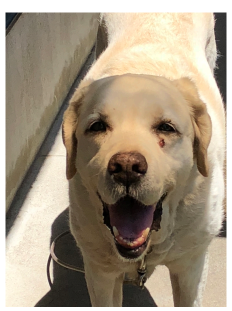
Weak-binding relation in the zero range limit



<u>Tomona Kinugawa</u> **Tetsuo Hyodo**

Department of Physics, Tokyo Metropolitan University

School for "Clustering as a window on the hierarchical structure of quantum systems"
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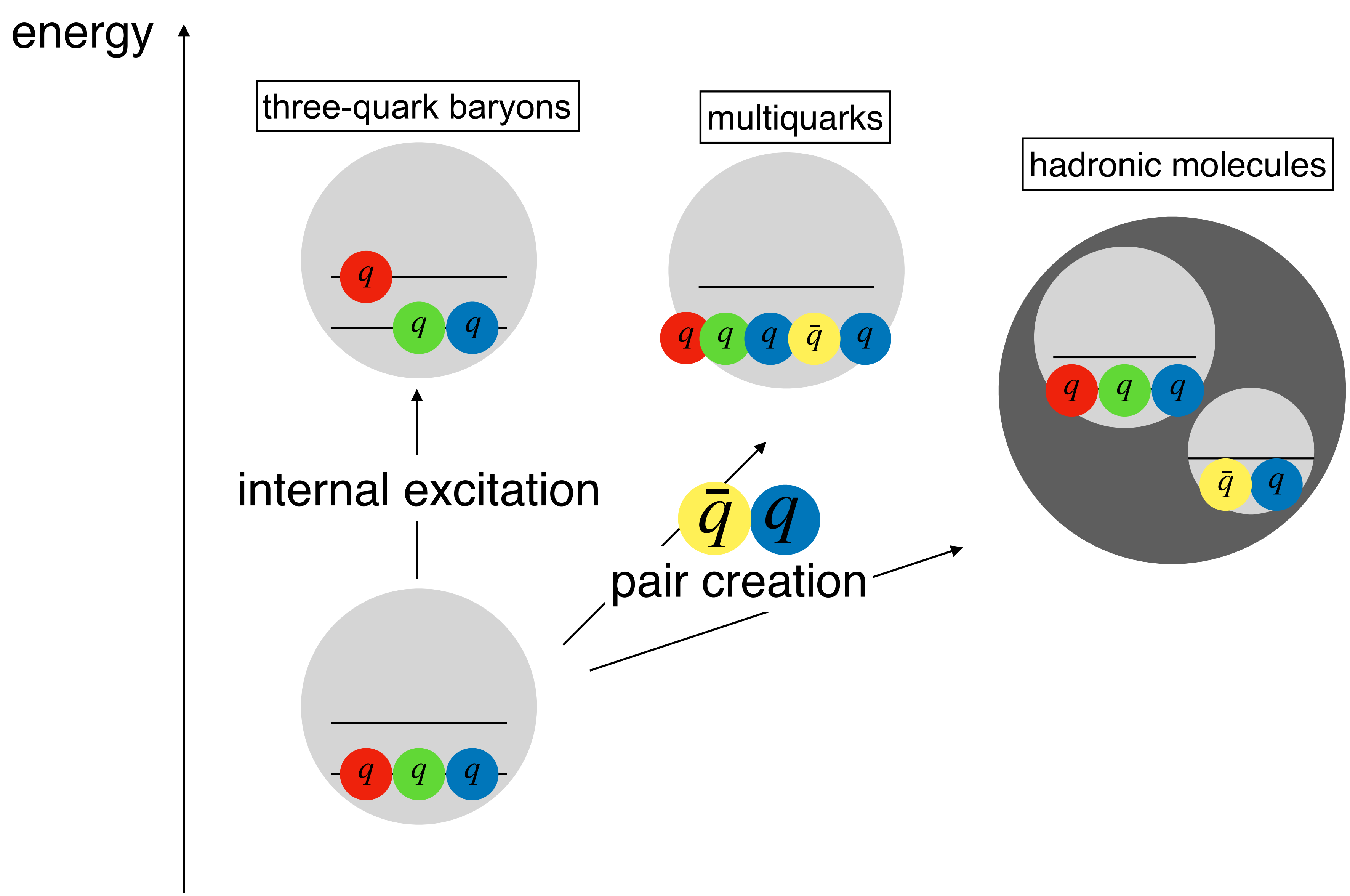
Background

