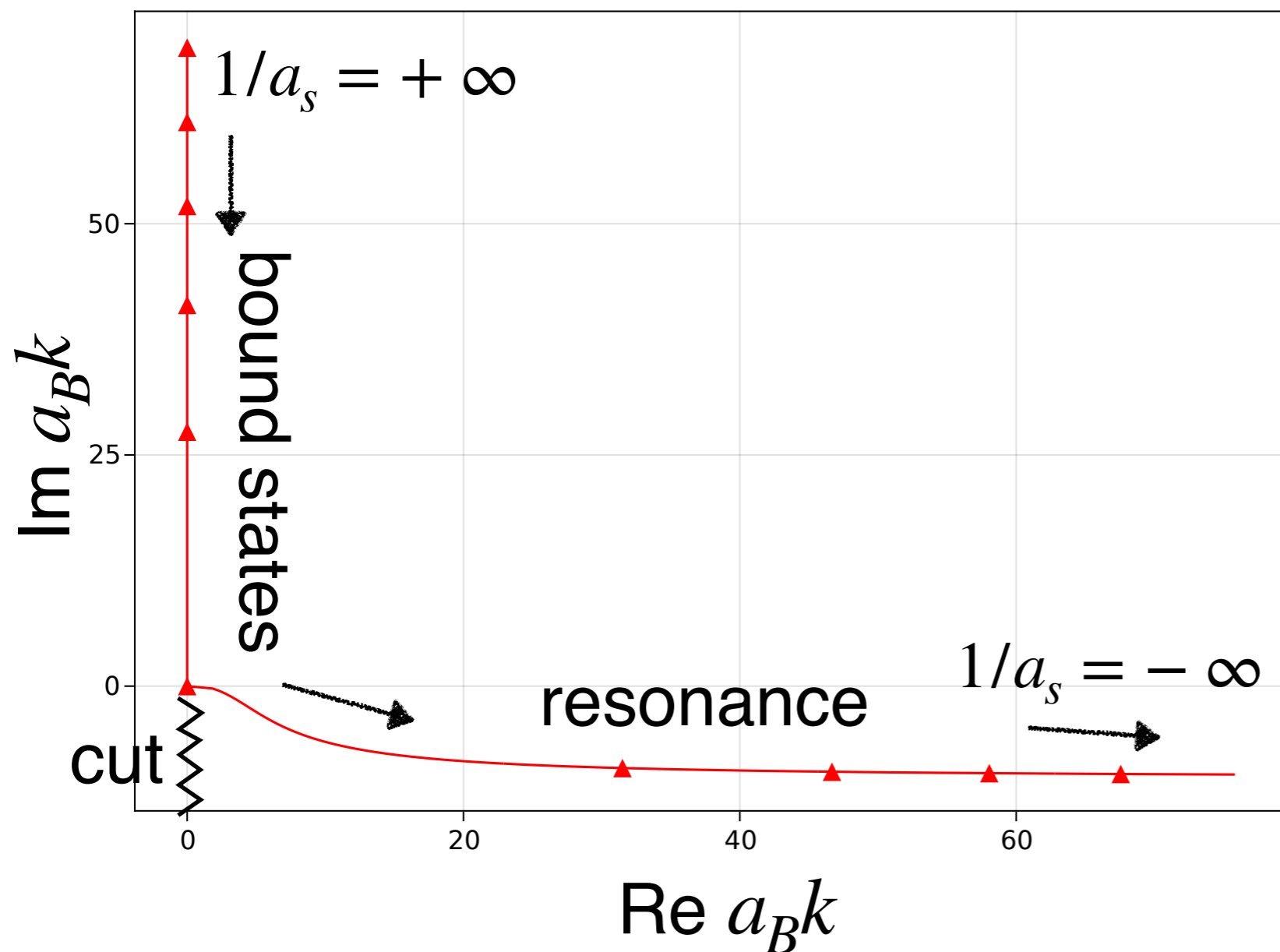
Pole trajectory (repulsive Coulomb)

9

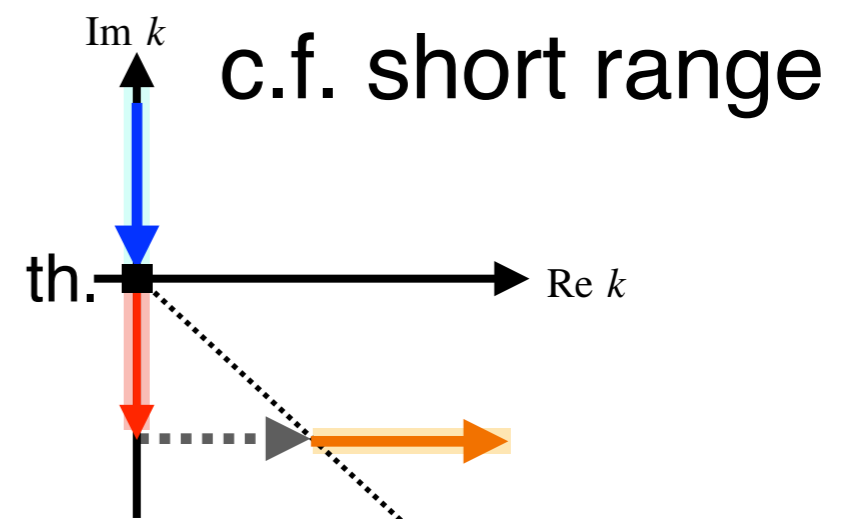
● pole trajectory in complex momentum k plane

- varying Coulomb scattering length a_s with fixed r_e and a_B

→ pole position (eigenmomentum) moves



- b.s. directory goes to resonance



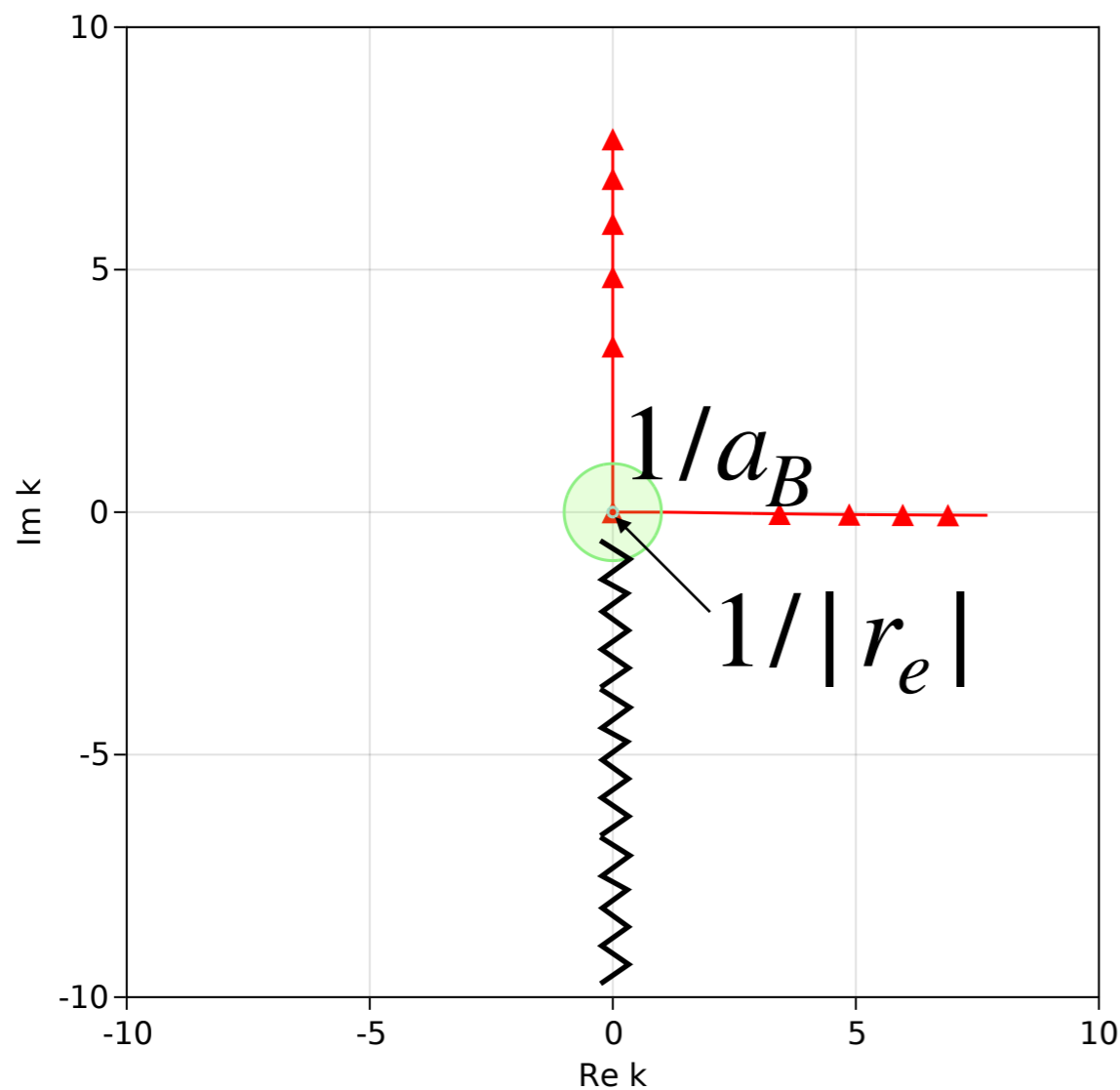
- $a_s \rightarrow \infty$ at threshold

- but no universality

\therefore radius of w.f. $< \infty$

Pole trajectory (repulsive Coulomb) 10

$$a_B = 1, r_e = -10$$

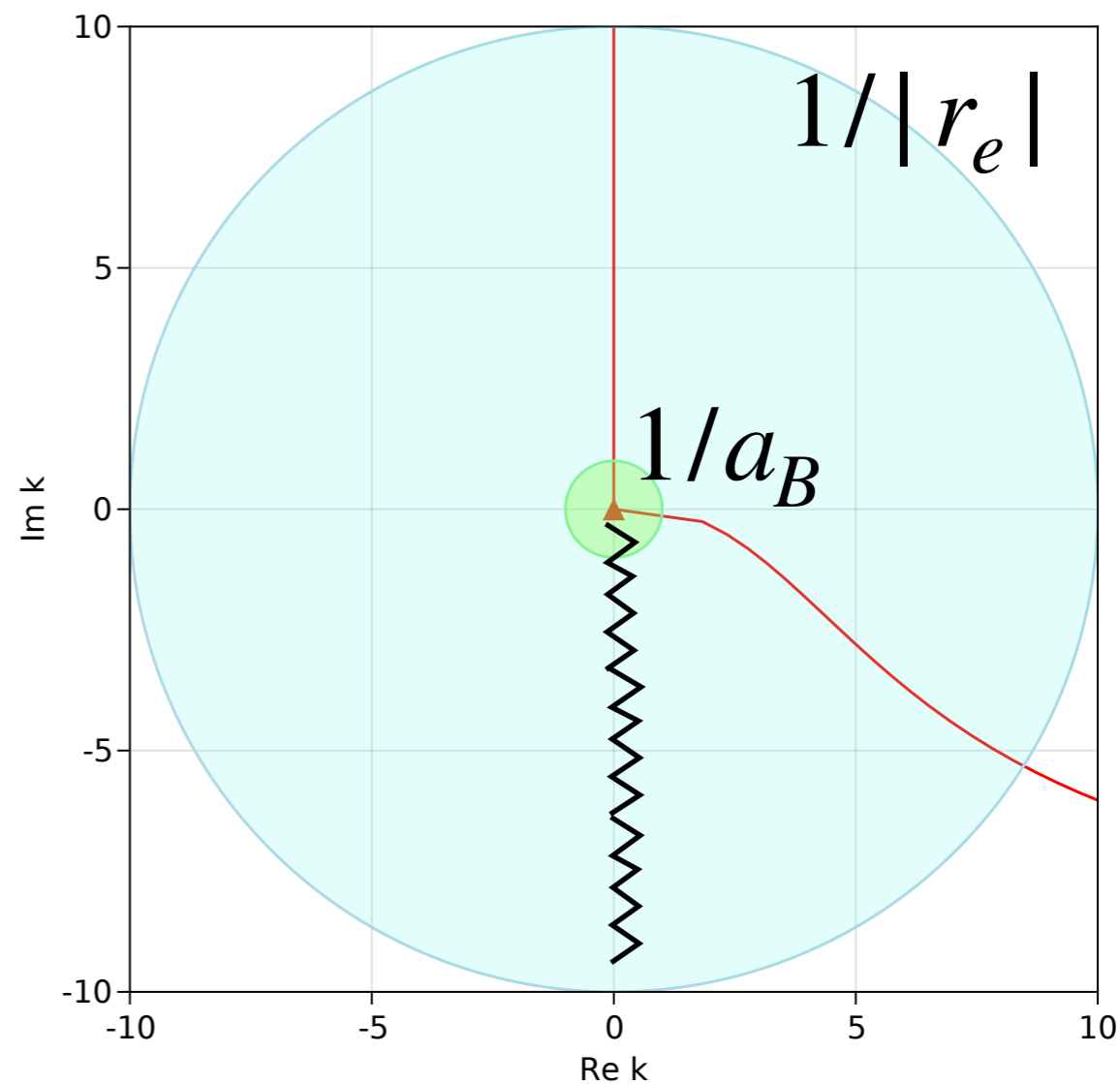


short range universality

\wedge

Coulomb dominant region

$$a_B = 1, r_e = -0.1$$



short range universality

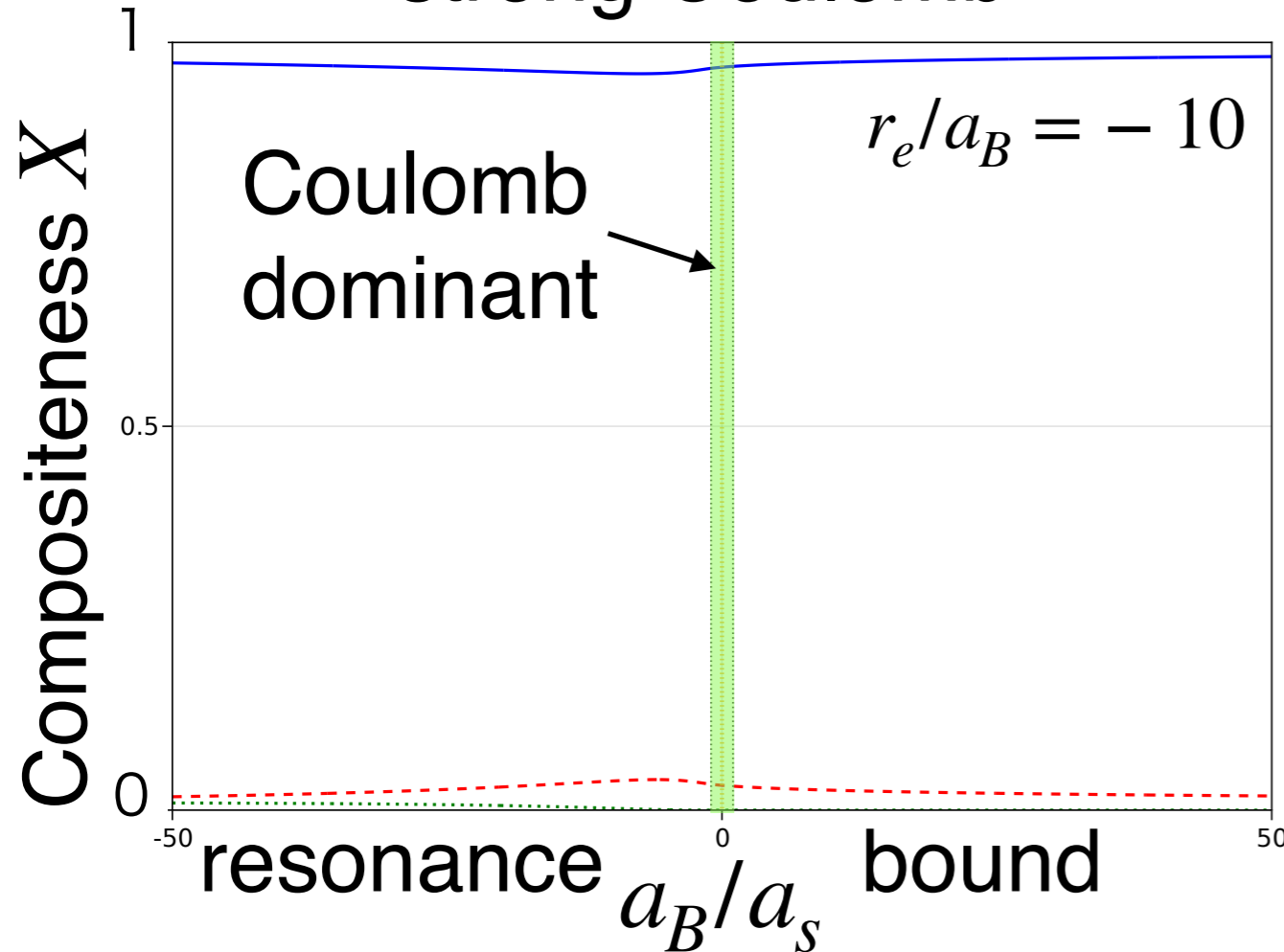
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Coulomb dominant region

