Study of Scattering Amplitude in the Complex Scaling Method

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Topics:
1. Continuum Level Density for Coupled-channel systems
2. Scattering T-Matrix Calculations in the Complex Scaling Method
Motivation and Purposes

• For unstable nuclei, or even for stable nuclei, a unified description of bound and resonant states is necessary.

• Besides the observation of the quantized bound and resonant states, the majority of information comes from scattering states.

• We show that the complex scaling method can provide us with promising tools for such a purpose.
Complex Scaling Method

- The **Complex Scaling Method (CSM)** is well known as a tool to investigate the resonant state.
- In the CSM, the coordinate is rotated in the complex plane and the wave function of resonance is changed into a **damping form**.

\[ U(\theta) : \quad r \rightarrow r \exp(i\theta) \]

\[ \theta : \text{scaling parameter} \]

**Resonance:** \( \exp(ikr) \rightarrow \alpha \exp(-(k_R \sin \theta - k_F \cos \theta) r) \rightarrow 0, \]
\[ k = k_R - i k_F, \quad (\tan \theta > k_F/k_R) \]

- Therefore, resonant states as well as bound states can be treated with squared integrable functions.
- Many calculations have been devoted to find resonant states of **cluster nuclei** or to calculate resonances of **three-body problems**.
Eigenvalues of a Complex Scaled Hamiltonian

- Carrying out the diagonalization of the complex scaled Hamiltonian, $H(\theta)$, we can gain information about the bound and resonant states.
- By contrast, the continuum spectra of the $H(\theta)$ are distributed on the $2\theta$-line from every threshold as shown in Fig. 1.

- In this talk, we show that rotated continuum states have important physical meanings in addition to bound and resonant states.

Keyword:
1. Basis function method
2. Extended completeness relation in CSM
3. The complex-scaled Green function (CLD, T-matrix)

![Fig.1: Schematic energy eigenvalue distribution for a complex scaled Hamiltonian.](image)
**Topic 1: Continuum Level Density**

- The continuum level density (CLD), \( \Delta(E) \), plays an important role in the study of scattering theory since the CLD relate the scattering \( S \)-matrix and the Green function as

\[
\Delta(E) = (2\pi)^{-1} \text{Im} \frac{d}{dE} \ln \det S(E), \quad \Delta(E) = -\frac{1}{\pi} \text{Im} \left[ \text{Tr} \left[ G(E) - G_0(E) \right] \right]
\]

[Ref: S. Shlomo, Nucl. Phys. A 539 (1992), 17.]

- CLD is constructed from two parts: the full Green function part, \( \rho(E) \), and the asymptotic Green function part, \( \rho_0(E) \)

\[
\Delta(E) = \rho(E) - \rho_0(E)
\]

\[
\rho(E) = -\frac{1}{\pi} \text{Im} \left\{ \text{Tr} \left[ \frac{1}{E - H} \right] \right\}, \quad \rho_0(E) = -\frac{1}{\pi} \text{Im} \left\{ \text{Tr} \left[ \frac{1}{E - H_0} \right] \right\}
\]
A. T. Kruppa and K. Arai studied CLD in a framework of a finite number ($M$) of basis function and the Strutinsky smoothing procedure.

\[ g_M(E) = \sum_{i}^{M} \delta(E - \epsilon_i) - \sum_{j}^{M} \delta(E - \epsilon_j^0) \]

Their calculation succeeded well. Their results depend on smoothing parameter. Only smoothed CLD is obtained (not “raw” CLD).

Continuum Level density in the complex scaling method


- **Density of state in CSM**

\[ \rho(E) = -\frac{1}{\pi} \text{Im} \left[ Tr \left\{ \frac{1}{E - H(\theta)} \right\} \right] \]

- **CSM**

\[ U(\theta) : \quad r \rightarrow r \exp(i\theta), \quad k \rightarrow k \exp(-i\theta) \]

\[ \theta : \text{scaling parameter} \]

- **Extended Completeness relation**

\[ 1 = \sum_B |\Phi_B(\xi)\rangle \langle \Phi_B^*(\xi')| + \sum_R |\Phi_R(\xi)\rangle \langle \Phi_R^*(\xi')| + \int dk_\theta |\Phi_{k_\theta}(\xi)\rangle \langle \Phi_{k_\theta}^*(\xi')| \]

New smoothing mechanism in the complex scaling method

\[ \rho(E) \approx \sum_B \delta(E - e_B) + \frac{1}{\pi} \sum_R \frac{\Gamma_R/2}{(E - E_R)^2 + \Gamma_R^2/4} + \frac{1}{\pi} \sum_C \frac{E_C^I}{(E - E_C^R)^2 + E_C^I^2} \]

\( \rho(E) \) is described by only the eigenvalue

- **Resonance** \( e_R = E_R - i \frac{\Gamma_R}{2} \)
- **Continuum** \( e_C = E_C^R - i E_C^I \)

**Basis function method**

**Ex. Gaussian or H.O. wfs**

**Artificial smoothing procedure is not needed.**
Numerical application of CLD: 
\(^4\text{He} = \{^3\text{H} + p\} + \{^3\text{He} + n\}\) system

\[V_c(r), V_d(r), :\text{obtained to reproduce the experimental phase shift}\]

[Ref: M. Teshigawara, doctor thesis, Hokkaido University, 1993]
We expand the relative wave function using the Gaussian basis functions. We use 30 basis functions for one channel. Therefore total basis number is $N=60$.