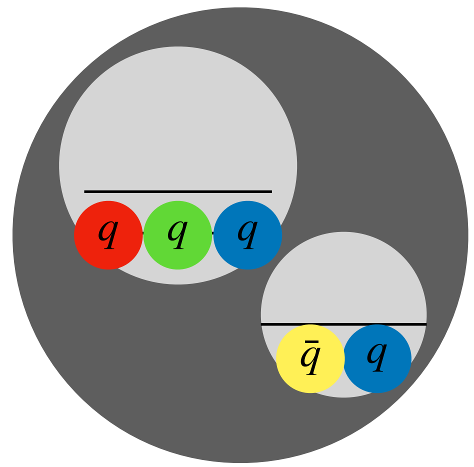
candidates for exotic hadrons
$\Lambda(1405)$, XYZ meson etc… → multiquarks
hadronic molecules

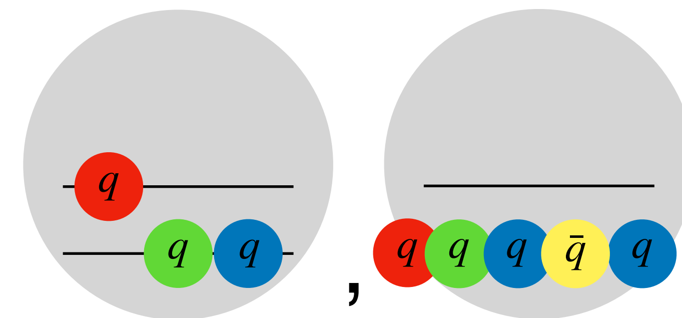


structure of hadrons ← model independent observable

Previous work

Hadron wave function





$$|\Psi\rangle = \sqrt{X}\,|\text{hadronic molecule}\rangle + \sqrt{1-X}\,|\text{others}\rangle$$

Compositeness (weight of hadronic molecule)

Weak-binding relation

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\mathrm{typ}}}{R}\right)\right\}$$

a_0 (scattering length)

$R \equiv (2\mu B)^{-1/2}$, B (binding energy)

R_{typ}(interaction range)

When $R \gg R_{\mathrm{typ}}$: observable(a_0, B) $\longrightarrow$ compositeness(X)

S. Weinberg, Phys. Rev. 137, B672 (1965); Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

Motivation

Weak-binding relation $a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\}$

Low-energy universality → $a_0 = R \ (R \to \infty)$

-Deviation by contributions from other channels ← $X \neq 1$

-Deviation by interaction range ← $R_{\text{typ}} \neq 0$

→ We study the range correction in the weak-binding relation by introducing the effective range r_e.

Effective range model

E. Braaten, M. Kusunoki, and D. Zhang, Annals Phys. 323, 1770 (2008), 0709.0499.

Single channel scattering of identical bosons with mass m :

$$\boxed{\mathscr{H}_{\text{int}} = \frac{1}{4}\lambda_0(\psi^\dagger\psi)^2 + \frac{1}{4}\rho_0 \nabla(\psi^\dagger\psi)\cdot\nabla(\psi^\dagger\psi)}$$

Off-shell T-matrix:

$$T(E,k,k') = T_1(E) + T_2(E)(k^2 + k'^2) + T_3(E)k^2k'^2,$$

$$\begin{pmatrix} T_1 & T_2 \\ T_2 & T_3 \end{pmatrix} = -i\begin{pmatrix} \lambda_0 & \rho_0 \\ \rho_0 & 0 \end{pmatrix} - i\begin{pmatrix} \lambda_0 & \rho_0 \\ \rho_0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{i}{E-q^2/m+i0^+} & \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{iq^2}{E-q^2/m+i0^+} \\ \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{iq^2}{E-q^2/m+i0^+} & \frac{1}{2}\int\frac{d^3q}{(2\pi)^2}\frac{iq^4}{E-q^2/m+i0^+} \end{pmatrix}\begin{pmatrix} T_1 & T_2 \\ T_2 & T_3 \end{pmatrix}.$$

cut off at Λ

Typical range $R_{\text{typ}} \sim 1/\Lambda$

On-shell scattering amplitude:

$$f(k) = \left[-\frac{8\pi}{m}\frac{\left(1+\frac{m}{12\pi^2}\Lambda^3\rho_0\right)^2}{N(k)} - \frac{2}{\pi}\Lambda - ik\right]^{-1},\ N(k) = \left[\lambda_0 - \frac{m}{20\pi^2}\Lambda^5\rho_0^2\right] + 2\rho_0\left(\frac{m}{24\pi^2}\Lambda^3\rho_0 + 1\right)k^2.$$