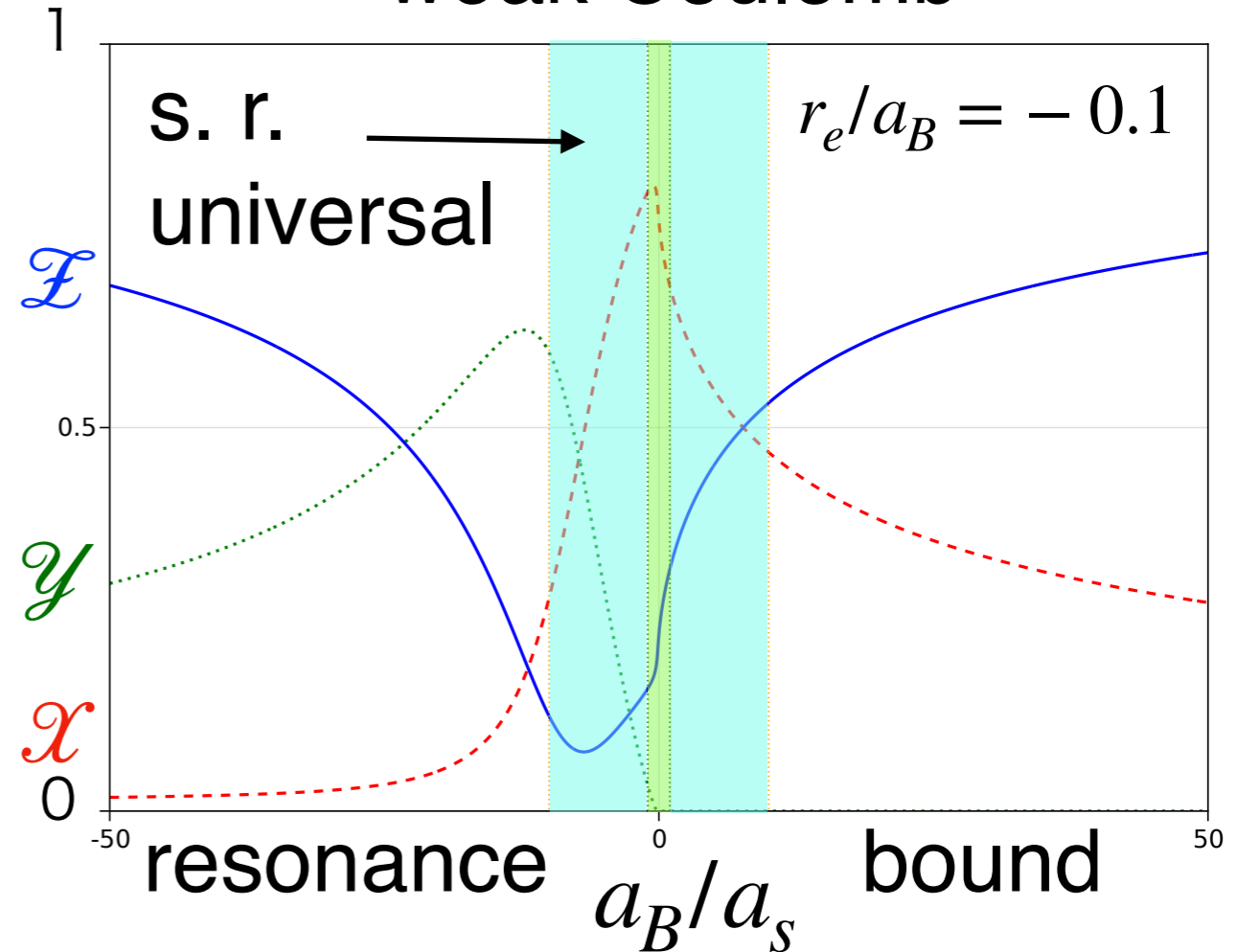
Compositeness (repulsive Coulomb)

11

strong Coulomb



weak Coulomb



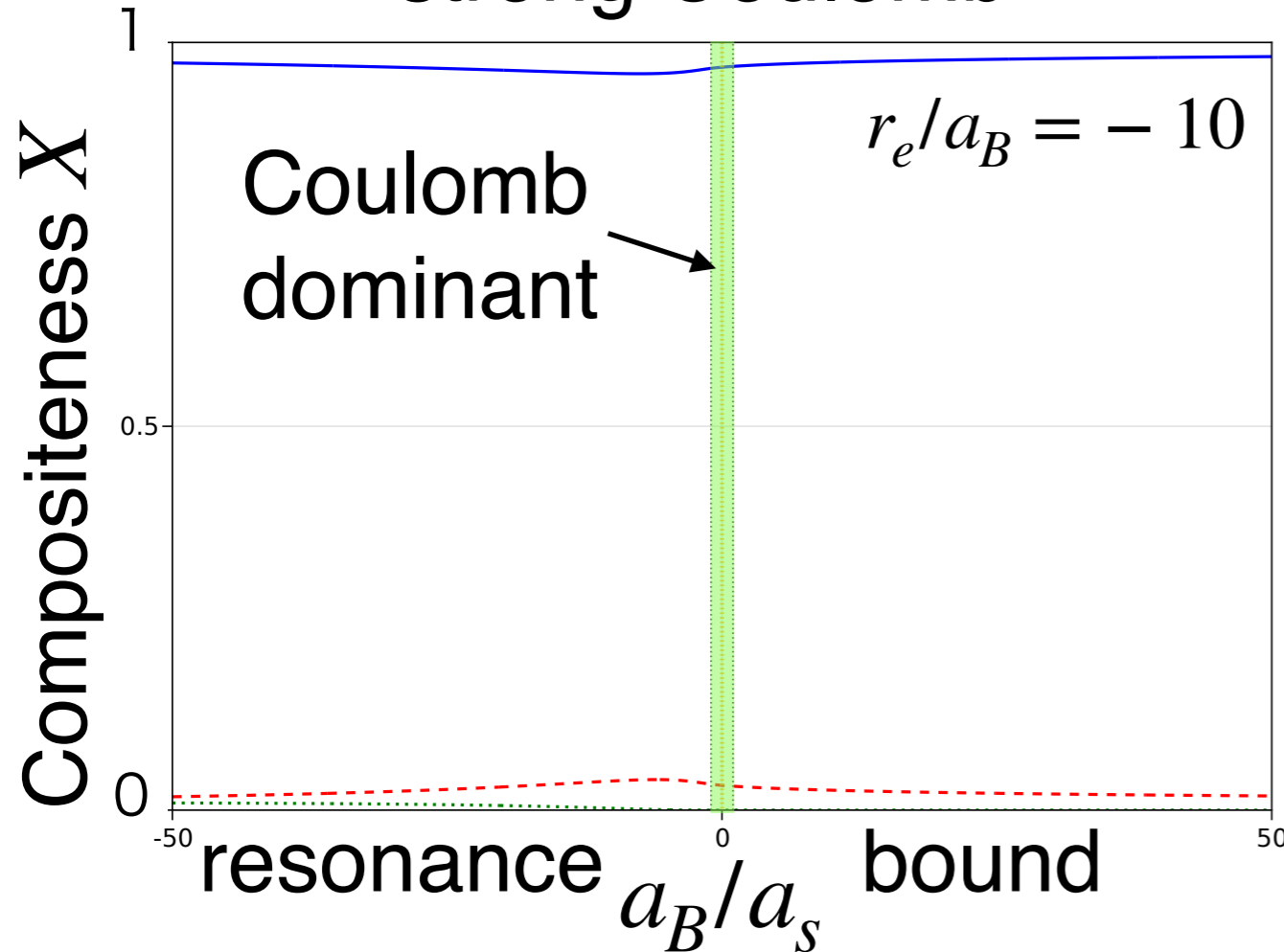
- complex compositeness $\leftarrow \mathcal{X}, \mathcal{Y}, \mathcal{F}$
- states with large $|1/a_s|$ are elementary \mathcal{F} dominant
- structure of bound states \approx resonances \because continuous X

T. Kinugawa and T. Hyodo,
arXiv:2403.12635 [hep-ph].

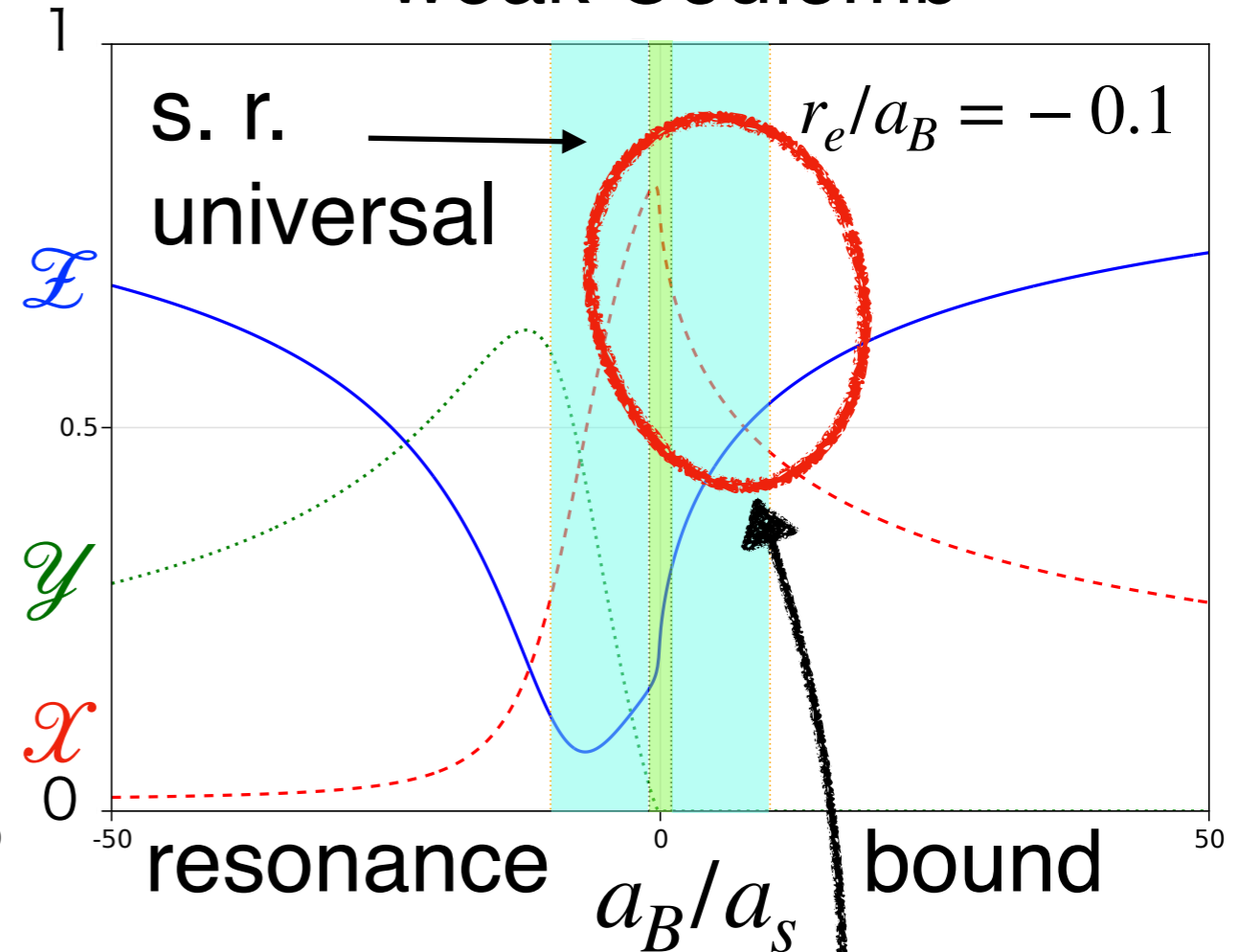
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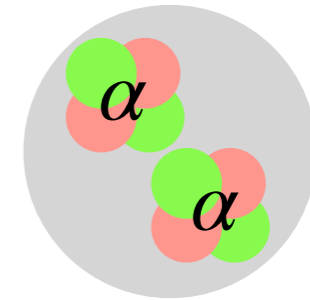
- complex compositeness $\leftarrow \mathcal{X}, \mathcal{Y}, \mathcal{F}$
- states with large $|1/a_s|$ are elementary \mathcal{F} dominant
- structure of bound states \approx resonances \because continuous X
- remnant of short range universality in $|r_e| \ll |a_B|$ case
- $X \rightarrow 1$ in $B \rightarrow 0$ limit in short range

T. Kinugawa and T. Hyodo,
arXiv:2403.12635 [hep-ph].

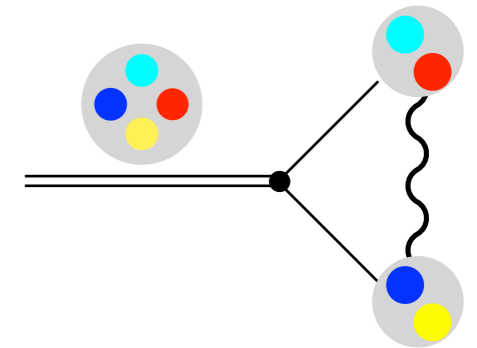
Summary



near-threshold bound states & resonances
with **Coulomb + short range** interaction



- bare state which couples to Coulomb scattering
- pole condition $\leftarrow a_s, r_e, a_B$



- repulsive Coulomb

bound \rightarrow resonance (does not become virtual states)

X is not necessary to be unity at threshold

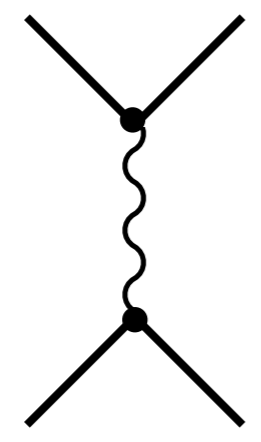
if Coulomb $<$ s.r., remnant of s.r. universality can be seen
nature of b.s. \approx nature of resonance



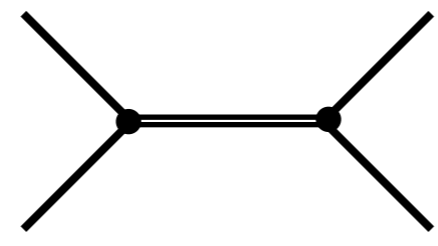
Back up

Coulomb+short range model

● model Coulomb



short range (s.r.)



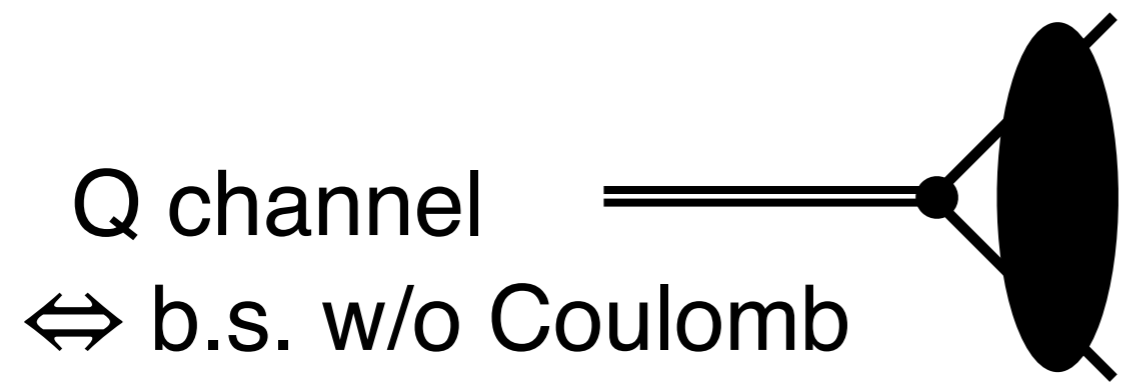
Weinberg, S. Phys. Rev. 137, 672–678 (1965);
 V. Baru, J. Haidenbauer, C. Hanhart,
 Y. Kalashnikova, A.E. Kudryavtsev, Phys. Lett.
 B 586, 53–61 (2004);
 T. Hyodo, Phys. Rev. C **90**, 055208 (2014) .

