The 163rd RIKEN RIBF Nuclear Physics Seminar

Pseudospin symmetry in nuclear single-particle spectra and its perturbative interpretation

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Main references:

H. Liang, P. Zhao, Y. Zhang, J. Meng, and N. Van Giai, *PRC* **83**, 041301(R) (2011)
H. Liang, S. Shen, P. Zhao, and J. Meng, *PRC* **87**, 014334 (2013)

Outline

Outline

Spin and pseudospin symmetries

- Spin symmetry (SS) breaking, i.e., remarkable spin-orbit splitting in $(n, l, j = l \pm 1/2)$

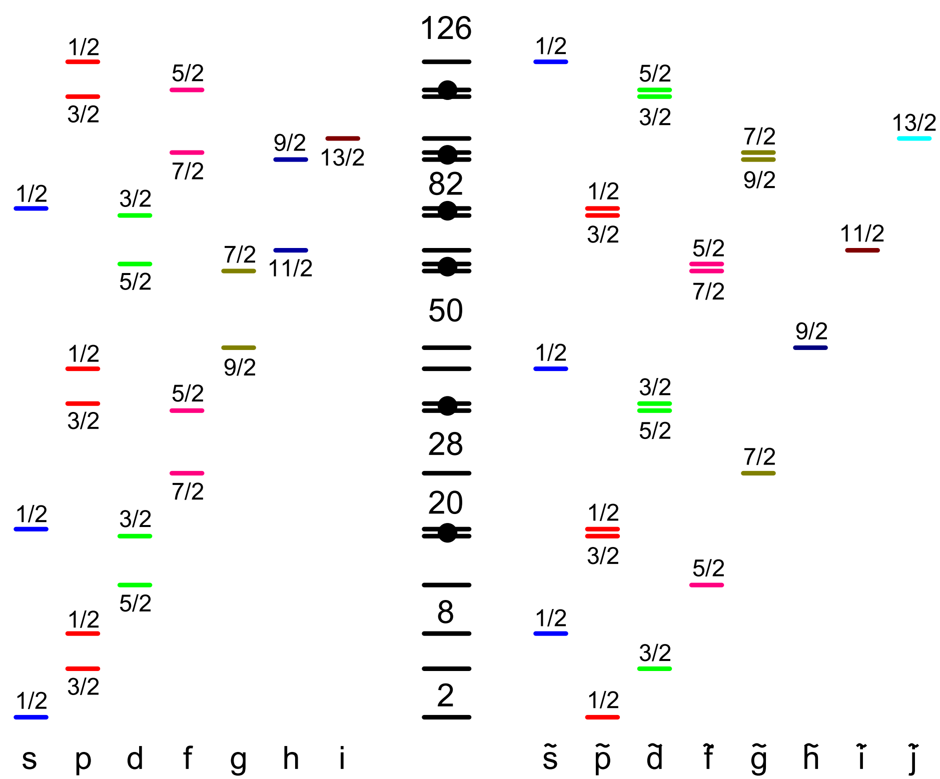
  Haxel:1949, Mayer:1949

- Pseudospin symmetry (PSS), i.e., near degeneracy in

$$\begin{cases} (n-1, l+2, j = l+3/2) \\ (n, l, j = l+1/2) \end{cases}$$

  by defining

$$(\tilde{n} = n-1, \tilde{l} = l+1, j = \tilde{l} \pm 1/2)$$



In shell model scheme

- No spin-orbit coupling ⇒ total spin S a good quantum number ⇒ LS extreme ×
- No pseudo s.o. coupling ⇒ total spin $\tilde{S}$ a good quantum number ⇒ $\tilde{L}\tilde{S}$ extreme

Hecht & Adler, *NPA* **137**, 129 (1969); Arima, Harvey, Shimizu, *PLB* **30**, 517 (1969)

From spin scheme to pseudospin scheme

- From spin scheme to pseudospin scheme

$$H\psi = E\psi \quad \text{with} \quad H = \frac{\mathbf{p}^2}{2M} + V(r) + W(r)\mathbf{l}\cdot\mathbf{s}$$

$$(UHU^\dagger)U\psi = EU\psi \quad \text{with} \quad UHU^\dagger = \frac{\mathbf{p}^2}{2M} + \tilde{V}(r) + \tilde{W}(r)\tilde{\mathbf{l}}\cdot\tilde{\mathbf{s}}$$

- Special ratio for v_{sl}/v_{ll}, e.g., $U_r = \mathbf{s}\cdot\hat{\mathbf{r}}$

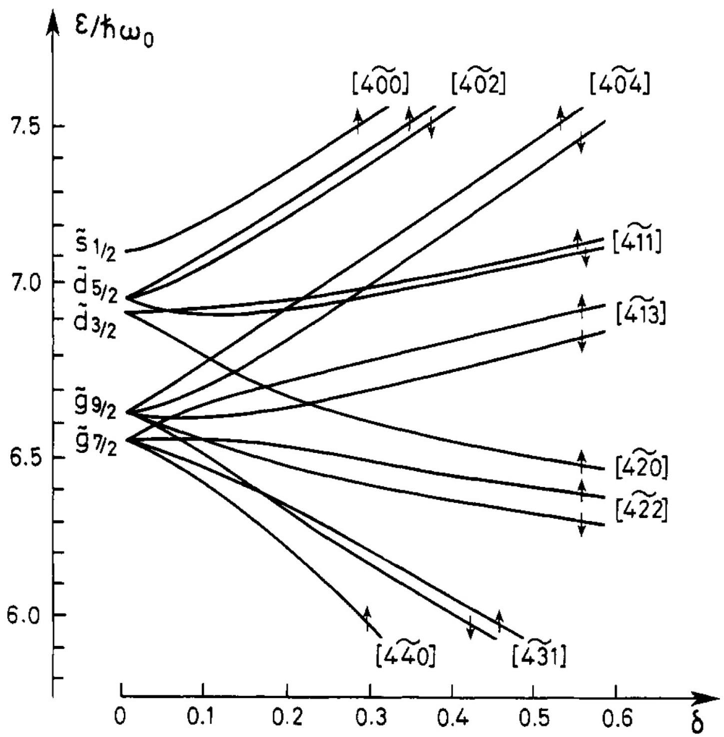
$$\begin{aligned} H &= H_{\rm HO} + v_{ll}\mathbf{l}^2 + v_{ls}\mathbf{l}\cdot\mathbf{s} \\ \tilde{H} &= \tilde{H}_{\rm HO} + v_{ll}\tilde{\mathbf{l}}^2 + (4v_{ll} - v_{ls})\tilde{\mathbf{l}}\cdot\tilde{\mathbf{s}} \end{aligned}$$