Effective range model

E. Braaten, M. Kusunoki, and D. Zhang, Annals Phys. 323, 1770 (2008), 0709.0499.

We obtain the scattering length a_0 and effective range r_e from low-energy behavior of $f(k; \lambda_0, \rho_0, \Lambda)$.

$$\rightarrow \quad a_0 = a_0(\lambda_0, \rho_0, \Lambda), \; r_e = r_e(\lambda_0, \rho_0, \Lambda)\,.$$

a_0 and r_e are the functions of the bare parameters and Λ.

Renormalization:

the bare parameters λ_0, ρ_0 are adjusted as functions of Λ so that a_0 and r_e are independent of Λ.

$$\rightarrow \quad f(k) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 + \mathcal{O}\left(\frac{1}{\Lambda}\right) - ik\right]^{-1}$$

$$\to \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - ik\right]^{-1} \quad (\Lambda \to \infty)$$

Zero range limit

Effective range model

Properties of the effective range model:

-Single channel: $|\text{hadronic molecule}\rangle$ only $\Leftrightarrow X = 1$

-Zero range limit: $\Lambda \to \infty \Leftrightarrow R_{\text{typ}} = 1/\Lambda \to 0$

$$\Rightarrow a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right\} \to R \ ?$$

Renormalized scattering amplitude ($\Lambda \to \infty$):

$$1/f(k = i/R) = 0$$

$$a_0 = R\frac{2r_e/R}{1-(r_e/R-1)^2} = R\left[1 + \mathcal{O}\left(\left|\frac{r_e}{R}\right|\right)\right] \Rightarrow a_0 \neq R$$

→ range correction in the weak-binding relation form r_e

Improved weak-binding relation

Weak-binding relation $a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$

interaction range: $R_{\rm typ}$ $\rightarrow$ $R_{\rm int} \sim 1/\Lambda$

Redefinition of $R_{\rm typ}$:

$$R_{\rm typ} = \max\left\{R_{\rm int}, R_{\rm eff}\right\},$$

$$R_{\rm eff} = \max\left\{|r_e|, \frac{|P_s|}{R^2}, \cdots\right\}.$$

$$f(k) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 - \frac{P_s}{4}k^4 + \cdots - ik\right]^{-1}$$

Length scale in the effective range expansion except for a_0

Numerical calculation

Weak-binding relation $a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right)\right\}$

Estimation with correction terms ($\xi \equiv R_{\rm typ}/R$): Y. Kamiya and T. Hyodo, PTEP 2017, 023D02 (2017).

Central value: $X_c = \dfrac{a_0/R}{2 - a_0/R}$

$$X_{\rm upper}(\xi) = \frac{a_0/R + \xi}{2 - a_0/R - \xi},\ X_{\rm lower}(\xi) = \frac{a_0/R - \xi}{2 - a_0/R + \xi}.$$

Weak-binding relation works when…

$X_{\rm lower} < X_{\rm exact} < X_{\rm upper}$

→ Validity condition

$(X_{\rm upper} - X_c)/X_c < 0.1$ and $(X_c - X_{\rm lower})/X_c < 0.1$

→ Precision condition

Numerical calculation

Effective range model ($\Lambda < \infty$)

$$f(k;\lambda_0,\rho_0,\Lambda) = \left[-\frac{1}{a_0} + \frac{r_e}{2}k^2 + \mathcal{O}\left(R_{\rm int}\right) - ik\right]^{-1}$$ (two length scales r_e and $R_{\rm int}$)

$$1/f(k = i/R) = 0$$

- $r_e \neq 0$ (range correction): $\xi_{r_e} = |r_e/R|$ Uncertainty from r_e

  $r_e < 0$ (effective range model)