● Hamiltonian

W. Domcke, Atom. Mol. Phys. 16 359 (1983);
 H. Feshbach, Annals Phys. 19 287-313 (1962).

R. Higa, H.-W. Hammer, and U. van
 Kolck, Nuclear Physics A 809 (2008).

$$\hat{H} = \begin{pmatrix} \hat{H}_{PP} & \hat{H}_{PQ} \\ \hat{H}_{QP} & \hat{H}_{QQ} \end{pmatrix} = \begin{pmatrix} \text{loop with vertical oval} & \text{triangle with vertical oval} \\ \text{triangle with vertical oval} & \text{loop with vertical oval} \end{pmatrix}$$



Coulomb+short range model

● Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle$ $|\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$

$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

● effective Hamiltonian (channel eliminating)

$$\hat{H}_{P\text{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\text{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$

Coulomb+short range model

● Schrödinger equation $\hat{H}|\Psi\rangle = E|\Psi\rangle \quad |\Psi\rangle = \begin{pmatrix} |P\rangle \\ |Q\rangle \end{pmatrix}$

$$\hat{H}_{PP}|P\rangle + \hat{H}_{PQ}|Q\rangle = E|P\rangle$$

$$\hat{H}_{QQ}|Q\rangle + \hat{H}_{QP}|P\rangle = E|Q\rangle$$

● effective Hamiltonian (channel eliminating)

$$\hat{H}_{P\text{ch}}|P\rangle = E|P\rangle \quad \hat{H}_{P\text{ch}} = \hat{H}_{PP} + \hat{H}_{PQ}(E - \hat{H}_{QQ})^{-1}\hat{H}_{QP}$$

$$= \hat{H}^0 + \hat{V}_P \quad = \hat{V}_Q$$

$$\rightarrow \hat{H}_{P\text{ch}} = \hat{H}^0 + (\hat{V}_P + \hat{V}_Q)$$

\hat{H}^0 : free Hamiltonian \hat{V}_P : pure Coulomb interaction

\hat{V}_Q : short range interaction

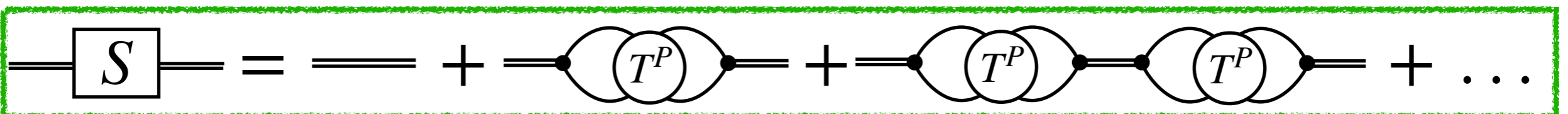
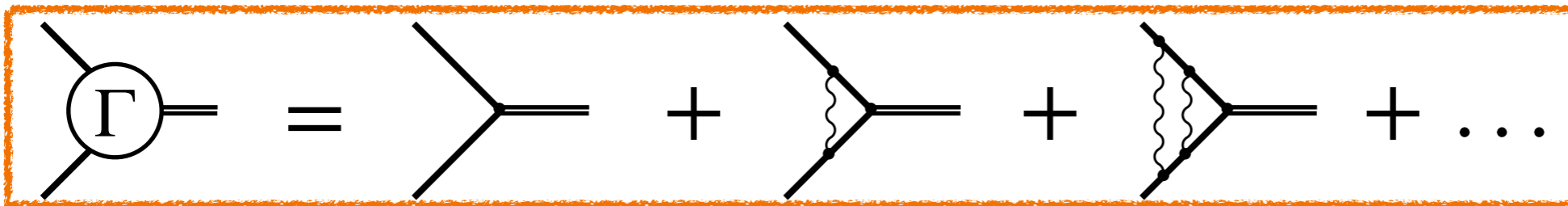
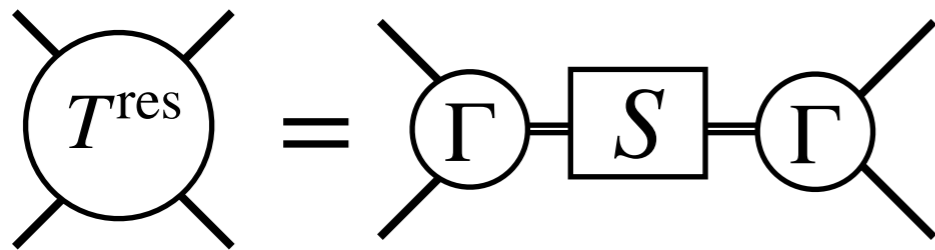
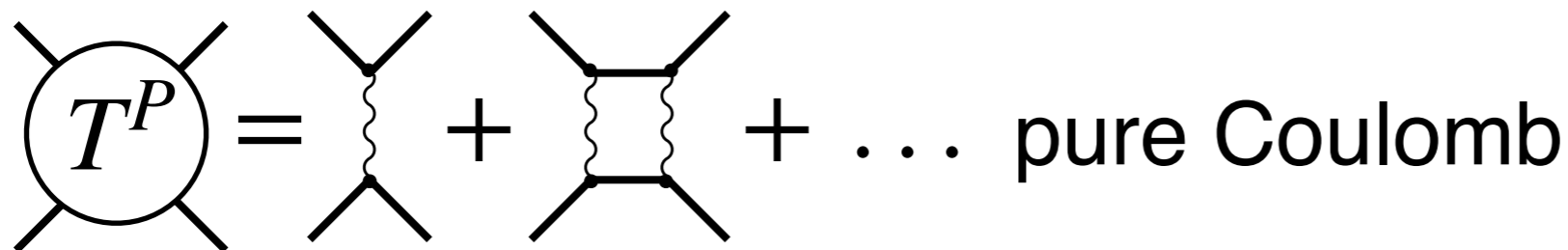
Coulomb+short range model

● T-matrix H. Feshbach, Annals Phys. 19 287-313 (1962); R. Higa, H.-W. Hammer, and U. van W. Domcke, Atom. Mol. Phys. 16 359 (1983); Kolck, Nuclear Physics A 809 (2008).

Lippmann-Schwinger eq. : $\hat{T} = [(\hat{V}_P + \hat{V}_Q)^{-1} - \hat{G}^0]^{-1}$

↔ Feshbach method : $\hat{T} = \hat{T}^P + \hat{T}^{\text{res}}$

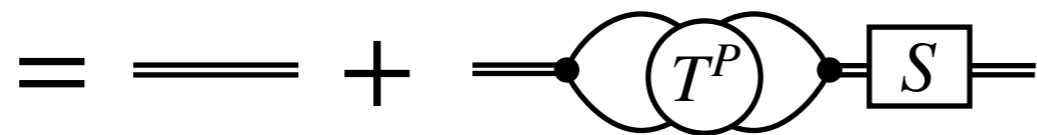
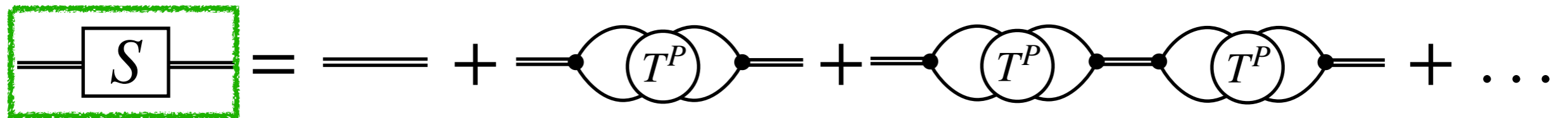
$$\hat{T}^P = [\hat{V}_P^{-1} - \hat{G}^0]^{-1}, \quad \hat{T}^{\text{res}} = \hat{T}^P \hat{V}_P^{-1} [\hat{V}_Q^{-1} - \hat{G}_P]^{-1} \hat{V}_P^{-1} \hat{T}^P$$



Coulomb+short range model

● pole condition $\hat{T} = \hat{T}^P + \hat{T}^{\text{res}}$

pole of $T(k, k')$ \Leftrightarrow pole of $T^{\text{res}}(k, k')$ $\Leftrightarrow [V_Q^{-1} - G_P]^{-1} = \infty$



$$H_{QP}G_P H_{PQ}$$

$\text{---} \boxed{S} \text{---} = S(E), \text{---} = (E - \varepsilon_d)^{-1}, \text{---} \text{---} \text{---} \text{---} \text{---} = F(E)$

$\rightarrow S(E) = (E - \varepsilon_d)^{-1} + (E - \varepsilon_d)^{-1} F(E) S(E)$

$= [E - \varepsilon_d - F(E)]^{-1}$

\rightarrow pole condition : $E - \varepsilon_d - F(E) = 0$

bare state energy

self energy

Coulomb+short range model

● self energy $F(E)$ in low-energy limit

W. Domcke, Atom. Mol. Phys. 16 359 (1983).

- attractive Coulomb

$$F(k) = \frac{A}{2\pi} \left[c - \frac{1}{2}ia_Bk + \log(-ia_Bk) + \psi \left(1 - \frac{i}{a_Bk} \right) \right]$$

- repulsive Coulomb

$$F(k) = -\frac{A}{2\pi} \left[c + \frac{1}{2}ia_Bk + \log(-ia_Bk) + \psi \left(1 + \frac{i}{a_Bk} \right) \right]$$

A : constant with dimension of energy

c : dimensionless constant

$\psi(x) = \frac{d}{dx} \log(\Gamma(x))$: digamma function

Coulomb+short range model

● pole condition in low-energy limit

- Coulomb scattering length a_s and effective range r_e

$$(\text{amplitude})^{-1} = -\frac{1}{a_s} + \frac{r_e}{2}k^2 + \mathcal{O}(k^4) - ik + 2 \log(-ik) + 2\psi \left(1 + \frac{i}{k} \right) + \dots,$$

$$\longrightarrow a_s = -a_B \left[\frac{4\pi}{A} \varepsilon_d \pm 2c \right]^{-1}, \quad r_e = -\frac{4\pi}{A a_B \mu}$$

R. Higa, H.-W. Hammer, and U. van Kolck, Nuclear Physics A 809 (2008).

- pole condition with a_s and r_e H. A. Bethe, Phys. Rev. 76, 38-50 (1949).

$$-\frac{1}{a_s} + \frac{r_e}{2}k^2 - ik \pm \frac{2}{a_B} \left[\log(-ia_B k) + \psi \left(1 + \frac{i}{a_B k} \right) \right] = 0$$

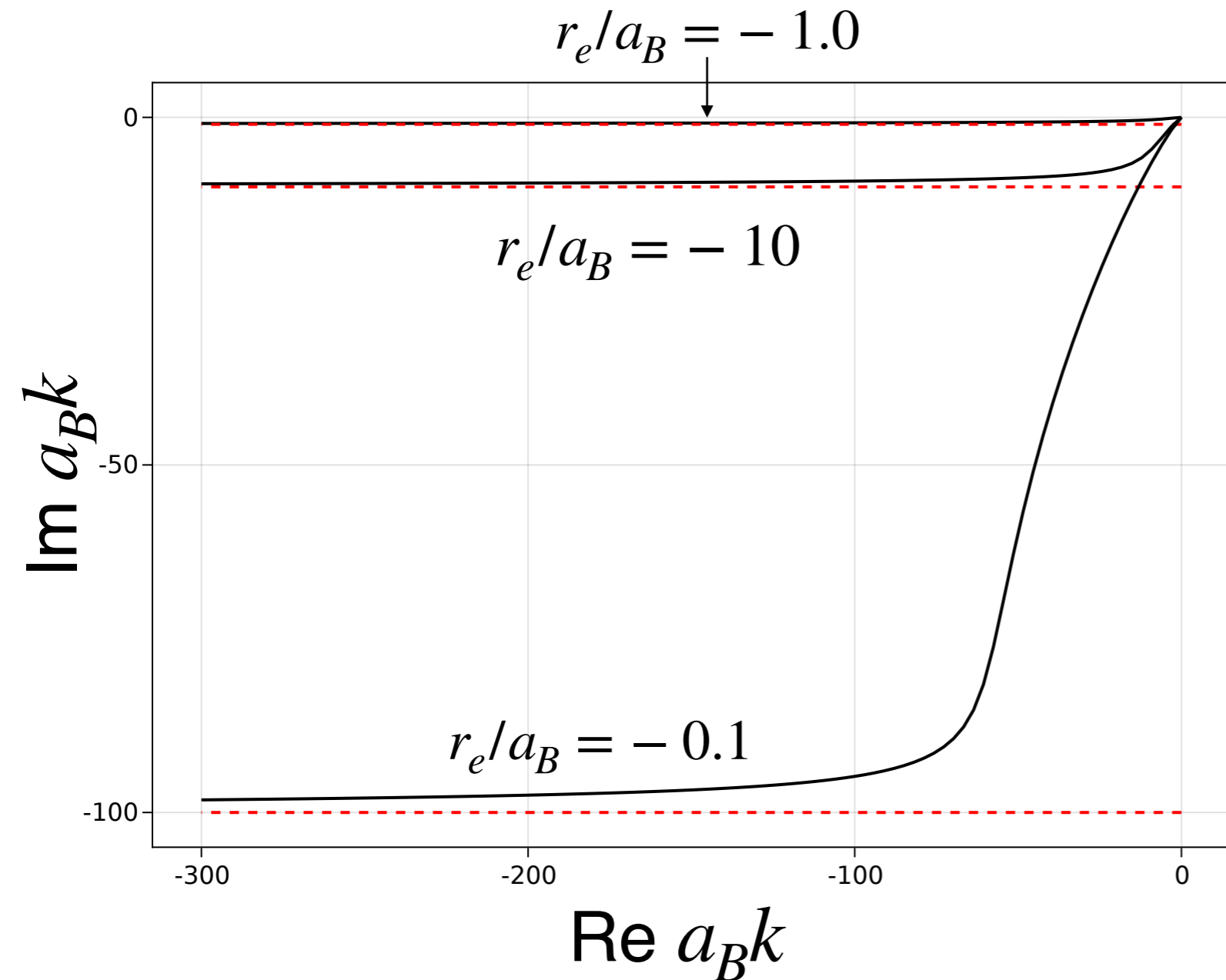
● compositeness X T. Hyodo, Phys. Rev. C 90, 055208 (2014).

$$X = 1 - \frac{1}{1 - \frac{d}{dE} F(E)} \text{ self energy}$$

far from threshold (repulsive Coulomb) 20

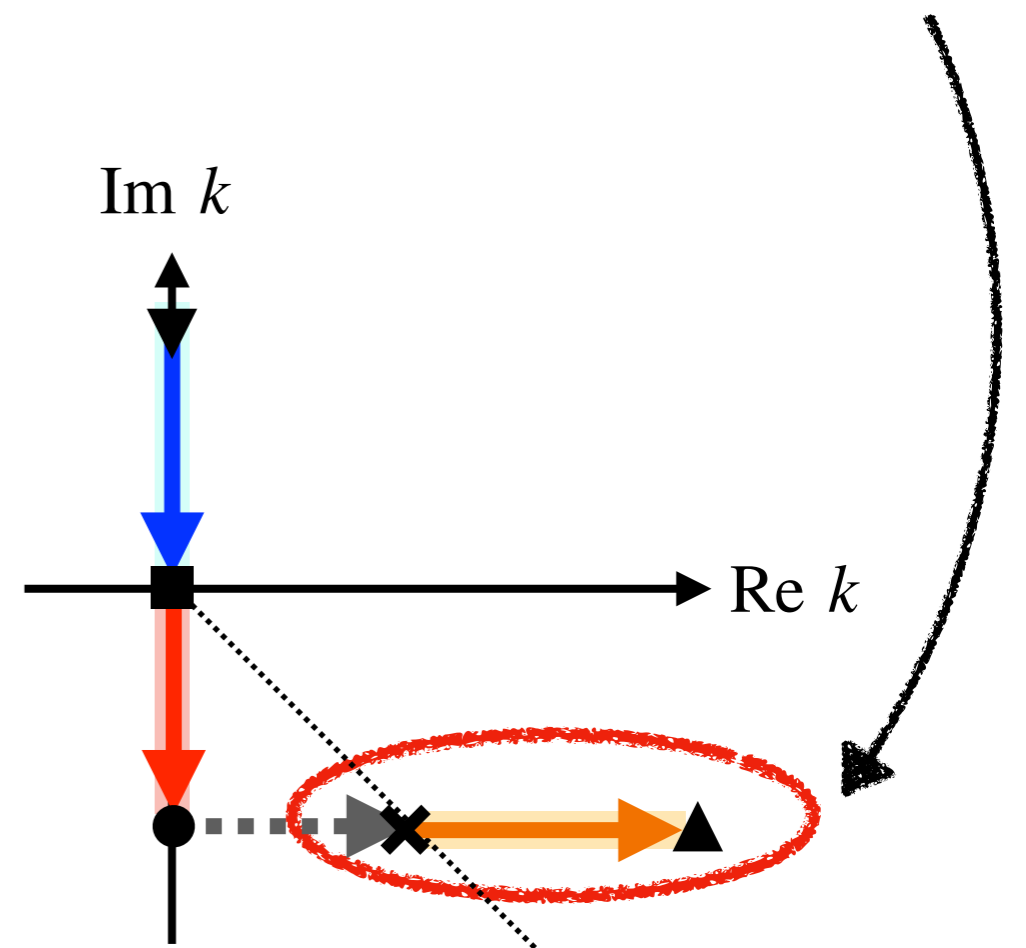
● imaginary part of eigenenergy in complex momentum k plane