⋆ Parameters for the modified oscillator potential.
Bohr, Hamamoto, Mottelson, *Phys. Scr.* **26**, 267 (1982)

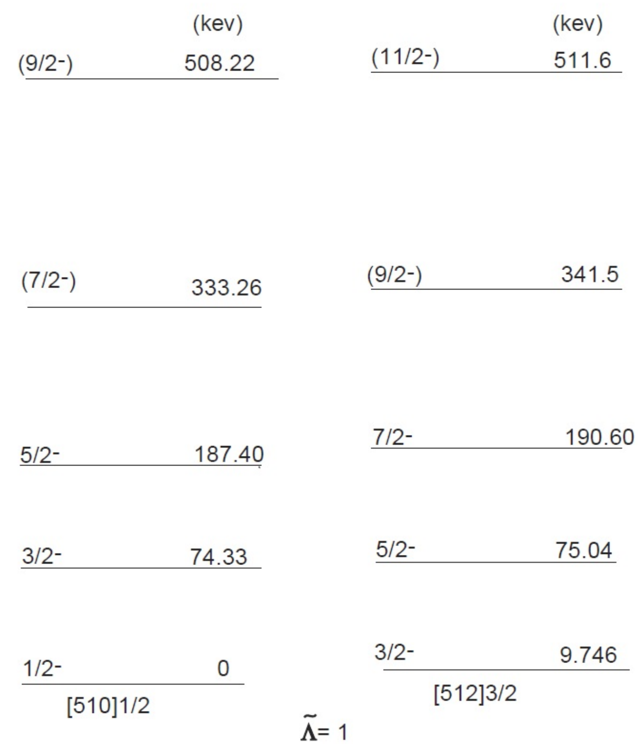
Region	$-v_{ls}$	$-v_{ll}$	$-\tilde{v}_{ls}$
$50 < Z < 82$	0.127	0.0382	0.026
$82 < N < 126$	0.127	0.0268	-0.019
$82 < Z < 126$	0.115	0.0375	0.035
$126 < N$	0.127	0.0206	-0.045

PSS in deformed nuclei

- Single-particle states: $[Nn_z\Lambda]\Omega$ & $[Nn_z\Lambda+2]\Omega+1 \Rightarrow [\widetilde{Nn_z\Lambda}]$ with $\tilde{N}=N-1, \tilde{\Lambda}=\Lambda+1, \Omega=\tilde{\Lambda}\pm 1/2$
- Rotational bands: from $\tilde{\Lambda}\Omega IM$ coupling to $\tilde{\Lambda}\tilde{R}IM$ coupling



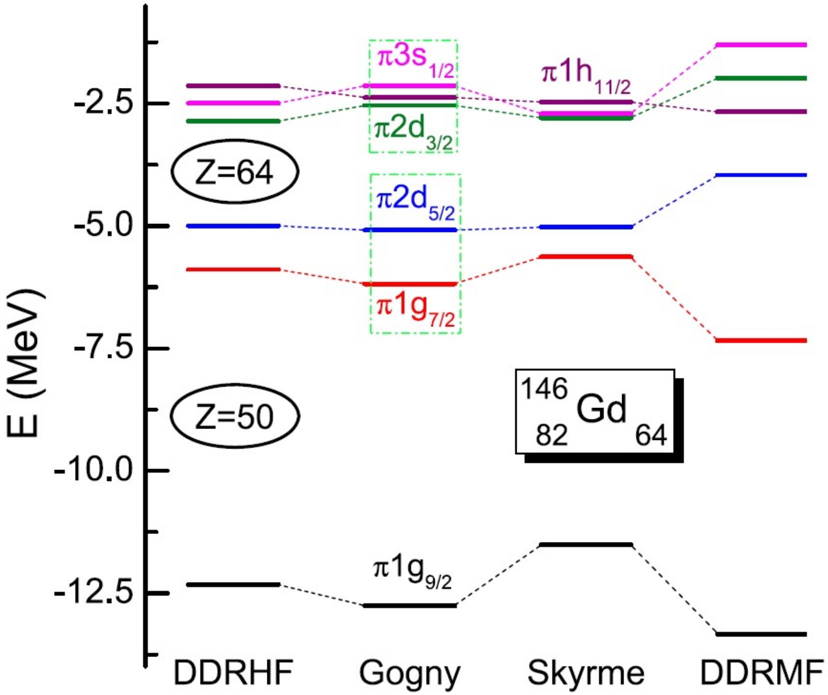
Bohr, Hamamoto, Mottelson, *Phys. Scr.* **26**, 267 (1982)



⋆ g.s. & neighboring bands in ^{187}Os

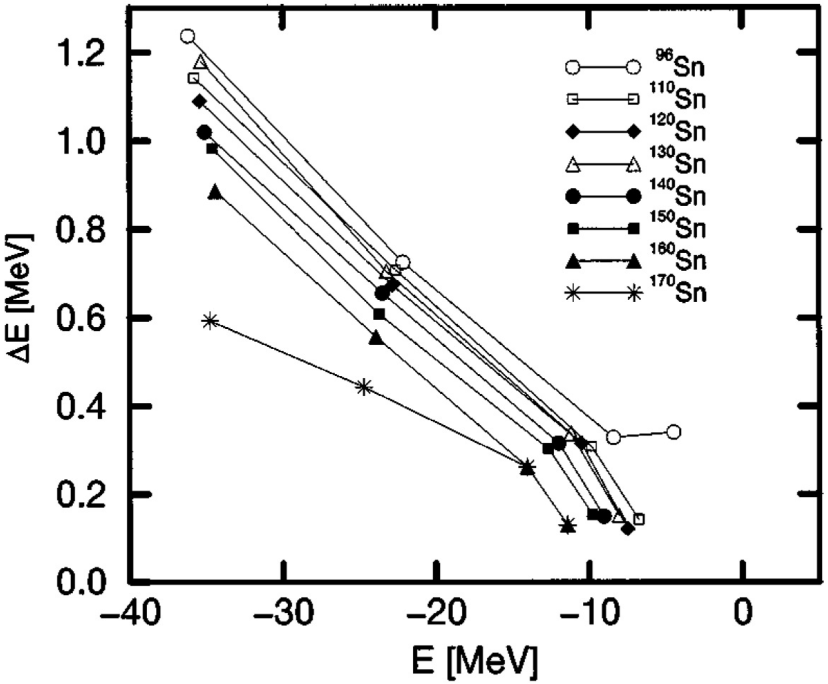
Data: Bruce *et al.*, *PRC* **56**, 1438 (1997)