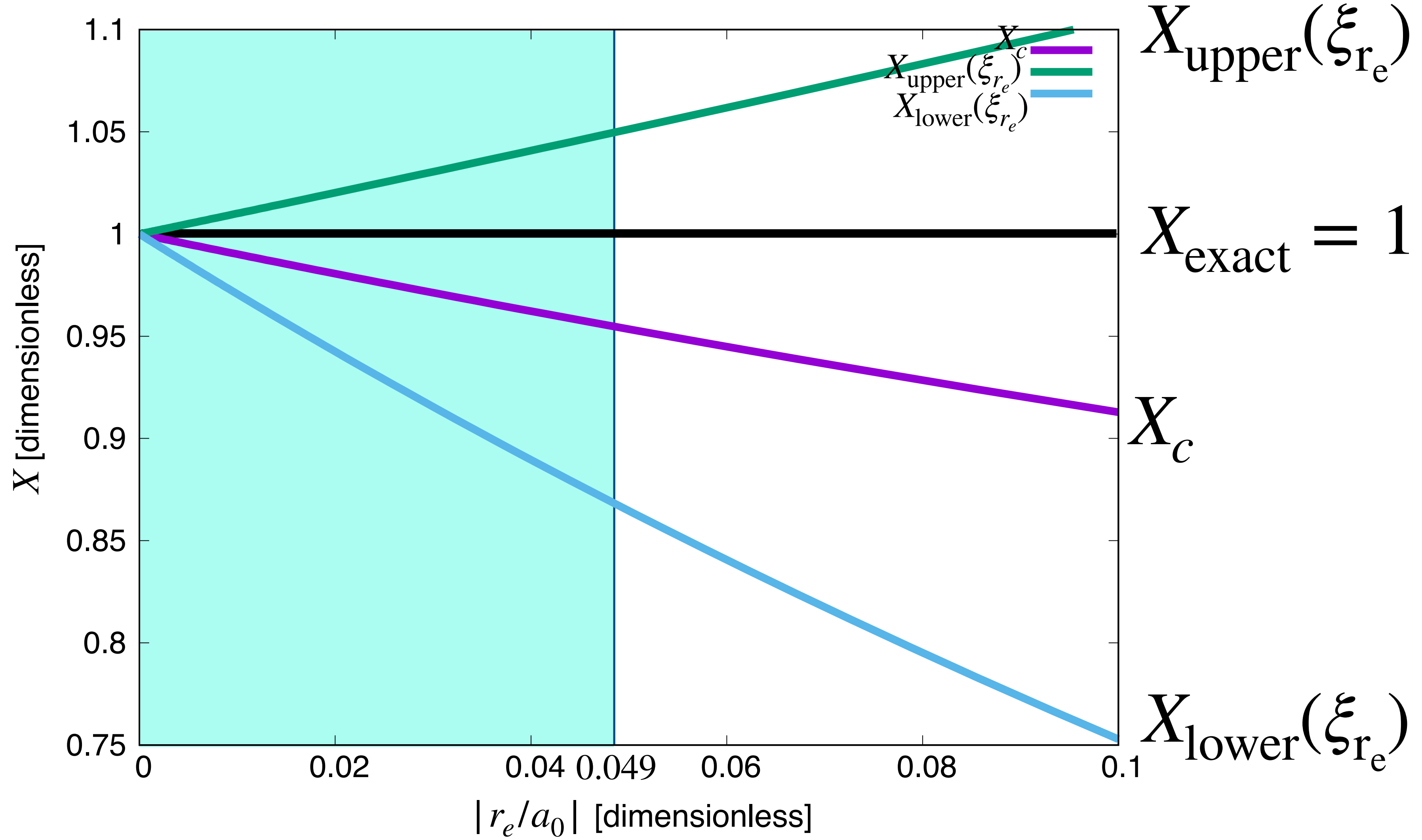
- $R_{\rm int} = 1/\Lambda \neq 0$: $\xi_{\rm int} = R_{\rm int}/R$. Uncertainty from $R_{\rm int}$

- $X_{\rm exact} = 1$

We search for the regions of r_e and $R_{\rm int}$ in which validity and precision conditions are satisfied.

Numerical calculation

Estimated X_c and uncertainty from r_e ($R_{\rm int} = a_0/1000$)



-$X_{\rm exact} = 1$ is always contained.