- far from threshold in $1/a_s \rightarrow -\infty$ limit



- $\text{Im } k \rightarrow 1/r_e$

→ trajectory close to that of resonance in ERE

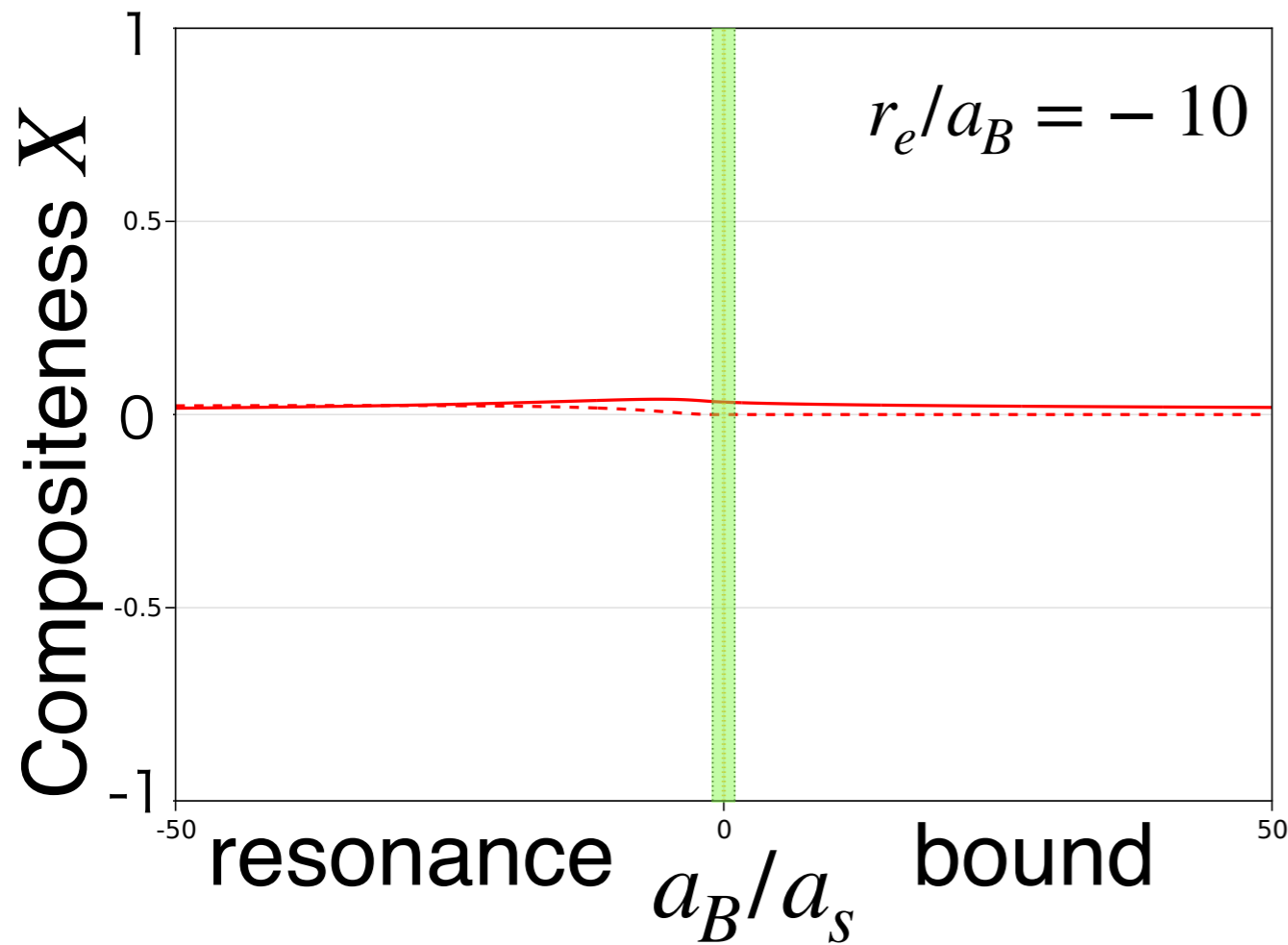


dotted lines : a_B/r_e

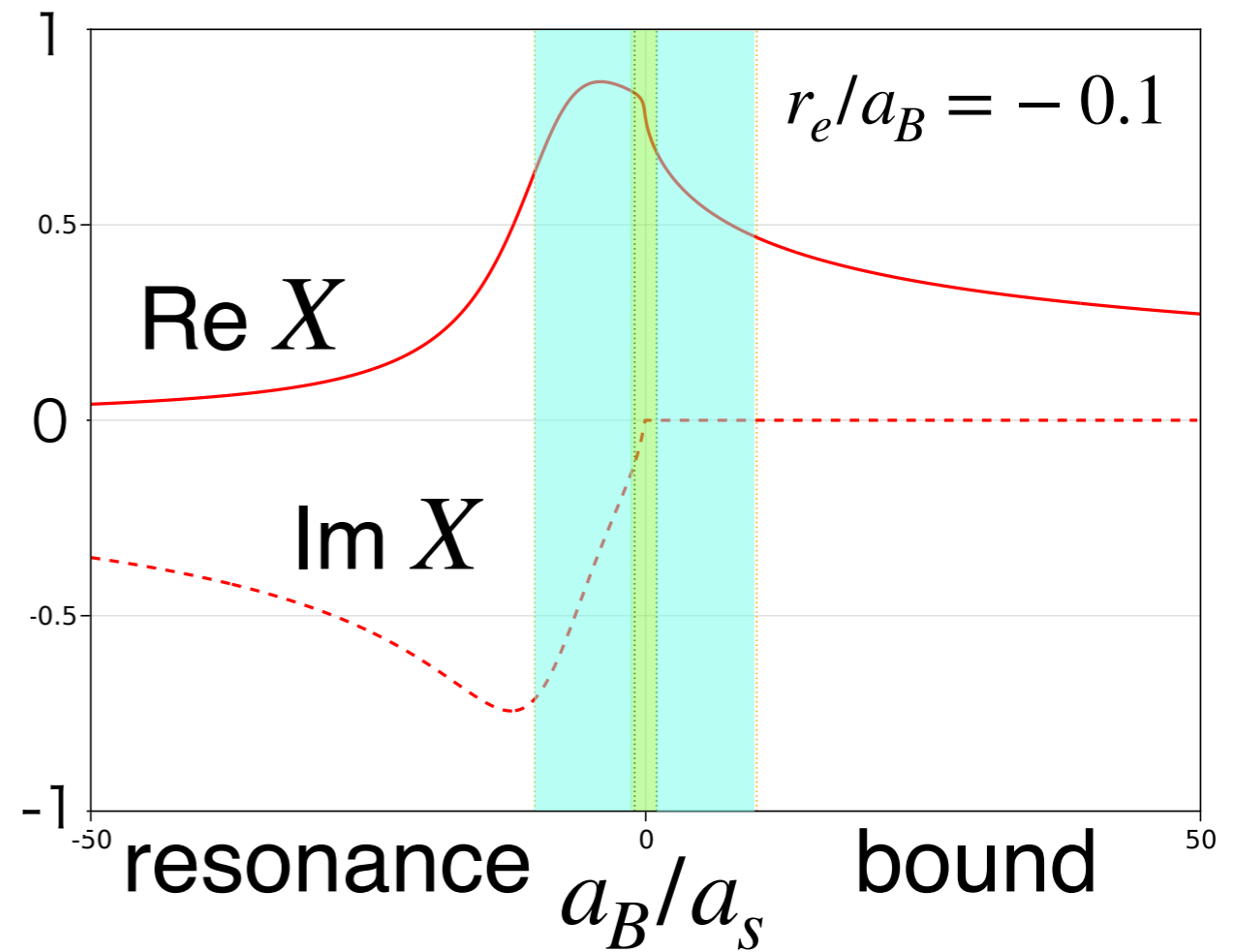
Compositeness (repulsive Coulomb)

21

strong Coulomb



weak Coulomb

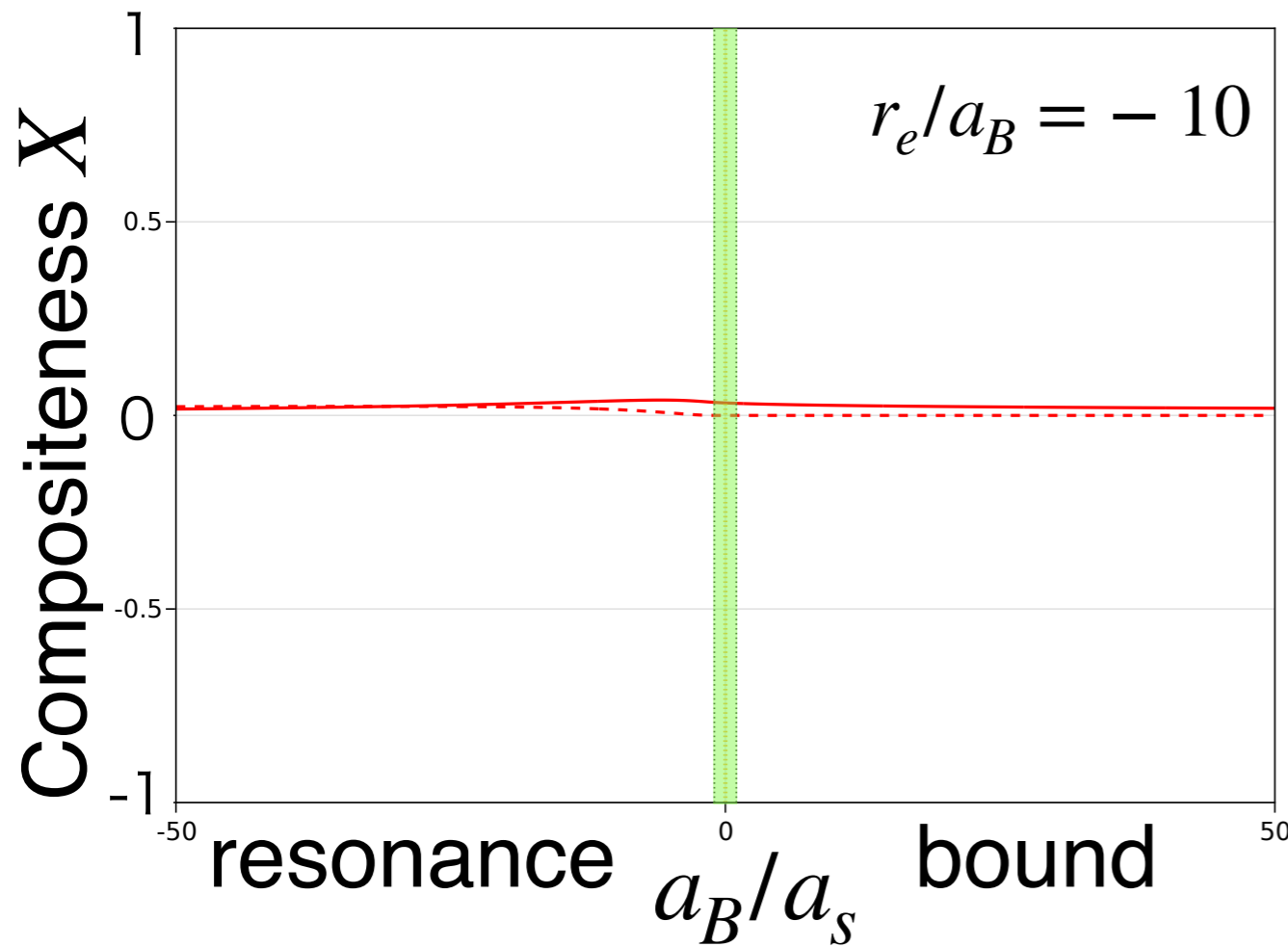


- $\pm 1/a_B$: Coulomb force dominant region
- $\pm 1/|r_e|$: short range universal region

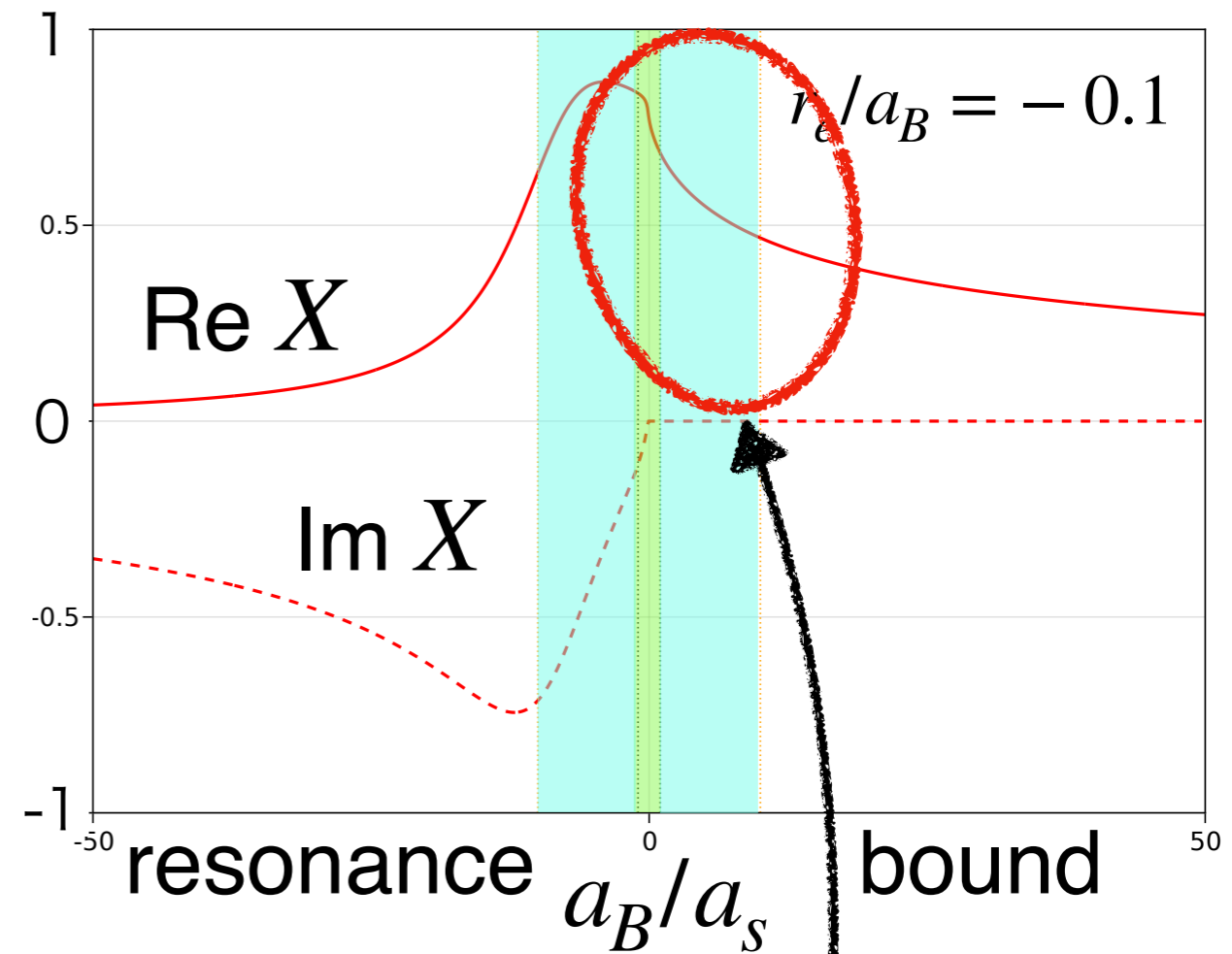
Compositeness (repulsive Coulomb)