PSS in shell structure evolutions



⋆ Proton single-particle energies for ^{146}Gd

Long, Nakatsukasa, Sagawa, Meng, Nakada, Zhang, *PLB* **680**, 428 (2009)



⋆ Pseudospin-orbit splitting in Sn isotopes

Meng, Sugawara-Tanabe, Yamaji, Arima *PRC* **59**, 154 (1999)

- Splitting of both spin and pseudospin doublets play important roles in the shell structure evolutions.
- It is a fundamental task to explore the origin of SS and PSS, as well as the mechanism of their breaking.

Outline

PSS — a relativistic symmetry

- PSS — a relativistic symmetry in Dirac Hamiltonian
  a. $\tilde{l}$ is the orbital angular momentum of the lower component of the Dirac spinor Ginocchio, *PRL* **78**, 436 (1997)
  b. the condition that $S(\mathbf{r}) + V(\mathbf{r}) = 0$ is suggested as the exact PSS limit by reducing the Dirac equation to the Schrödinger-like equation Ginocchio:1997
  c. $S(\mathbf{r}) + V(\mathbf{r}) = \mathrm{Constant}$ can be approximately fulfilled in exotic nuclei with highly diffuse potentials Sugawara-Tanabe:1998,2000, Meng:1998,1999

- Dirac equation: (local potentials, no tensor potential, spherical symmetry)

$$\begin{pmatrix} \Sigma(r) + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta(r) - M \end{pmatrix} \begin{pmatrix} G(r) \\ F(r) \end{pmatrix} = E \begin{pmatrix} G(r) \\ F(r) \end{pmatrix},$$

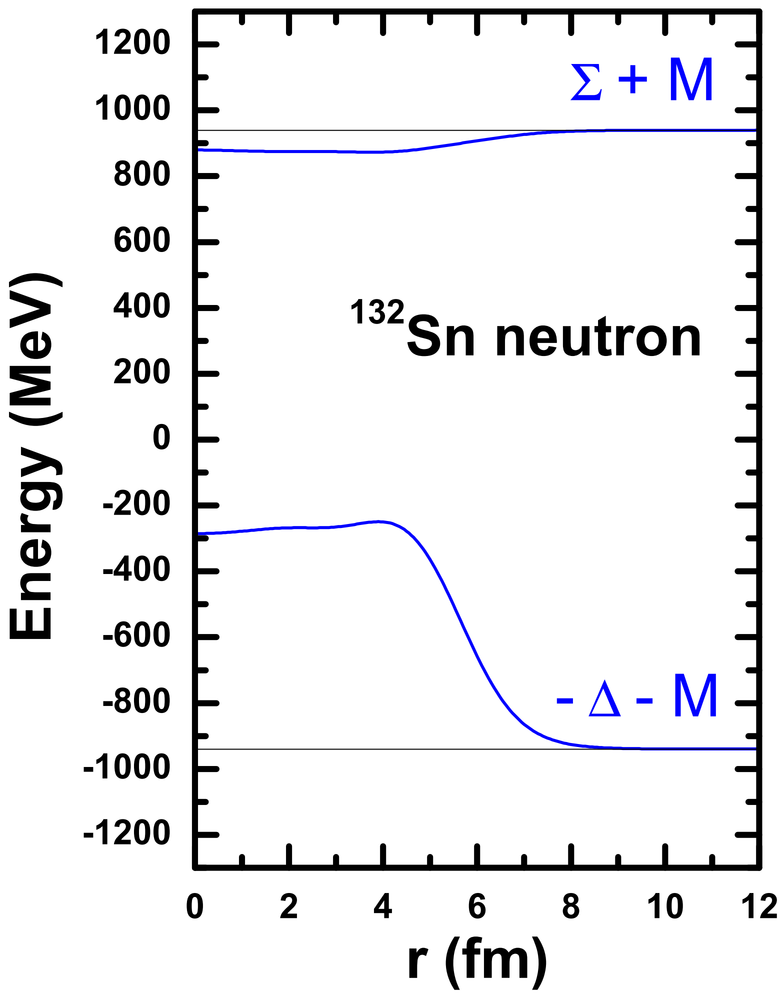
where

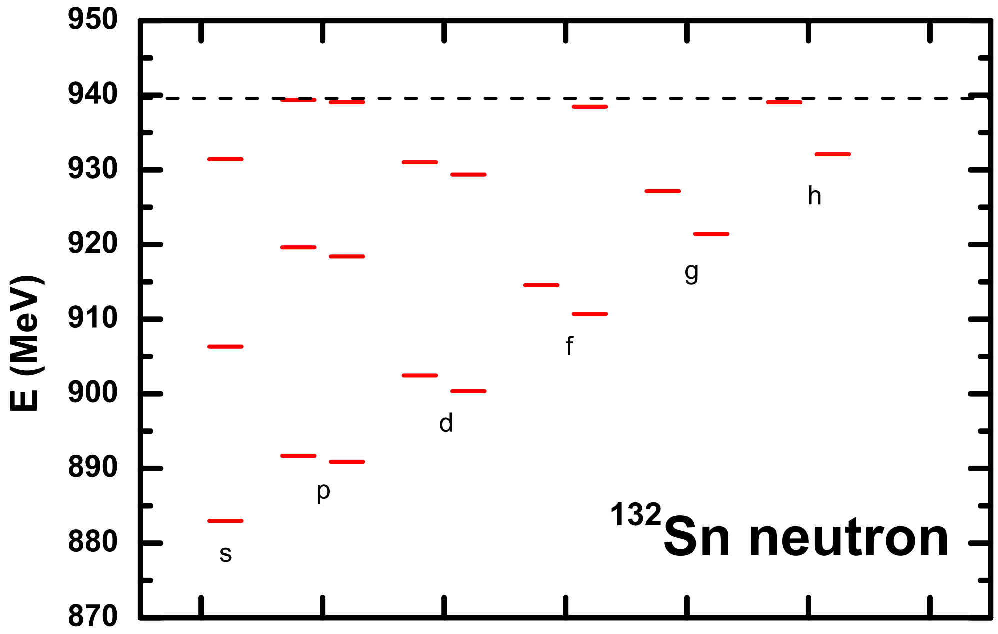
$$\Sigma(r) = S(r) + V(r), \qquad \Delta(r) = S(r) - V(r),$$