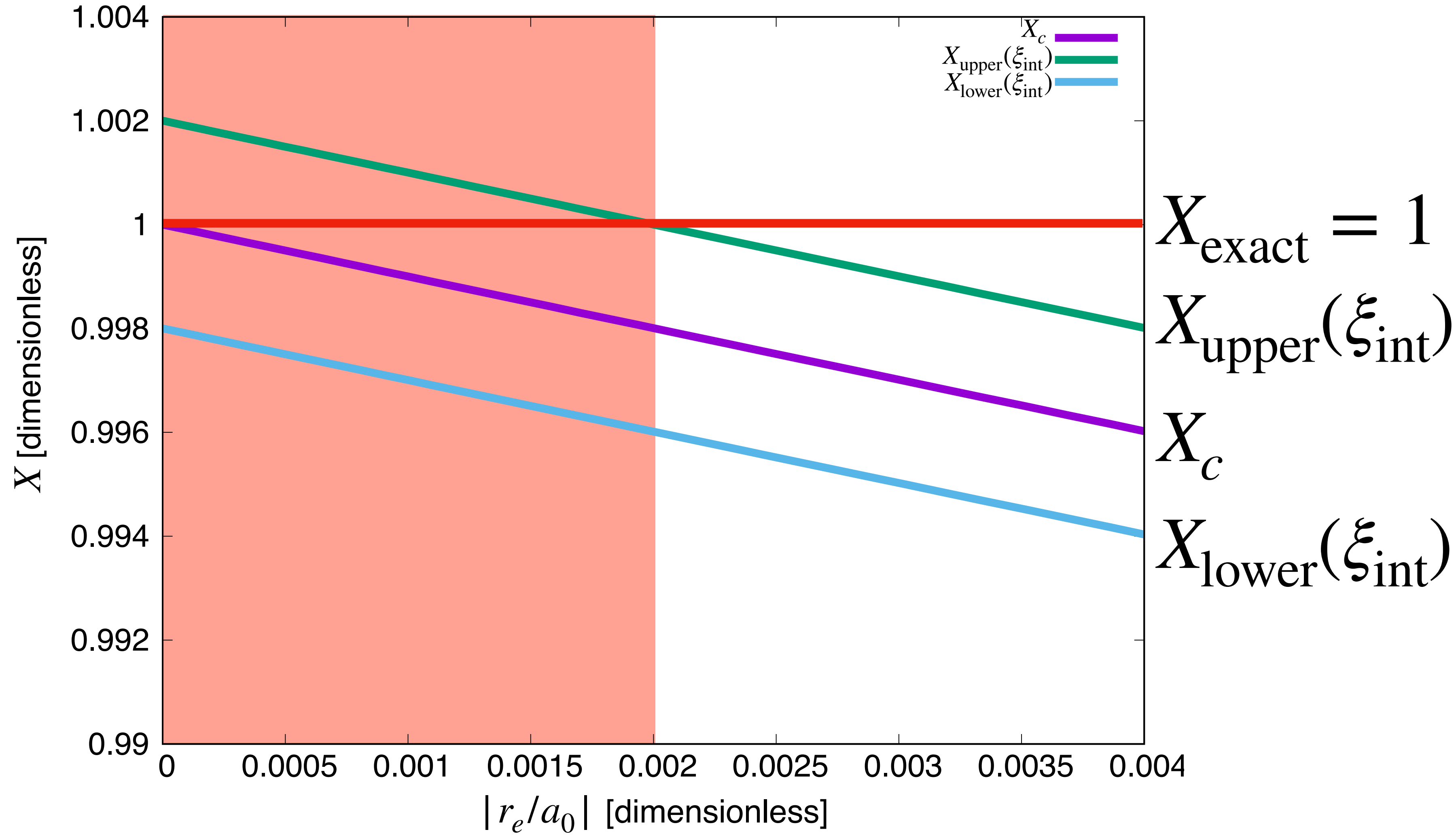
→ The validity condition is always satisfied.

-The uncertainty is smaller than 10 % when $|r_e| \lesssim 0.049 a_0$.

→ The precision condition is satisfied in this region.

Numerical calculation

Estimated X_c and uncertainty from R_{int} ($R_{\text{int}} = a_0/1000$)