21

strong Coulomb



weak Coulomb

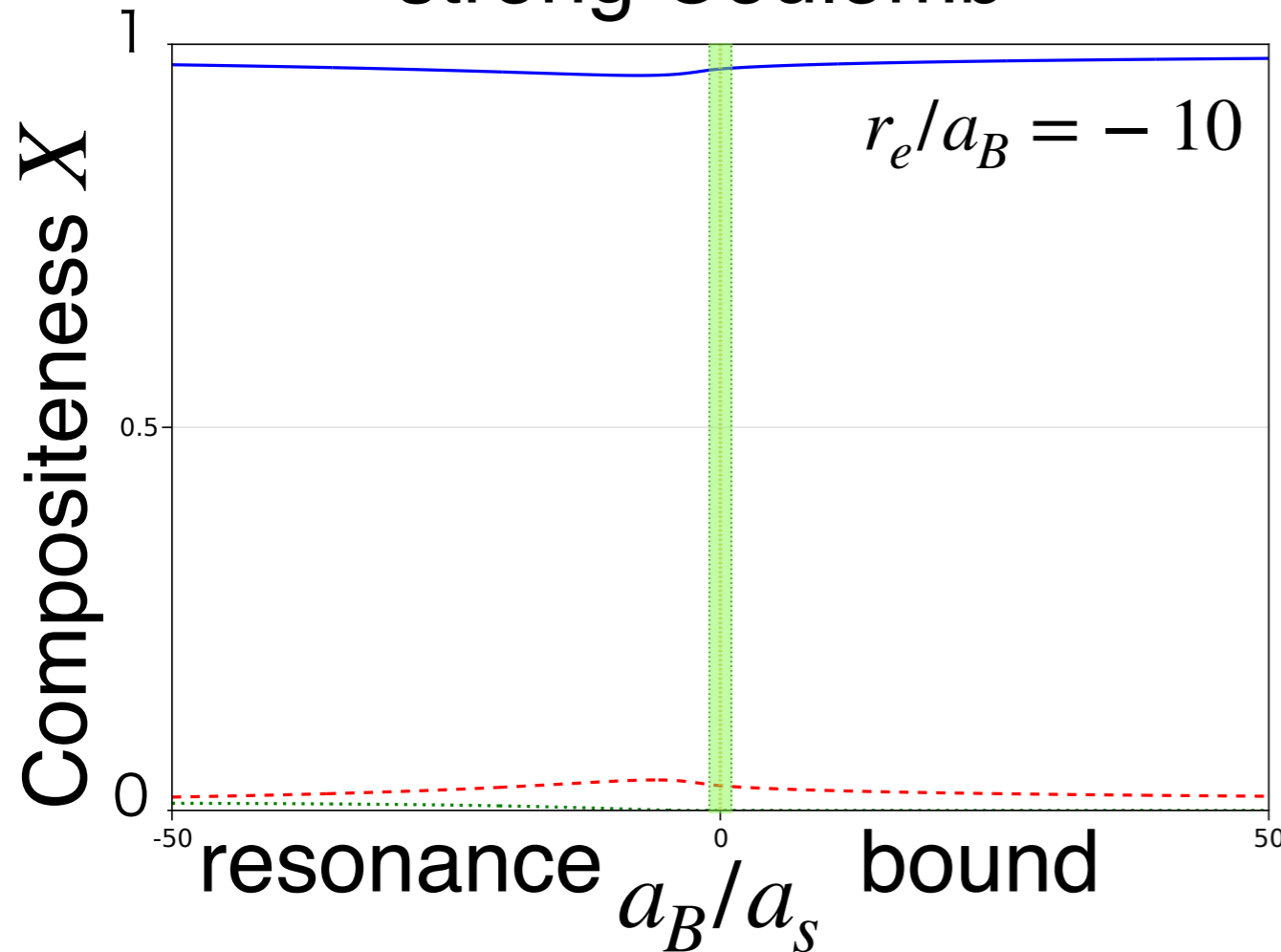


- $\pm 1/a_B$: Coulomb force dominant region
- $\pm 1/|r_e|$: short range universal region
- remnant of short range universality in $|r_e| \ll |a_B|$ case
 $X \rightarrow 1$ in $B \rightarrow 0$ limit in short range

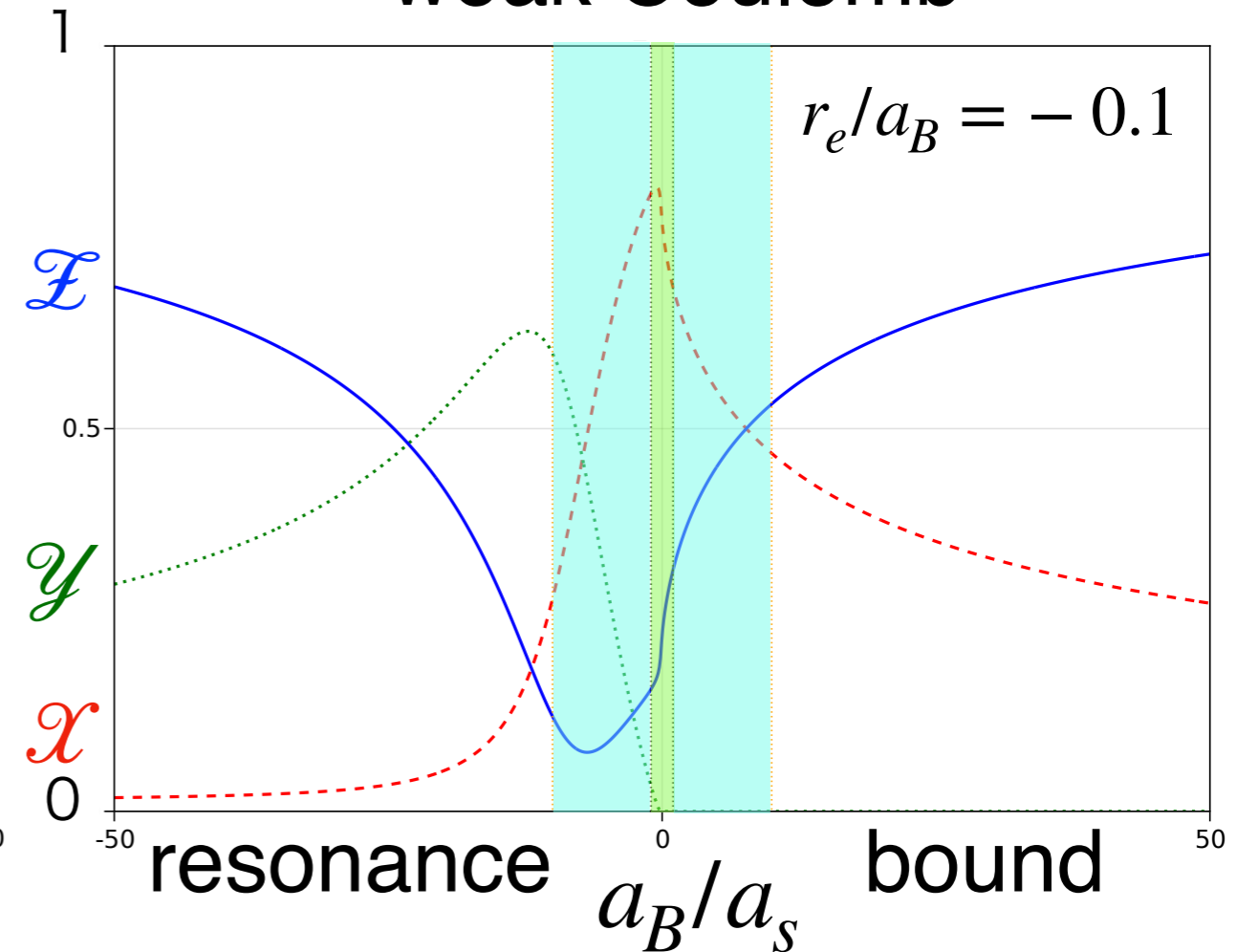
Compositeness (repulsive Coulomb)

22

strong Coulomb



weak Coulomb



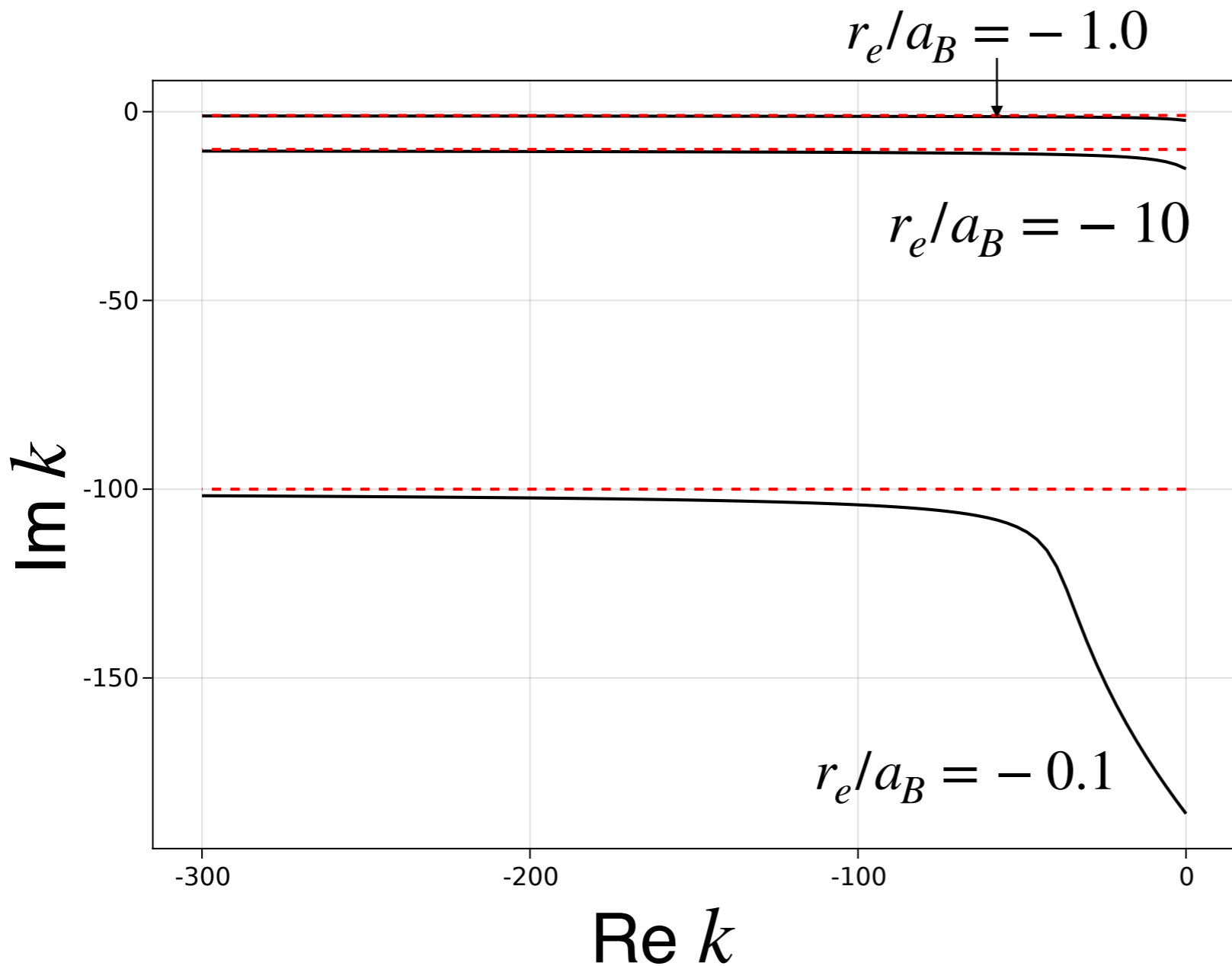
- compositeness of resonances $\leftarrow X, Y, Z$
- all states are interpretable \because no virtual states
- states with large $|1/a_s|$ are elementary Z dominant
- nature of bound states = nature of resonances
 $\because X$ is continuous across threshold

T. Kinugawa and T. Hyodo,
arXiv:2403.12635 [hep-ph].

far from threshold (attractive Coulomb)²³

● imaginary part of eigenenergy in complex momentum k plane

- far from threshold in $1/a_s \rightarrow -\infty$ limit



- Im k of **resonance pole**

- Im $k \rightarrow 1/r_e$

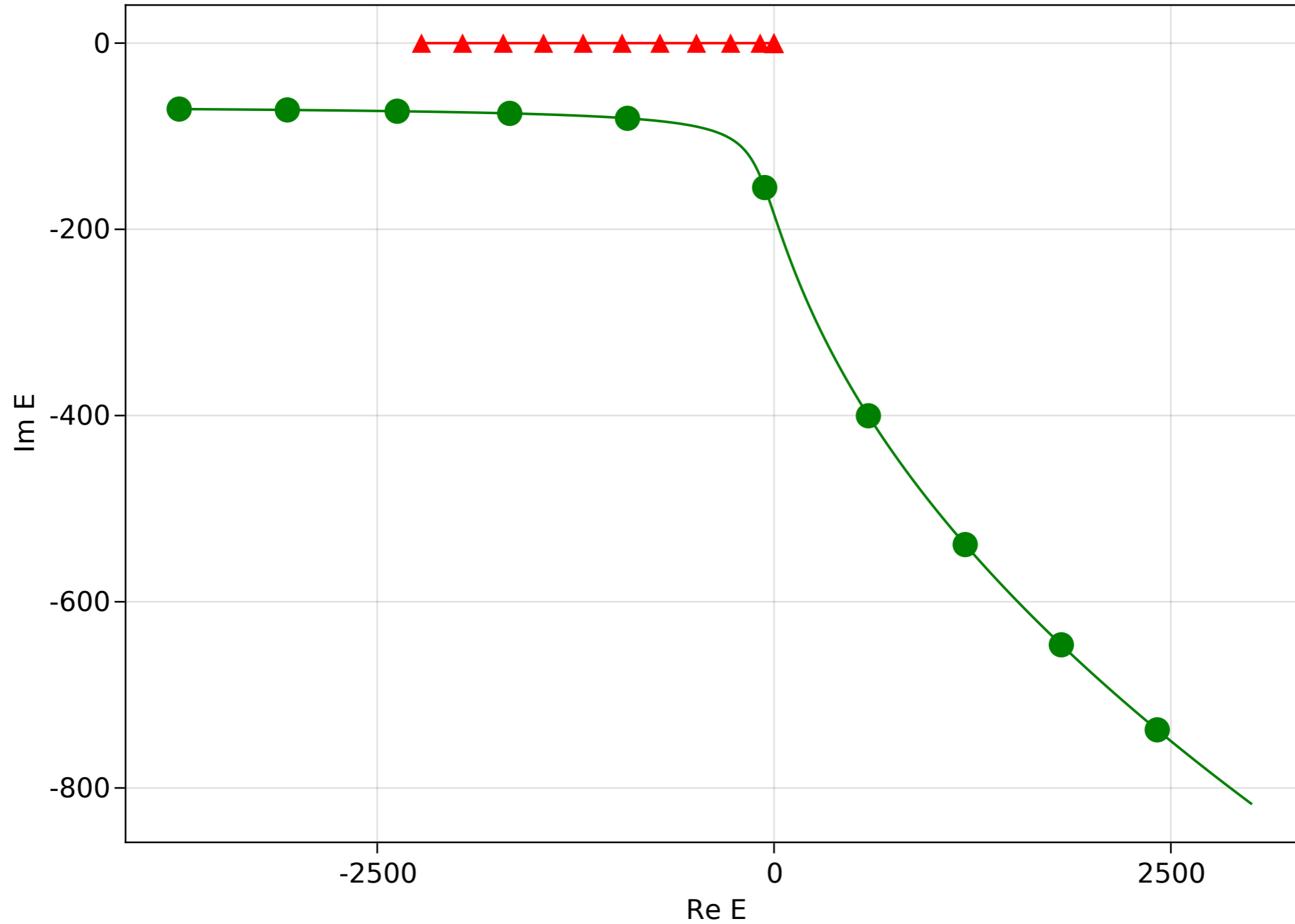
→ trajectory close to that of resonance in ERE