Description of smoothing behaviour is very successful in CSM with basis function method and slightly depends on the scaling parameter $\theta$. To remove $\theta$ dependence, we subtract the free density term $\rho_0(E)$.

$$\Delta(E) = \rho(E) - \rho_0(E)$$
Calculated continuum level density: $\Delta(E)$

Exact solution is calculated by solving the scattering problem and obtained by eigenphase shift:

$\Delta(E) = \frac{1}{\pi} \sum_j \frac{d\delta_j}{dE}$

$\theta$ independent!!

Absolute values are consistent with the exact calculation!!
Single channel calculation

\[ \Delta(E) = -\frac{1}{\pi} \text{Im} \left[ \text{Tr} \left[ G(E) - G_0(E) \right] \right] \quad \text{and} \quad \Delta(E) = \frac{1}{\pi} \frac{d\delta_l}{dE} \]

\[ \delta(E) = \pi \int_0^E dE' \Delta(E') dE' \]

Phase shift of $^5\text{He}$

Energy eigenvalue distributions
Three-body CLD (3α calculation)  

[Ref: C. Kurokawa et al.,]

• **Formalism**

\[
\Delta(E) = \rho_{3B}(E) - \rho_{3B}^0(E) - \left( \rho_{2B}(E) - \rho_{2B}^0(E) \right)
\]

\[
= -\frac{1}{\pi} \text{Im} \left\{ \text{Tr} \left[ \frac{1}{E - H_{3B}} - \frac{1}{E - H_{3B}^0} - \left( \frac{1}{E - H_{2B}} - \frac{1}{E - H_{2B}^0} \right) \right] \right\}
\]

• **Numerical application** : CLD is calculated using the eigenvalues

![Graph showing CLD calculations for different states](image)
**Topic 2: Scattering T-Matrix Calculations in the Complex Scaling Method**

By using Green-operator and complex scaling method

\[ t_{\alpha, \beta}(E) = \langle \Phi_\alpha | \int dr F_l(k_\alpha r) V(r) F_l(k_\beta r) | \Phi_\beta \rangle + e^{i\theta} \langle \Phi_\alpha | \int dr dr' F_l(k_\alpha re^{i\theta}) \hat{V}(re^{i\theta}) G^\theta(E; r, r') \hat{V}(r'e^{i\theta}) F_l(k_\beta r'e^{i\theta}) | \Phi_\beta \rangle \]

- Green's function

\[ G^\theta(E; r', r) = \langle r' | \frac{1}{E - H(\theta)} | r \rangle \]

- Using the Extended Completeness relation in the CSM

\[ 1 = \sum_B |\Phi_B(\xi)\rangle \langle \tilde{\Phi}_B(\xi')| + \sum_R N_\theta |\Phi_R(\xi)\rangle \langle \tilde{\Phi}_R(\xi')| + \int dk_\theta |\Phi_{k_\theta}(\xi)\rangle \langle \tilde{\Phi}_{k_\theta}(\xi')| \]
2\textsuperscript{nd} term is approximated as

\[ e^{i\theta} \left[ \sum_{B} \frac{d_{B}(\theta)d_{B}(\theta)}{E - E_{B}^{\theta}} + \sum_{R} \frac{d_{R}(\theta)d_{R}(\theta)}{E - E_{R}^{\theta}} + \sum_{C} \frac{d_{C}(\theta)d_{C}(\theta)}{E - E_{C}^{\theta}} \right] \]

where

\[ d_{i}(\theta) = \int dr \psi_{i}^{\theta} V(r e^{i\theta}) F_{i}(k r e^{i\theta}) \]

Basis function method
Numerical application: $^4\text{He} = \{^3\text{H}+p\} + \{^3\text{He}+n\}$ system

The $P$-wave phase shifts of the $^3\text{H}+p$ and $^3\text{He}+n$. The solid lines are determined by CS calculation and the circles are calculated by integrating the differential equations with the Runge-Kutta method.
Conclusions and summary

- We showed that the complex scaling method provides us with very useful tools to describe bound and unbound states in a unified way.
  - Resonant states are described with $L^2$ basis functions in the same manner as bound states.
  - Rotated continuum states have also an important role.
  - It is easily to apply to three-body systems.

- As applications, we showed the continuum level density (CLD) and scattering matrix calculations using only square integrable functions.
Application to $^5\text{He}=\alpha+n$ system

$$[T_{\text{rel}} + V_{\alpha n}(r) + \lambda |\phi_{PF}\rangle \langle \phi_{PF}| - E] \psi^J_{\text{rel}}(r) = 0$$

- $V_{\alpha n}(r)$: KKNN potential [3]
- Pauli principle is treated by OCM [4]
  \[ \lambda : 10^6 \text{ MeV}, \quad PF : 0s_{1/2} \]


Energy eigenvalue distribution of $s_{1/2}$ state
Gaussian basis; $N=30$, $b_0=0.2\text{fm}$, $\gamma=1.2$
\( \theta \)-dependency of phase shift of \( s_{1/2} \) state
Calculated phase shift of $s_{1/2}$, $p_{1/2}$ and $p_{3/2}$ state

\[ \delta \text{[deg.]} \]

Energy [MeV]
Check: Unitarity of calculated $S$-matrix

The quantity $||\text{det}(S_k(E))|| - 1|$ for different partial waves as the function of the energy. The CS calculation is carried out using 30 basis functions. $S_k(E)$ is the 2x2 $S$-matrix in the partial wave $k=2S+1L_J$. The complex scaling parameter is 20 degree.