and

$$\kappa = \mp(j + 1/2) \quad \text{for} \quad j_G = l_G \pm 1/2.$$

Potentials and single-particle spectrum of neutron

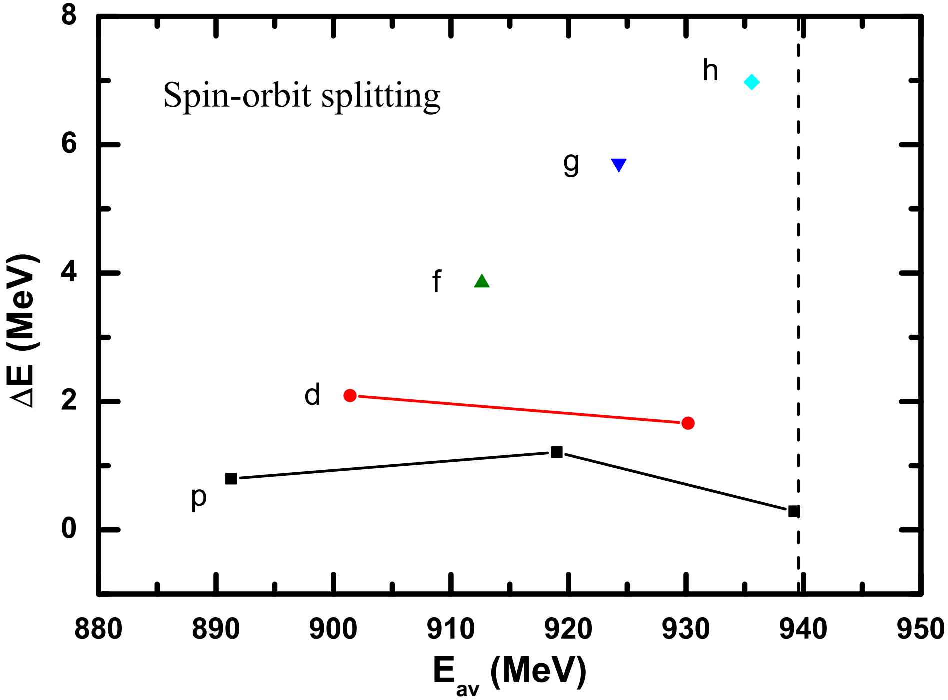


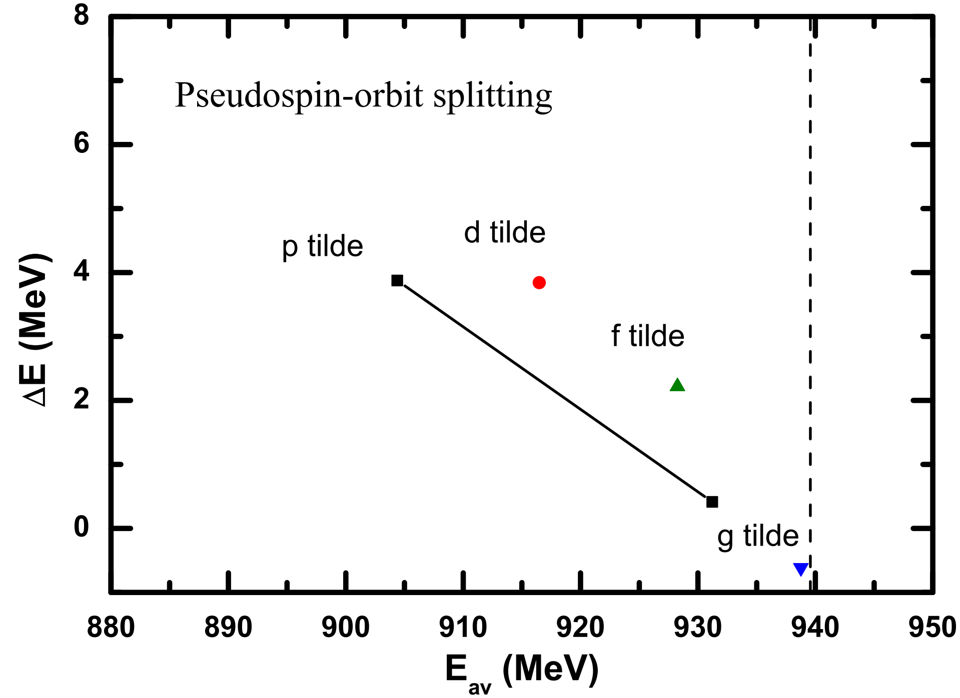


★ Neutron potentials in ^{132}Sn calculated by RMF theory with PK1 parameter.

★ Neutron spectrum of ^{132}Sn. For each pair of the spin doublets, the left level is with $j = l - 1/2$ and the right one with $j = l + 1/2$. The dashed line shows the continuum limit.