- same as repulsive case

dotted lines : $1/r_e$

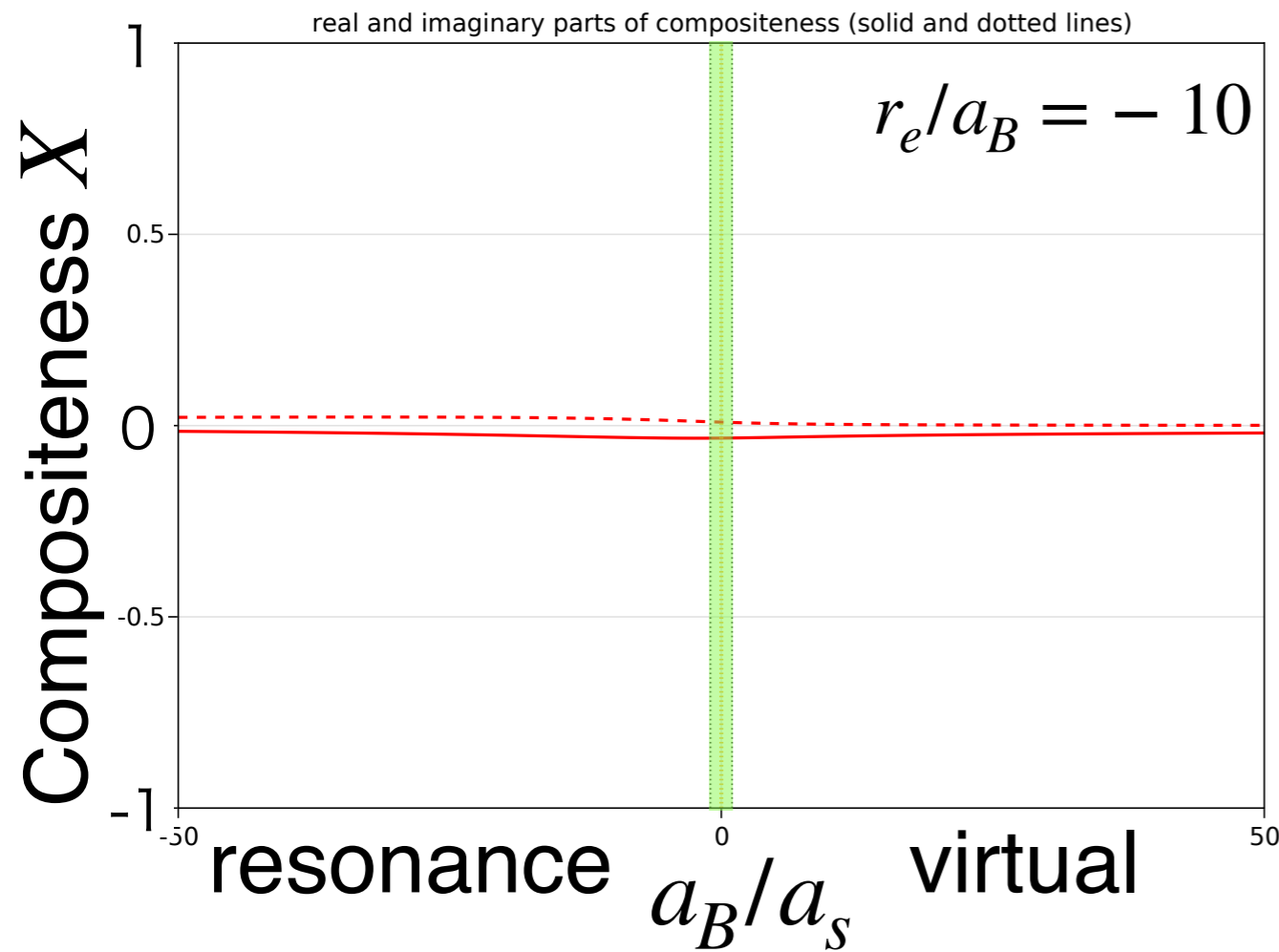
Attractive

pole trajectories in complex E plane

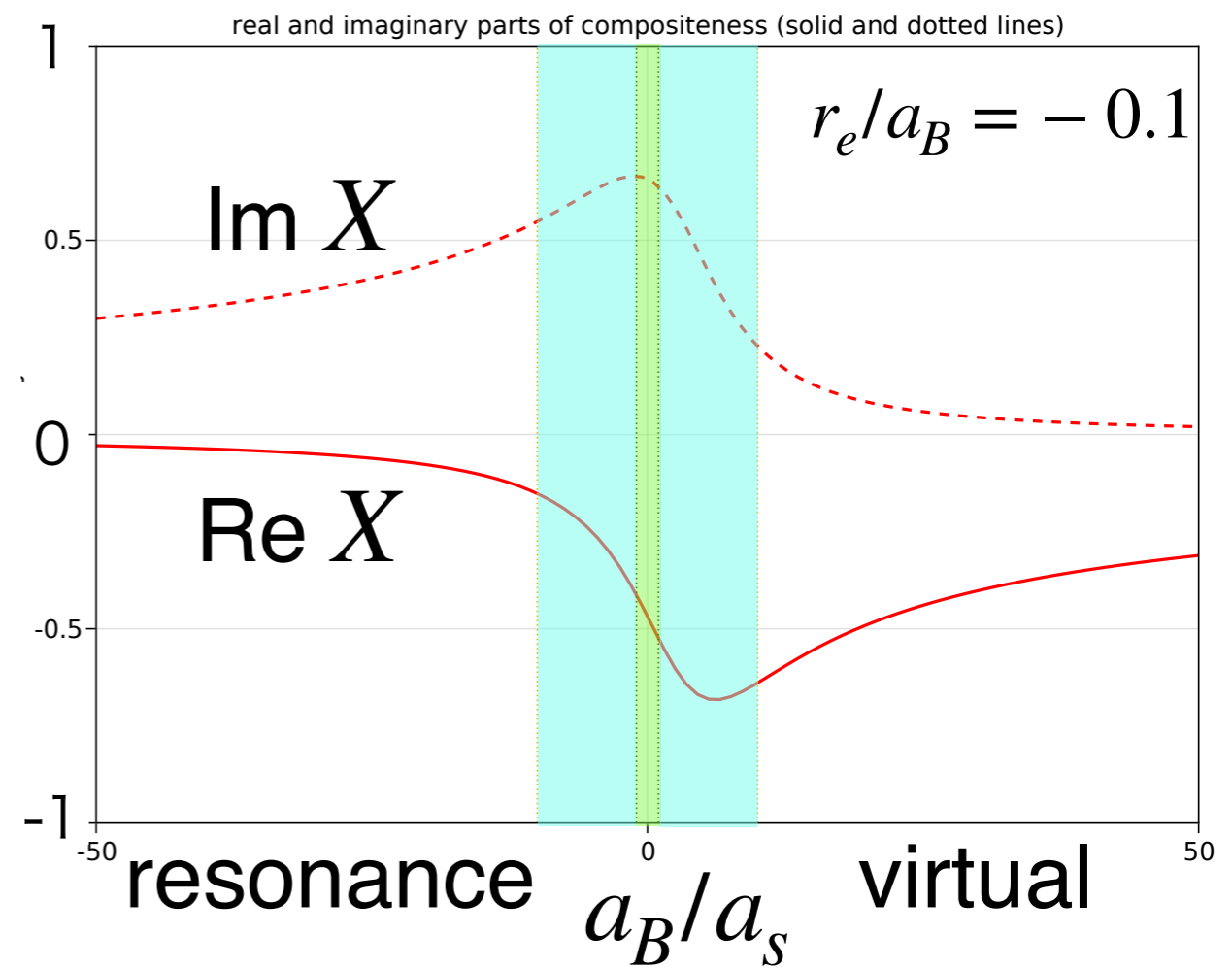


Compositeness (att. Coulomb resonance) 25

strong Coulomb



weak Coulomb



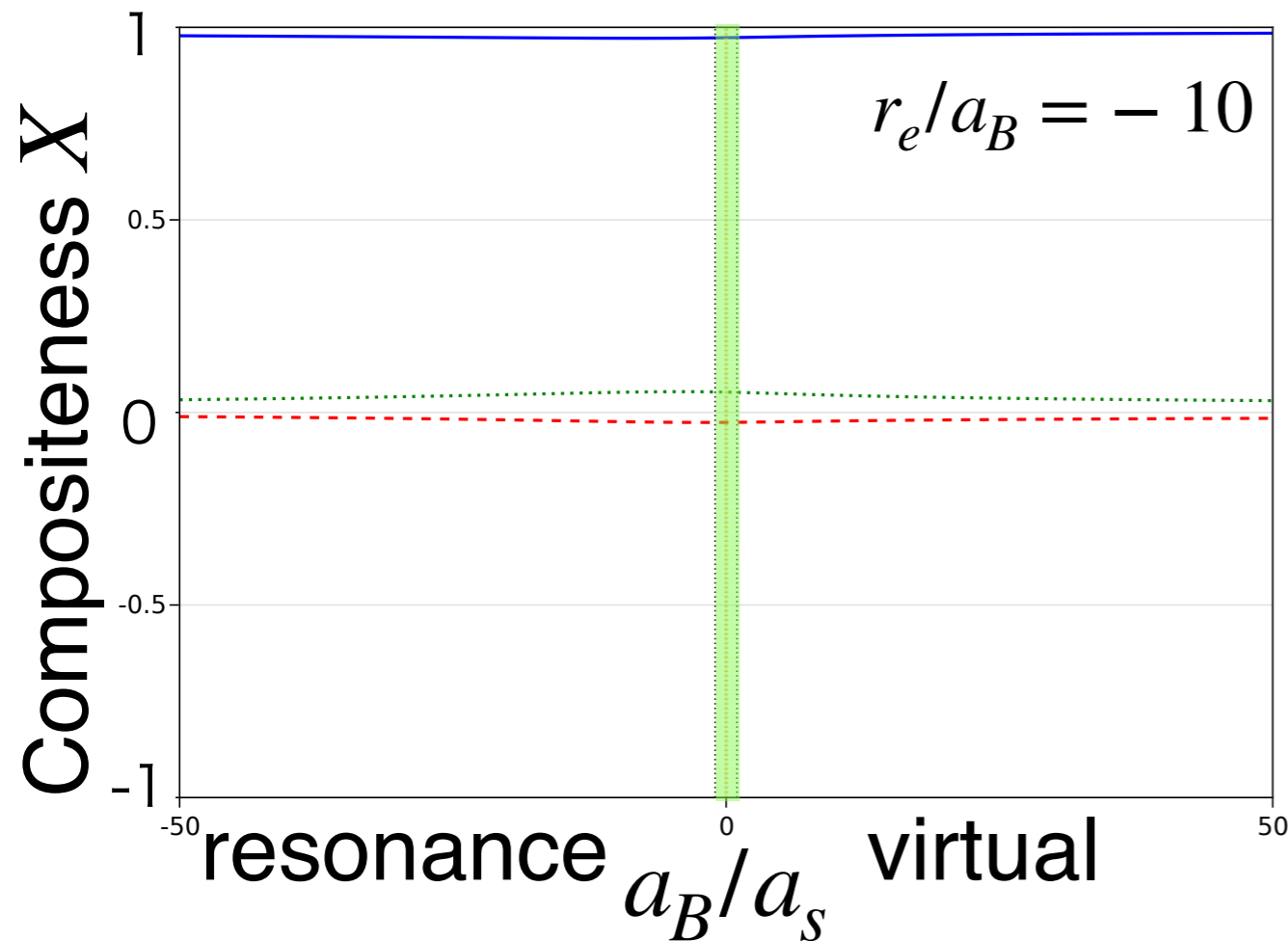
- $\pm 1/a_B$: Coulomb force dominant region

- $\pm 1/|r_e|$: short range universal region

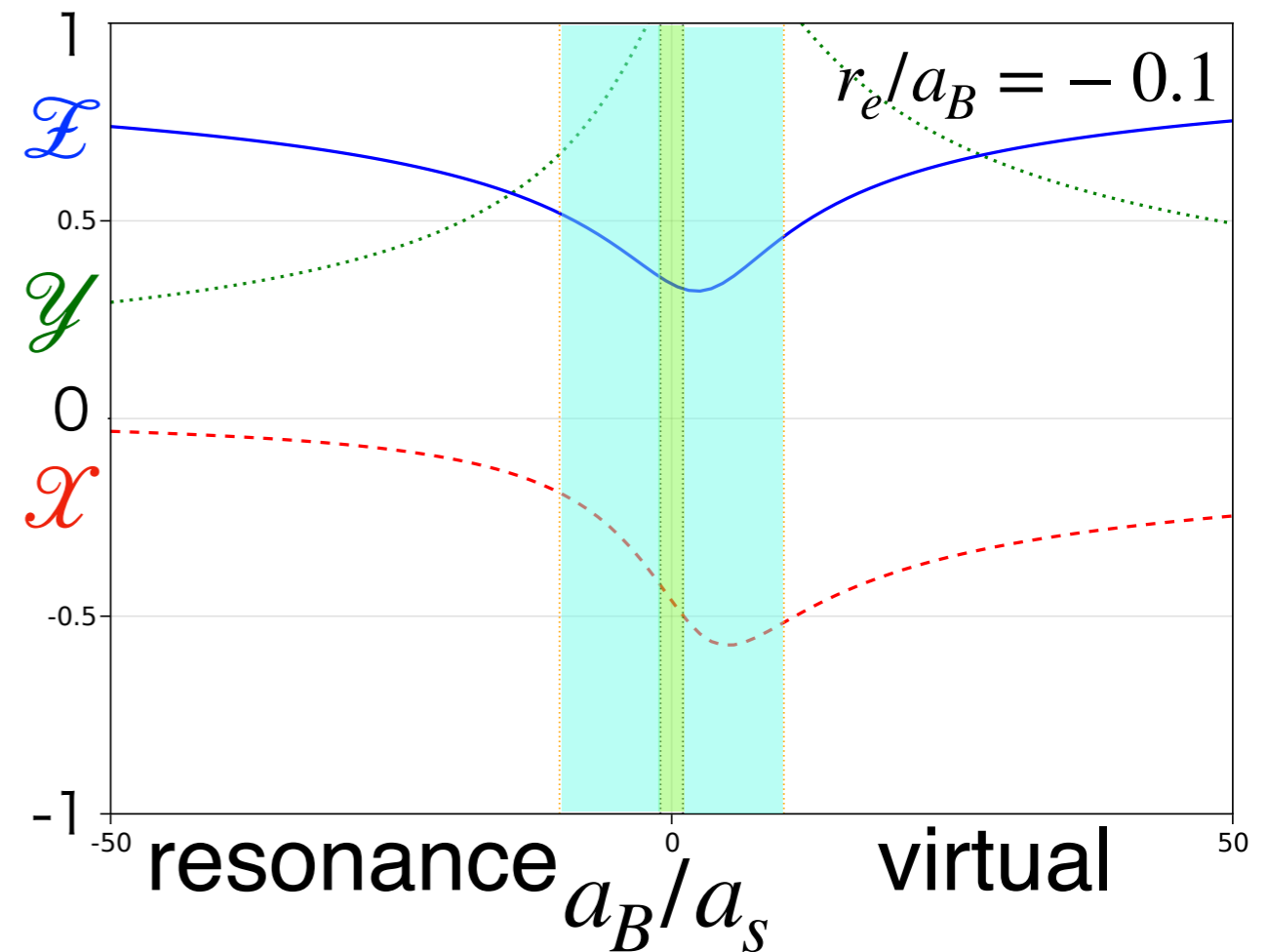
- compositeness of unstable resonances are complex $X \in \mathbb{C}$