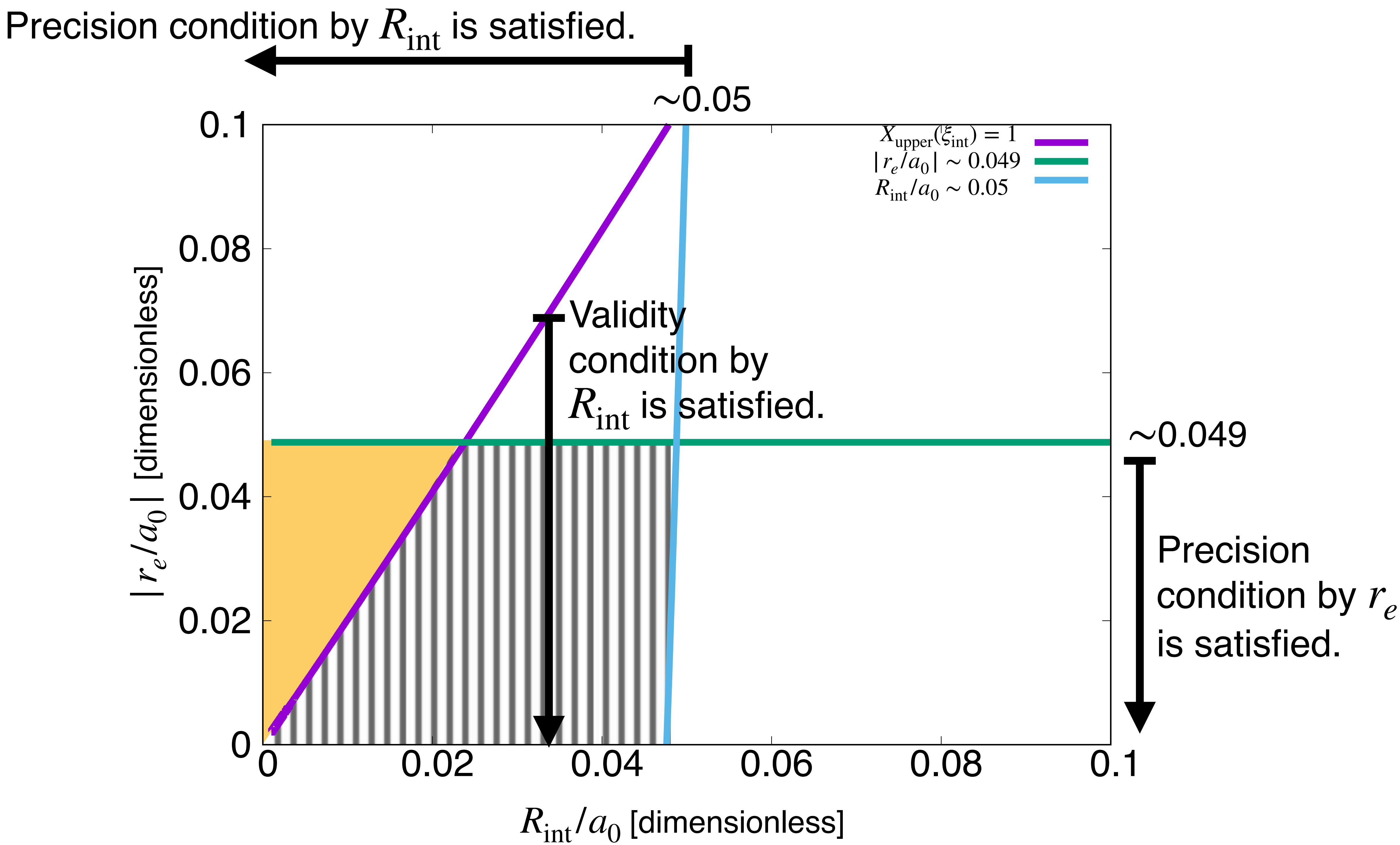
-$X_{\text{exact}} = 1$ is contained in $|r_e| \lesssim 0.002a_0$.

→ The validity condition is satisfied in this region.

- Uncertainty from R_{int} is not sensitive to $|r_e/a_0|$.

Numerical calculation

Validity and precision conditions in $R_{\rm int}/a_0$-$|r_e/a_0|$ plane



Only the improved weak-binding relation can be applied.

Conclusion and future prospect

- Weak-binding relation : observable → compositeness (X)

$$a_0 = R\left\{ \frac{2X}{1+X} + \mathcal{O}\left(\frac{R_{\rm typ}}{R}\right) \right\}$$

- We study the range correction in weak-binding relation from r_e.

- Improved weak-binding relation by redefinition of $R_{\rm typ}$:

$$R_{\rm typ} = \max\left\{ R_{\rm int}, R_{\rm eff} \right\}, \quad R_{\rm eff} = \max\{ |r_e|, \cdots \}$$

- Numerical calculations : effective range model ($\Lambda < \infty$)

→ There is the region where only the improved weak-binding relation can be applied.

- Apply the improved relation to hadron systems.