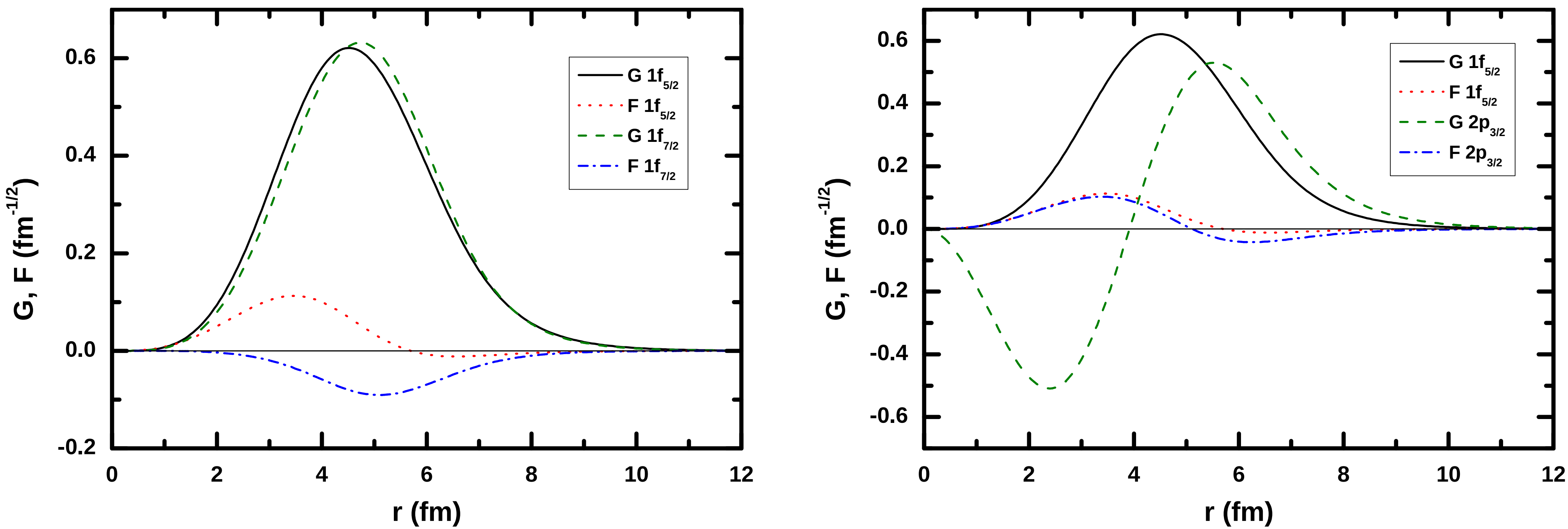
Spin-orbit and pseudospin-orbit splittings





★ Spin-orbit and pseudospin-orbit splittings in neutron spectrum of ^{132}Sn versus the average energy of a pair of spin doublets. The vertical dashed line shows the continuum limit.

Wave functions of spin and pseudospin doublets



★ Wave functions of spin doublets $1f$ and pseudospin doublets $1\tilde{d}$.

- The upper components of spin doublets $1f$ are similar to each other.
- The lower components of pseudospin doublets $1\tilde{d}$ are similar to each other.

Schrödinger-like equations

- Schrödinger-like equations

$$\left\{-\frac{1}{M_+}\frac{d^2}{dr^2}+\frac{1}{M_+^2}\frac{dM_+}{dr}\frac{d}{dr}+\left[(M+\Sigma)+\frac{1}{M_+}\frac{\kappa(\kappa+1)}{r^2}+\frac{1}{M_+^2}\frac{dM_+}{dr}\frac{\kappa}{r}\right]\right\}G=EG,$$

$$\left\{-\frac{1}{M_-}\frac{d^2}{dr^2}+\frac{1}{M_-^2}\frac{dM_-}{dr}\frac{d}{dr}+\left[(-M-\Delta)+\frac{1}{M_-}\frac{\kappa(\kappa-1)}{r^2}-\frac{1}{M_-^2}\frac{dM_-}{dr}\frac{\kappa}{r}\right]\right\}F=EF,$$

  with effective masses $M_+ = M+\Delta+E$, $M_- = E-M-\Sigma$.

- In analogy with Schrödinger equation,

$$V_{\rm CB}=\frac{1}{M_+}\frac{\kappa(\kappa+1)}{r^2},\qquad V_{\rm SOP}=\frac{1}{M_+^2}\frac{dM_+}{dr}\frac{\kappa}{r}.$$

$$V_{\rm PCB}=\frac{1}{M_-}\frac{\kappa(\kappa-1)}{r^2},\qquad V_{\rm PSOP}=-\frac{1}{M_-^2}\frac{dM_-}{dr}\frac{\kappa}{r}.$$

- $M_- = 0$ at finite r_0 $\Rightarrow$ there exist singularities in $V_{\rm PCB}$ and $V_{\rm PSOP}$
- Effective Hamiltonian is not Hermitian $\Rightarrow$ perturbation theory can NOT be applied
- No bound nuclei in PSS limit $S(\mathbf{r})+V(\mathbf{r})=\rm Constant$