Compositeness (att. Coulomb resonance) 26

strong Coulomb



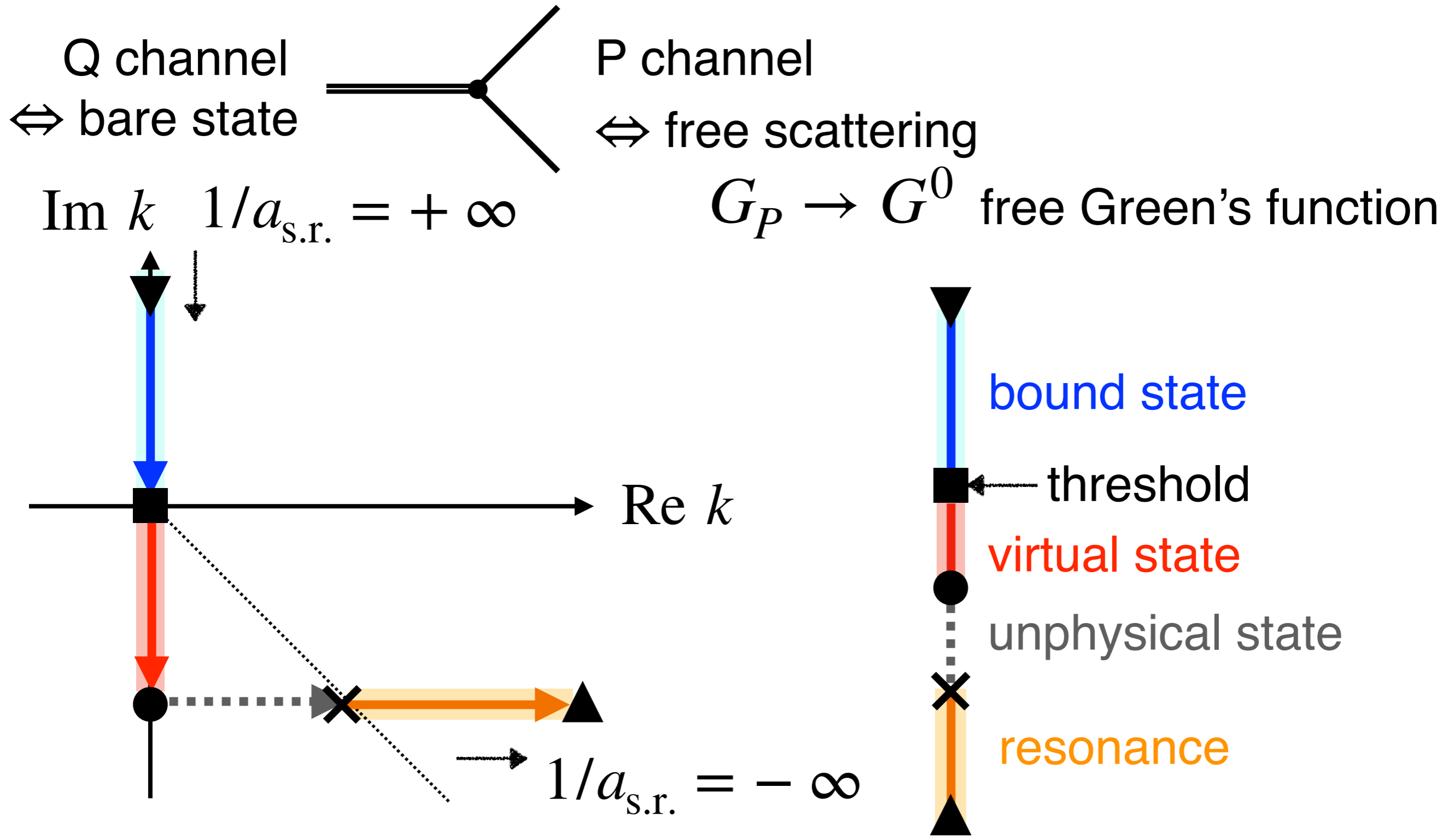
weak Coulomb



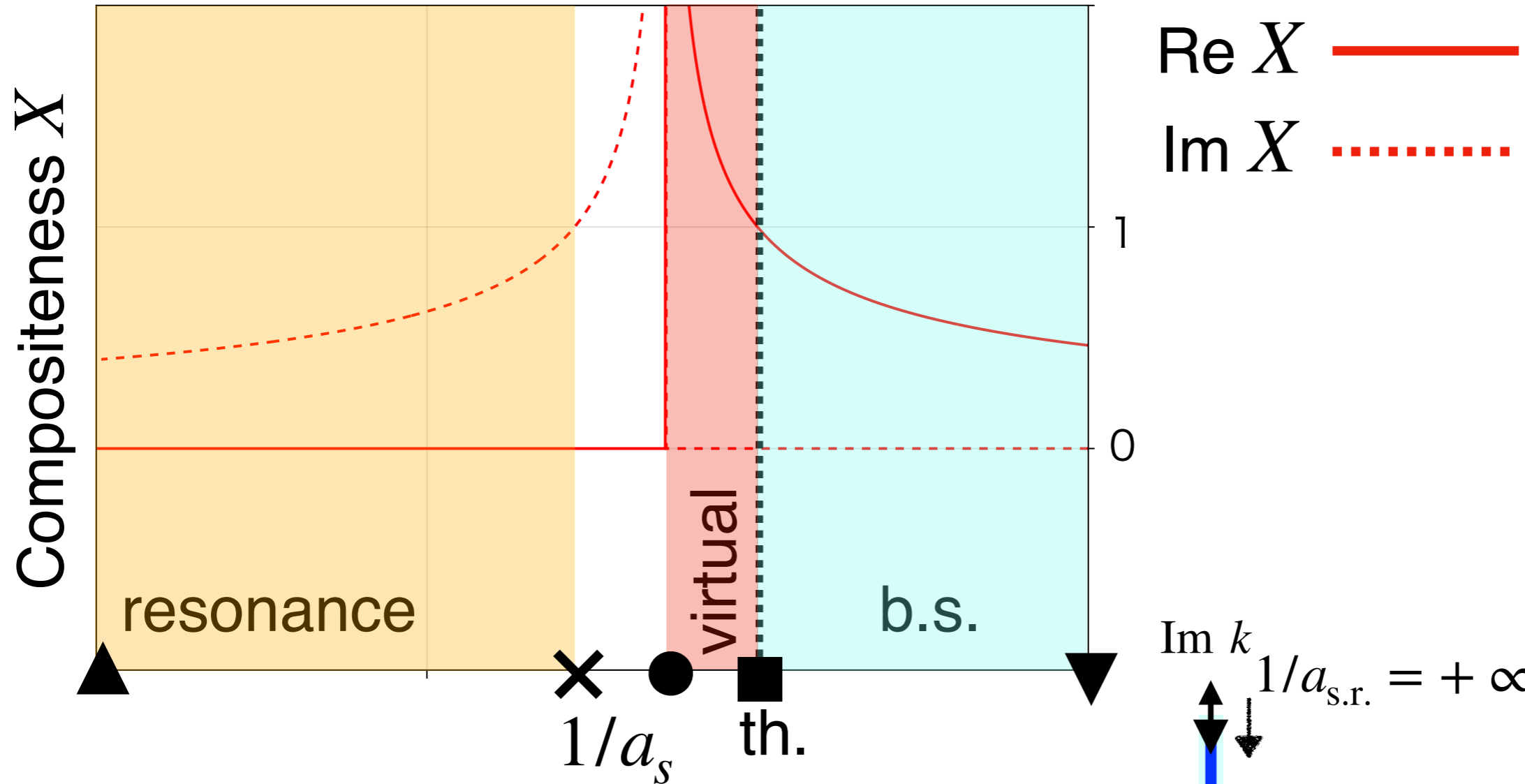
- $\mathcal{X} < 0 \rightarrow$ non-interpretable in this region
- but $\mathcal{X} \geq 0$ in far-threshold region with large $|1/a_s|$
- \rightarrow states are \mathcal{F} dominant with large bare state contribution

Pole trajectory (only w/ s.r.)

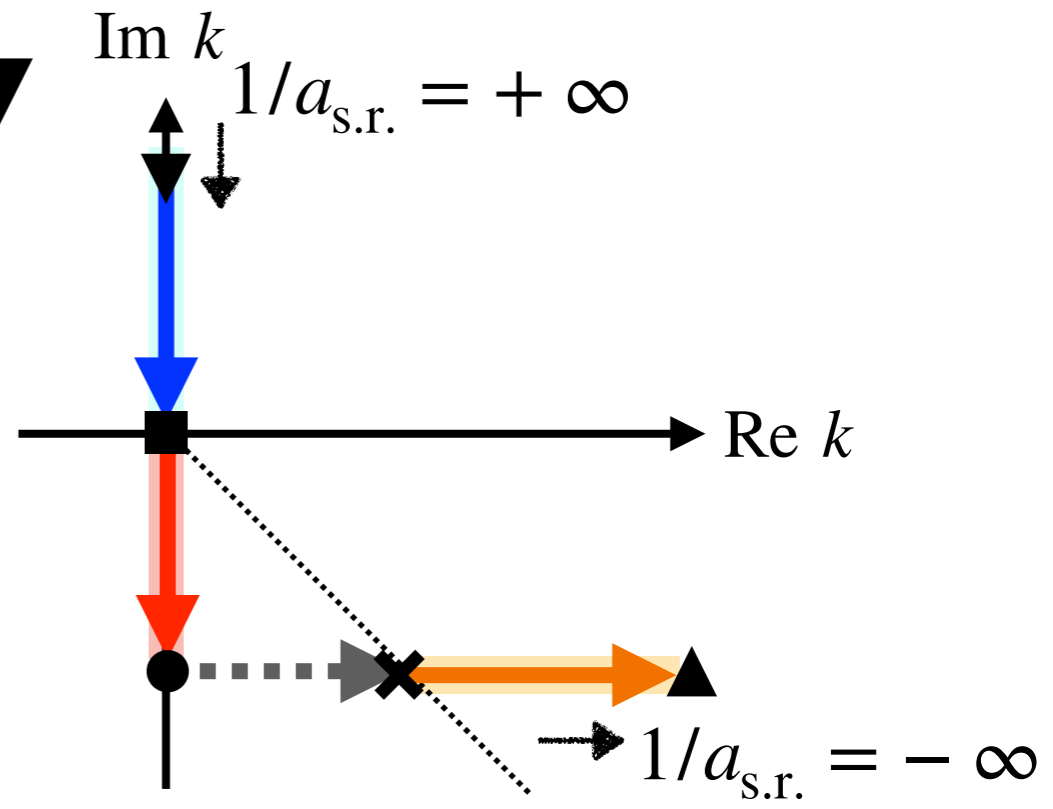
● pole trajectory in complex momentum k plane (No Coulomb)



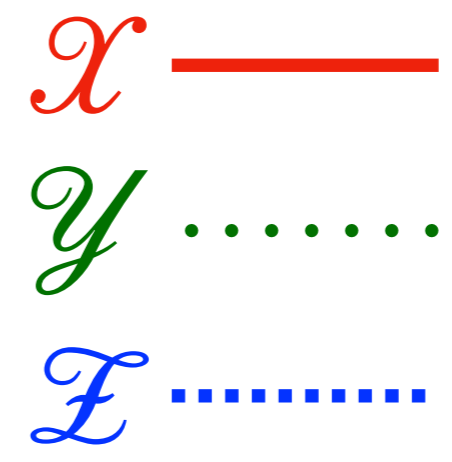
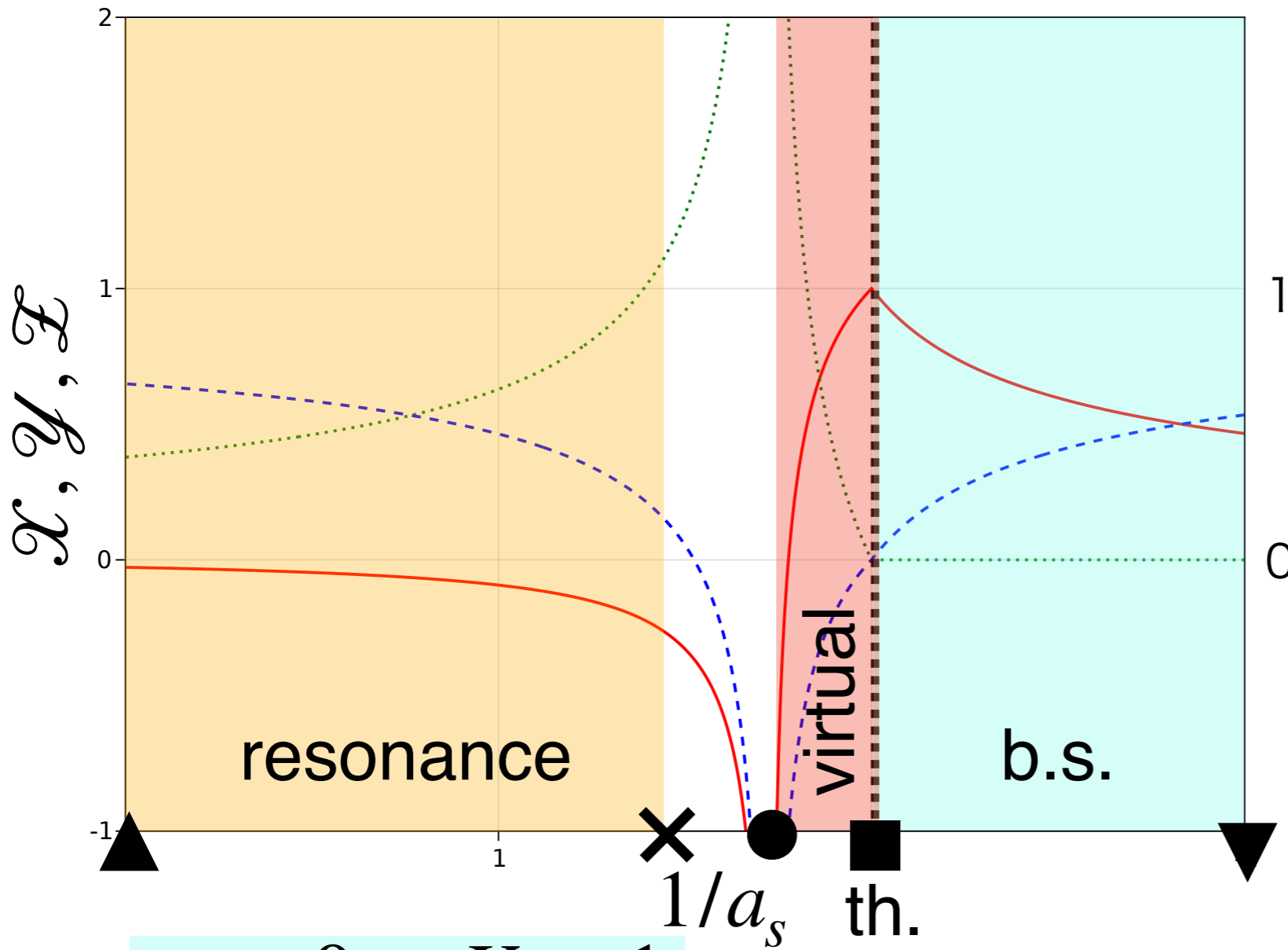
Compositeness (only w/ s.r.)



- b.s. : $0 \leq X \leq 1$
- virtual : $1 < X$
- divergence $X \rightarrow \infty$
- resonance : $\text{Im } X \leq 1$



$\mathcal{X}, \mathcal{Y}, \mathcal{F}$ (only w/ s.r.)

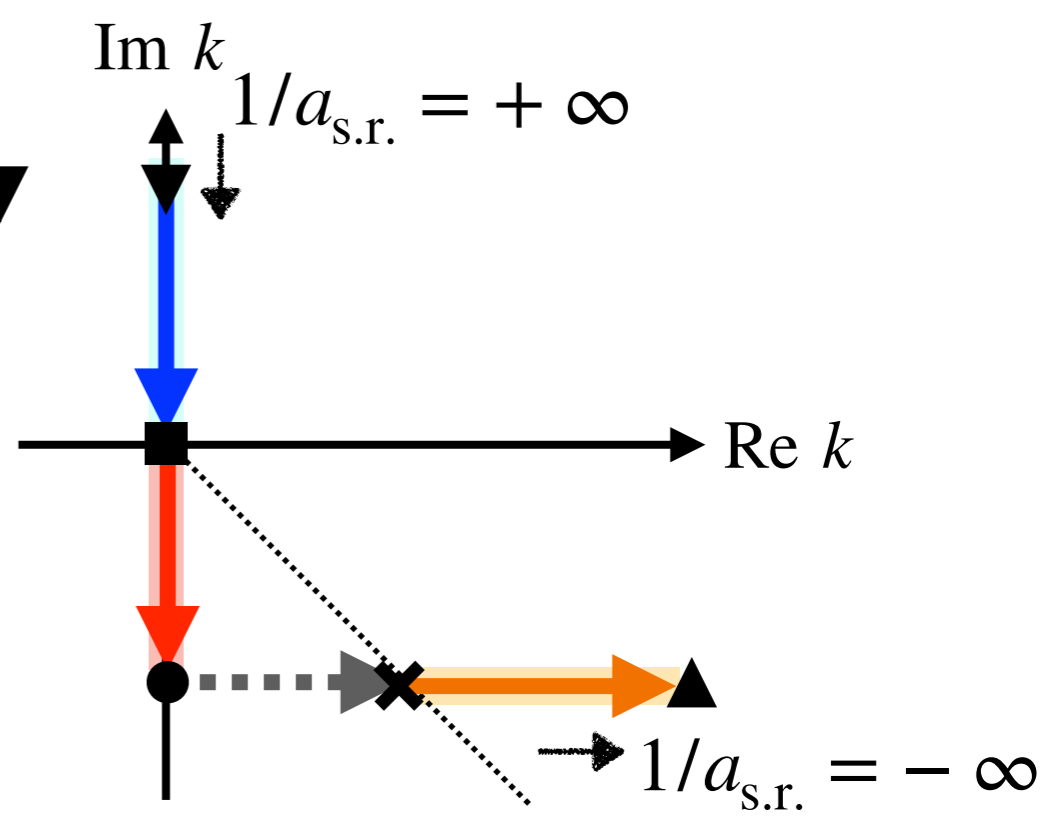


- b.s. : $0 \leq X \leq 1$

- virtual : $\mathcal{F} < 0$ (non-interpretable)

● divergence $\mathcal{X}, \mathcal{Y}, \mathcal{F} \rightarrow \infty$

- resonance : \mathcal{F} dominant



Complex compositeness

- probabilistic interpretation?

$$X \in \mathbb{C} \text{ and } X+Z = 1$$

- If $\text{Im } X$ is large, it seems that reasonable interpretation is impossible $\times \triangle$

- our proposal

i) \mathcal{X} : probability of certainly finding $|\text{composite}\rangle$

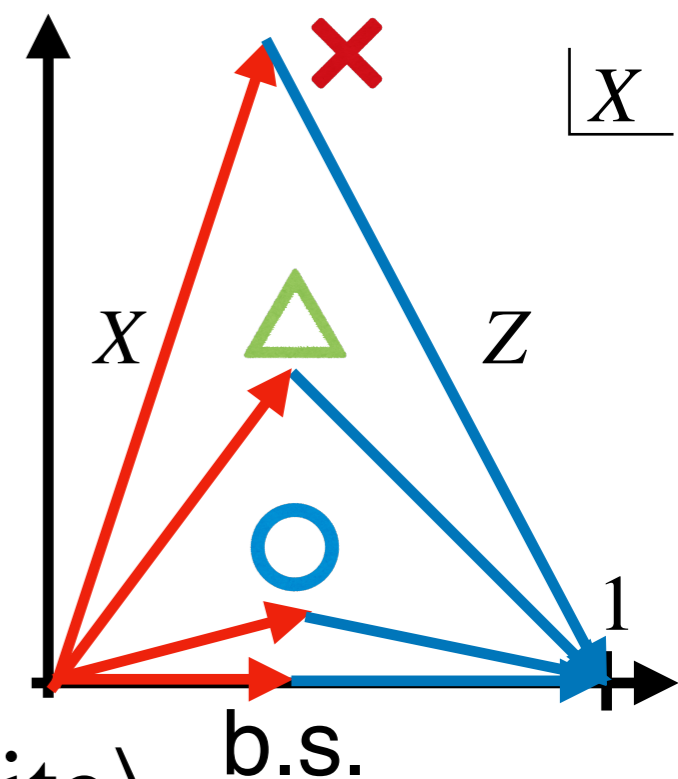
ii) \mathcal{F} : probability of certainly finding $|\text{elementary}\rangle$

iii) \mathcal{Y} : probability of uncertain identification

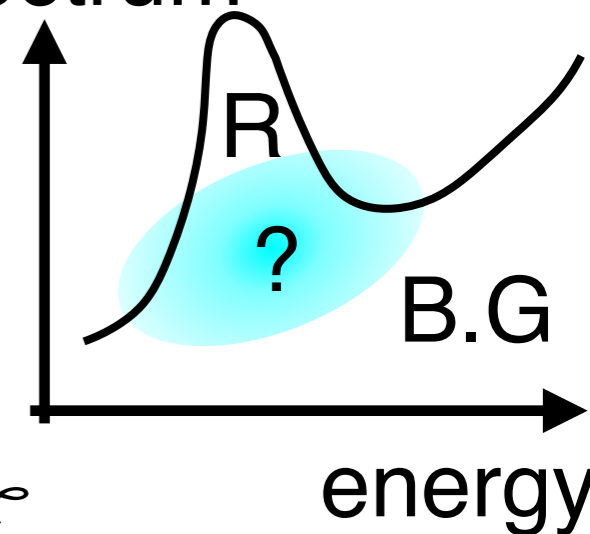
uncertain appears from T. Berggren, Phys. Lett. B 33, 547 (1970).

- finite lifetime (uncertainty in energy)
- separation from B.G.

complex X plane



spectrum



complex compositeness $X \in \mathbb{C} \longrightarrow \mathcal{X}, \mathcal{Y}, \mathcal{F}$

Definition

● conditions for sensible interpretation

- normalization : $\mathcal{X} + \mathcal{Y} + \mathcal{Z} = 1$ for probabilistic interpretation
- in bound state limit : $\mathcal{X} \rightarrow X, \mathcal{Z} \rightarrow Z$ and $\mathcal{Y} \rightarrow 0$

\mathcal{Y} characterizes uncertainty of resonance

● new interpretation

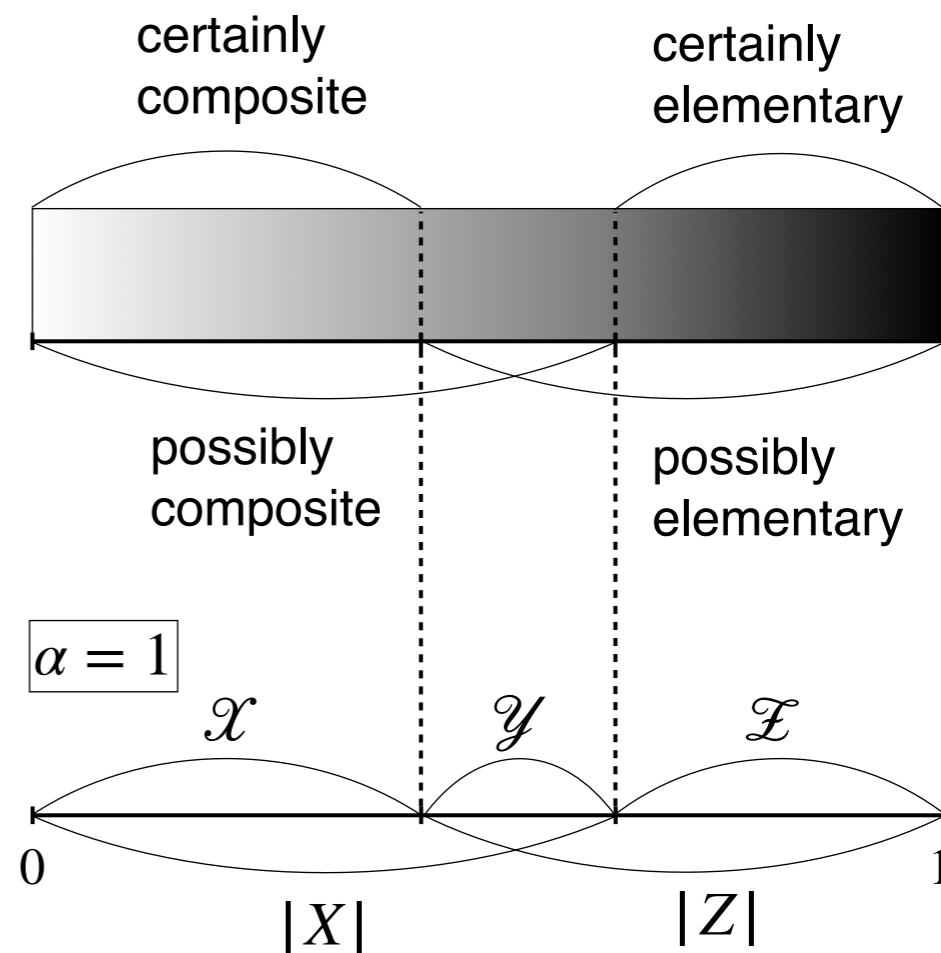
$$\mathcal{X} + \alpha \mathcal{Y} = |X|, \quad \mathcal{Z} + \alpha \mathcal{Y} = |Z|$$

$$\mathcal{X} = \frac{(\alpha - 1)|X| - \alpha|Z| + \alpha}{2\alpha - 1}$$

$$\mathcal{Z} = \frac{(\alpha - 1)|Z| - \alpha|X| + \alpha}{2\alpha - 1}$$

$$\mathcal{Y} = \frac{|X| + |Z| - 1}{2\alpha - 1}$$

α reflects uncertain nature of resonances



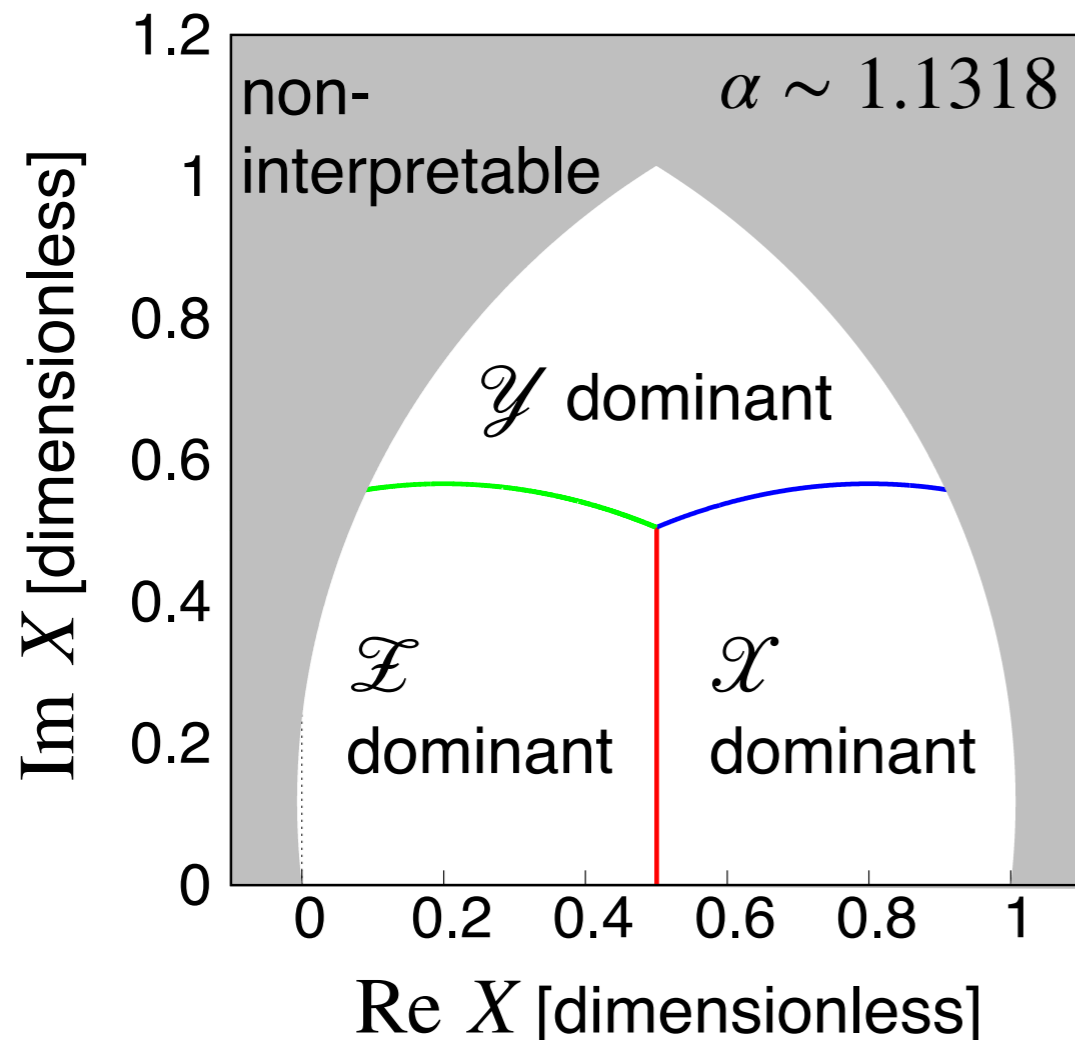
Definition

- if $\alpha > 1/2$, \mathcal{Y} is always positive but \mathcal{X}, \mathcal{Z} can be negative

	$\mathcal{X} > \mathcal{Y}, \mathcal{Z}$	composite dominant
$\mathcal{X} \geq 0$ and $\mathcal{Z} \geq 0$	$\mathcal{Z} > \mathcal{Y}, \mathcal{X}$	elementary dominant
	$\mathcal{Y} > \mathcal{X}, \mathcal{Z}$	uncertain

$\mathcal{X} < 0$ or $\mathcal{Z} < 0$

non-interpretable



our criterion for physical “state”

$$\Gamma \leq \text{Re } E$$

- exclude poles which we cannot regard as physical “state” from probabilistic interpretation

$\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ dominant regions
and
non-interpretable region

uncertainty in resonances

a single
measurement



sum of measurements of a bound states / resonances

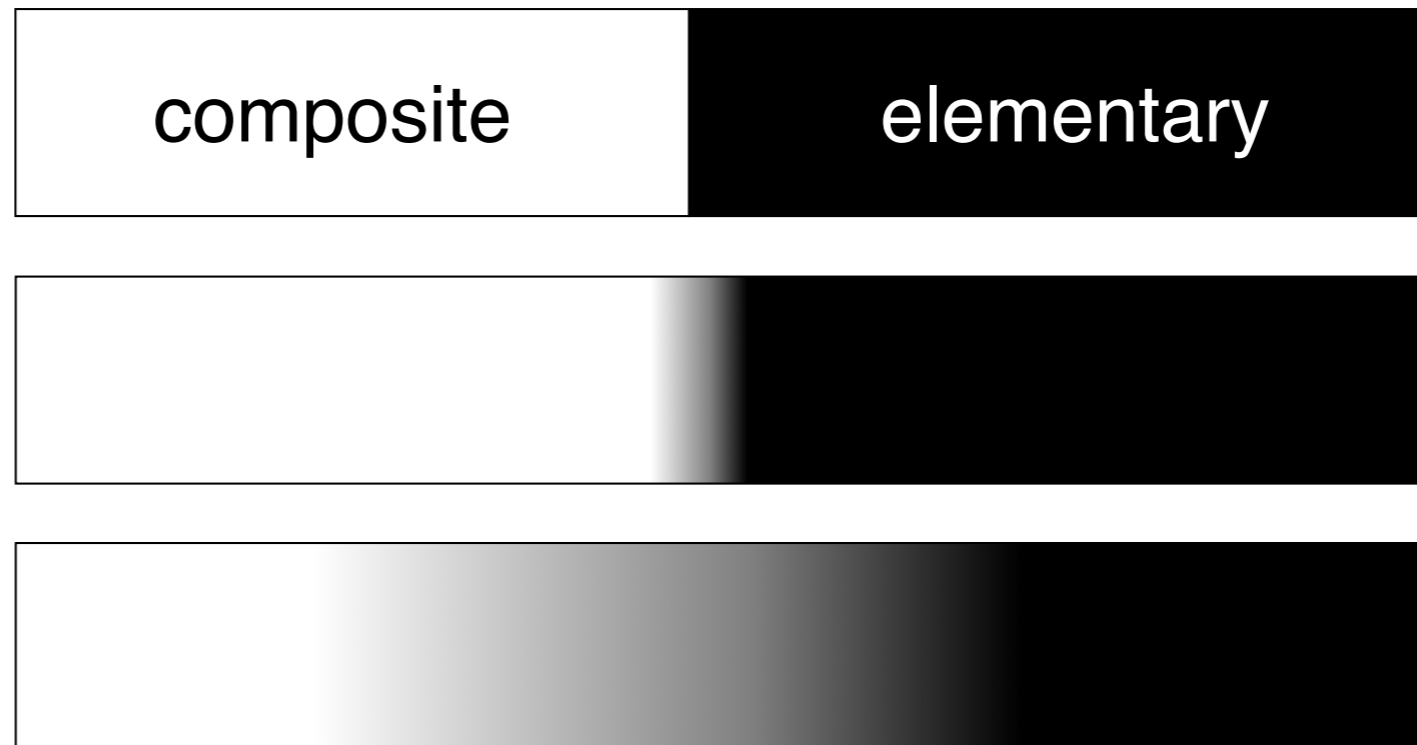
bound state

composite

elementary

narrow
resonance

broad
resonance



measurements