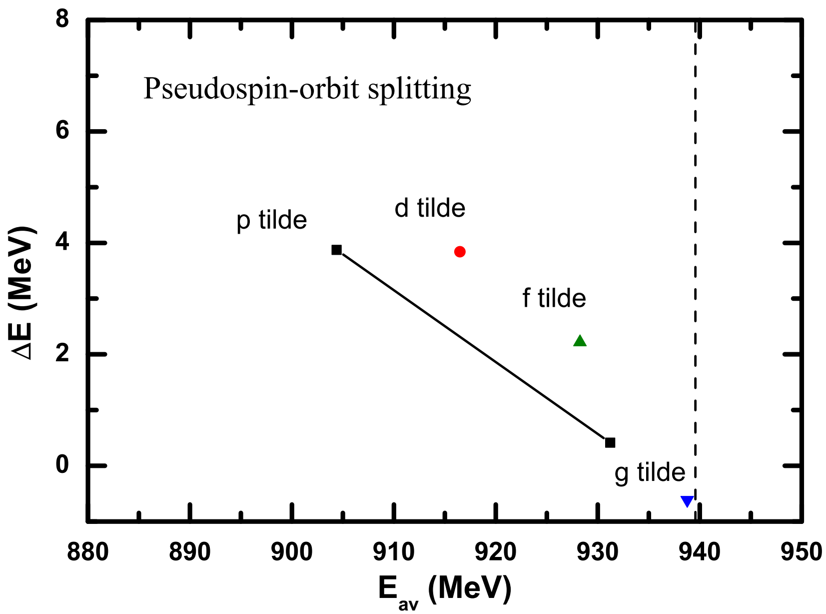
Still an open problem

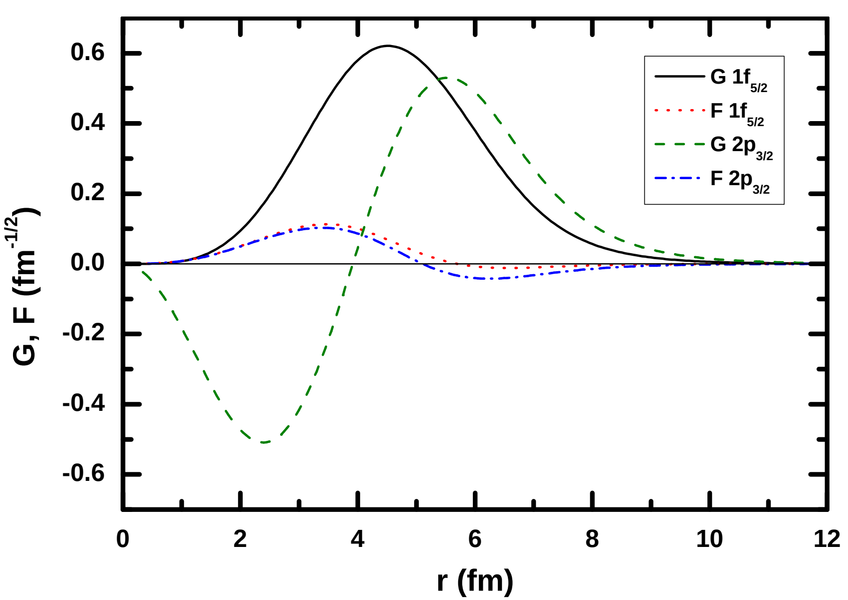
No bound states in the proposed PSS limit

⇒ PSS is a dynamical symmetry in nuclei Alberto:2001

⇒ The nature of PSS is nonperturbative Alberto:2002, Ginocchio:2011

⇒ PSS is an accidental symmetry in the relativistic framework Marcos:2008





- PSS can be understood qualitatively but not quantitatively?

Outline

Perturbation theory (I)

Rayleigh-Schrödinger perturbation theory

- Hamiltonian is split up as

$$H = H_0 + W.$$

- The eigenvalues and eigenfunctions of H_0 are known as

$$H_0\psi_n^0 = E_n^0\psi_n^0.$$

- The eigenvalues and eigenfunctions of H can be expressed as

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \cdots ,$$

and

$$\Psi = \sum_m a_m\psi_m^0, \quad \text{with} \quad a_m = a_m^{(0)} + a_m^{(1)} + a_m^{(2)} + \cdots .$$

Perturbation theory (II)

- For the state k,
  a. 0th order approximation
  $$E^{(0)} = E_k^0, \qquad a_m^{(0)} = \delta_{mk}.$$
  b. 1st order approximation
  $$E^{(1)} = W_{kk}, \qquad a_m^{(1)} = \begin{cases} \dfrac{W_{mk}}{E_k^0 - E_m^0}, & m \neq k, \\ 0, & m = k. \end{cases}$$
  c. 2nd order approximation
  $$E^{(2)} = \sum_n{}' \frac{W_{kn} W_{nk}}{E_k^0 - E_n^0}, \qquad a_m^{(2)} = \begin{cases} -\dfrac{W_{kk} W_{mk}}{(E_k^0 - E_m^0)^2} + \sum_n{}' \dfrac{W_{mn} W_{nk}}{(E_k^0 - E_n^0)(E_k^0 - E_m^0)}, & m \neq k, \\ -\dfrac{1}{2} \sum_n{}' \dfrac{W_{kn} W_{nk}}{(E_k^0 - E_n^0)^2}, & m = k. \end{cases}$$
  d. 3rd order approximation
  $$E^{(3)} = \sum_n{}' a_n^{(2)} W_{kn}.$$

Perturbation theory (III)

- Whether $\left|\frac{W_{mk}}{E_k^0 - E_m^0}\right| \ll 1$ or not determines whether the perturbation term is small as well as the speed of convergency, since the eigenvalues and the eigenfunctions are expanded in powers of these quantities.

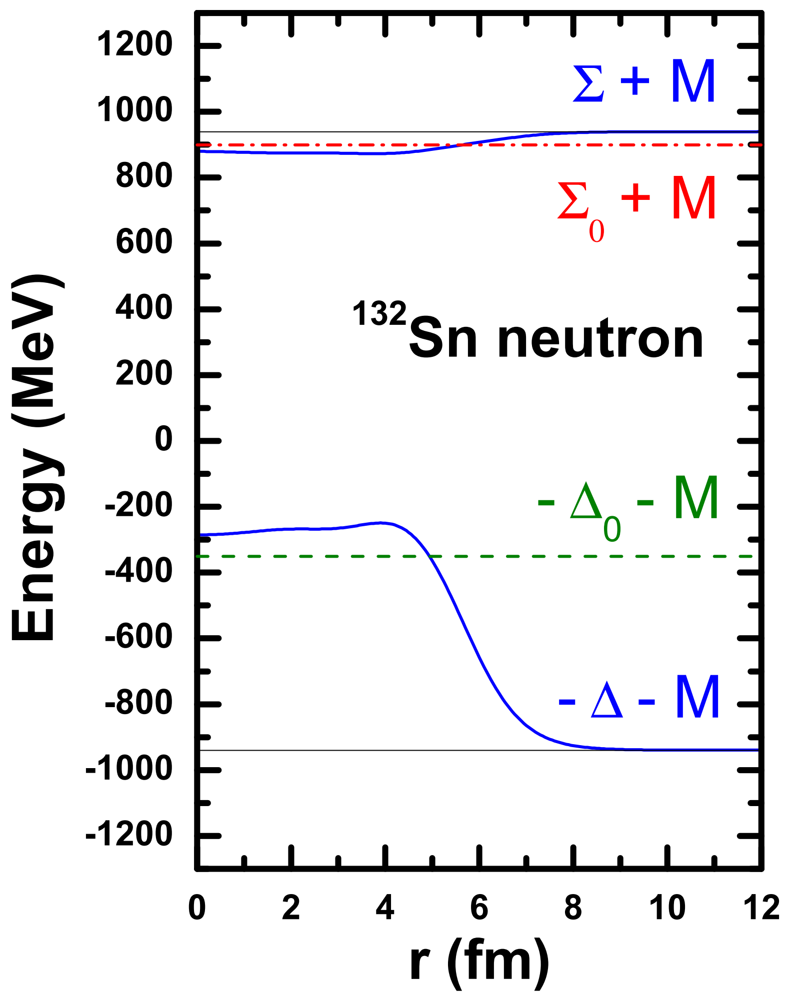
- Dirac Hamiltonian is divided into two parts, followed by [Ginocchio:1997,2005]

$$H = H_0^{\mathrm{SS}} + W^{\mathrm{SS}} = \begin{pmatrix} \Sigma + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta_0 - M \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \Delta_0 - \Delta \end{pmatrix},$$

$$H = H_0^{\mathrm{PSS}} + W^{\mathrm{PSS}} = \begin{pmatrix} \Sigma_0 + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta - M \end{pmatrix} + \begin{pmatrix} \Sigma - \Sigma_0 & 0 \\ 0 & 0 \end{pmatrix},$$

with constant numbers Δ_0 and Σ_0.

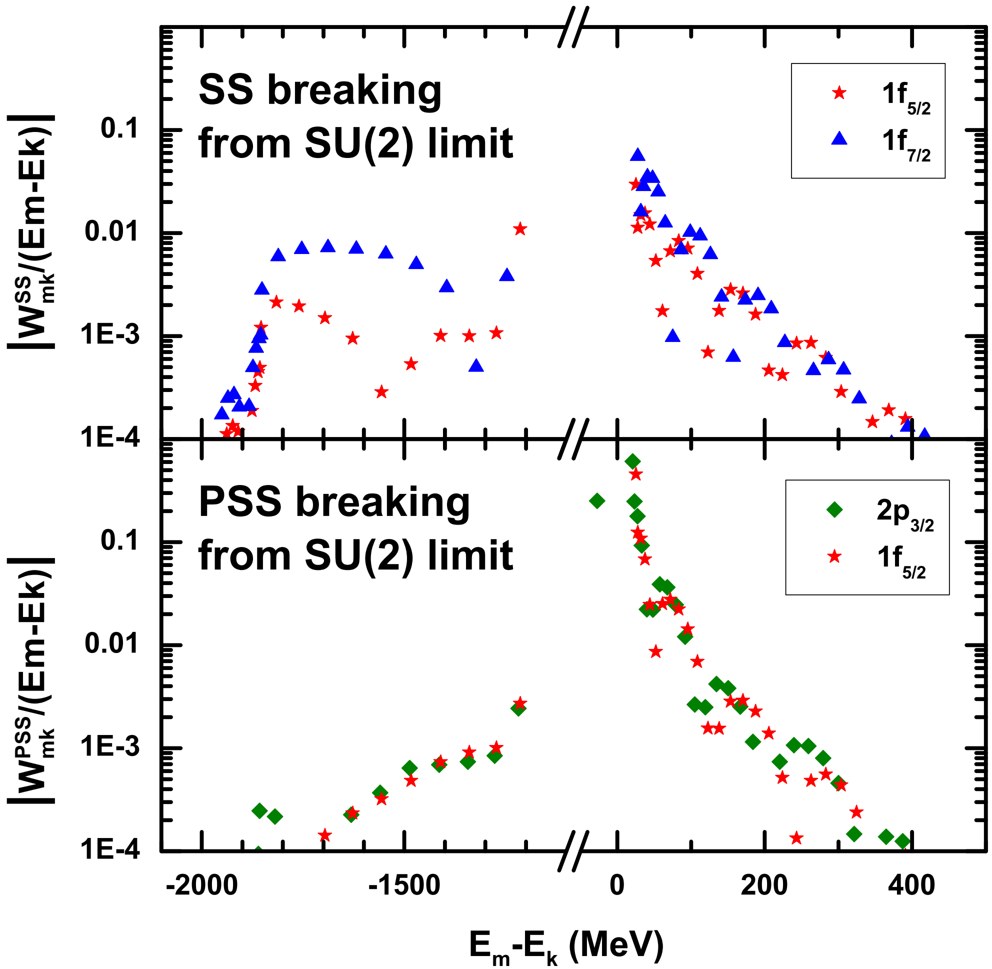
Potentials for the symmetry limits



- ★ single-particle potentials for neutrons in ^{132}Sn
- ★ solid lines: self-consistent potentials in realistic nucleus obtained by RMF with PK1
- ★ dashed line: potential $-\Delta_0 - M$ in $H_0^{\rm SS}$
- ★ dash-dotted line: potential $\Sigma_0 + M$ in $H_0^{\rm PSS}$

Liang, Zhao, Zhang, Meng, Giai, *PRC* **83**, 041301(R) (2011)

Validity of perturbation theory



- Even though it is clearly shown that

$$|\Sigma - \Sigma_0| \ll |\Delta_0 - \Delta|,$$

it should be noticed that

$$W^{\rm PSS}_{mk} = \langle G_m | (\Sigma - \Sigma_0) | G_k \rangle ,$$
$$W^{\rm SS}_{mk} = \langle F_m | (\Delta_0 - \Delta) | F_k \rangle .$$

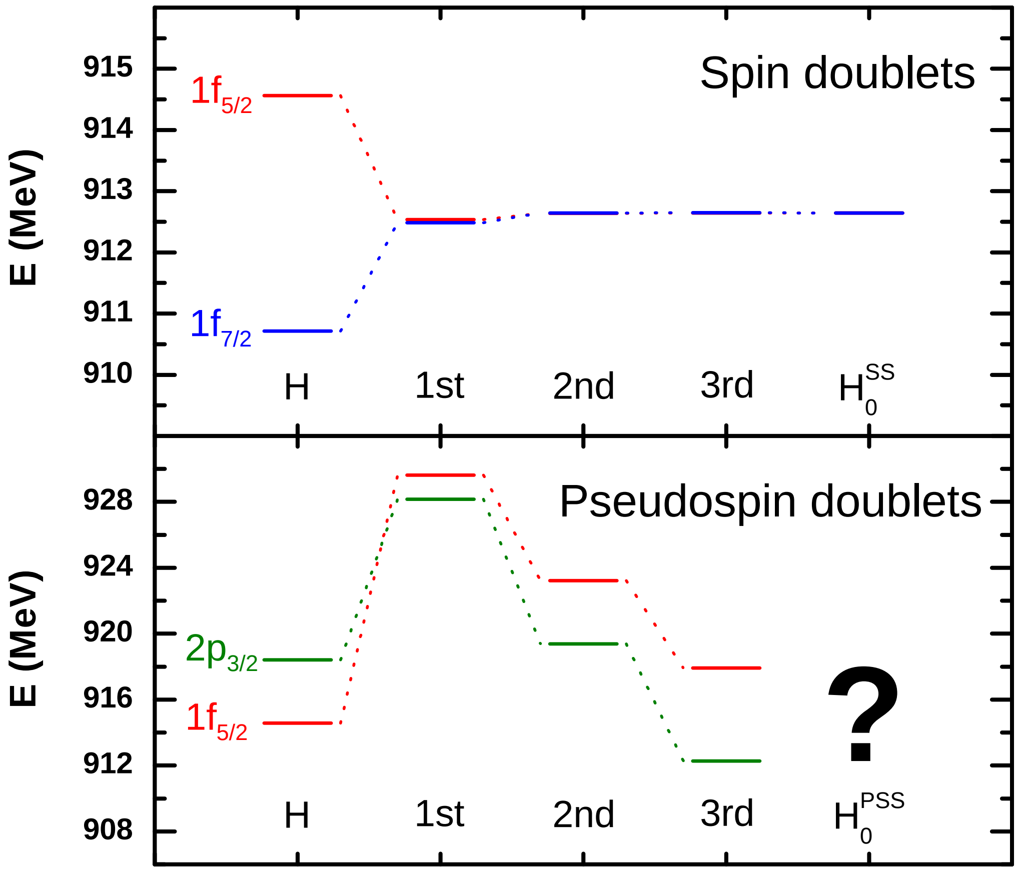
Therefore

$$W^{\rm PSS}_{mk} \gg W^{\rm SS}_{mk}.$$

Liang, Zhao, Zhang, Meng, Giai, *PRC* **83**, 041301(R) (2011)

- SS: the biggest perturbations $\sim 0.06 \Rightarrow$ valid
- PSS: the biggest perturbations $\sim 0.6 \Rightarrow$ questionable

Restoration of symmetries in single-particle energies



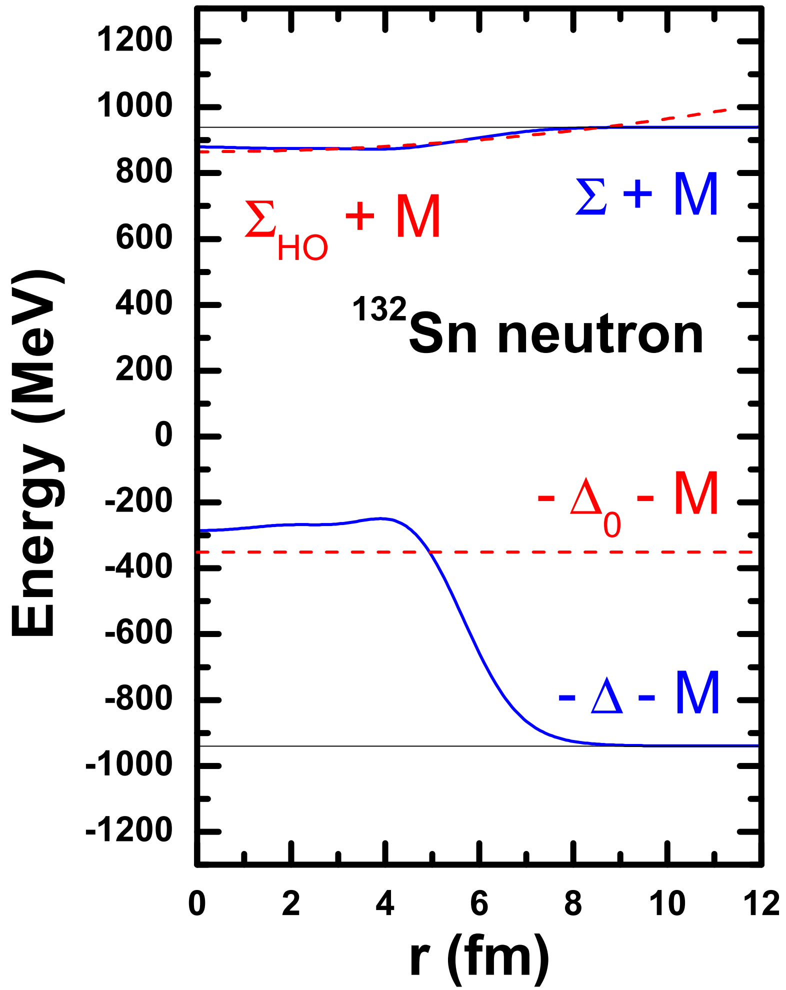
★ Single-particle energies of spin doublets $k = 1f$ (upper panel) and pseudospin doublets $k = 1\tilde{d}$ (lower panel) obtained by RMF theory, and by the first, second, and third order perturbation calculations, as well as those at the exact symmetry limits.

Liang, Zhao, Zhang, Meng, Giai, *PRC* **83**, 041301(R) (2011)

- SS: the energy degeneracy can be well restored by the 2nd order perturbation calculations.
- PSS: the restoration of the energy degeneracy cannot be restored up to the 3rd order perturbation calculations.

Potentials for the RHO symmetry limit

- Symmetry breaking from (relativistic) harmonic oscillator



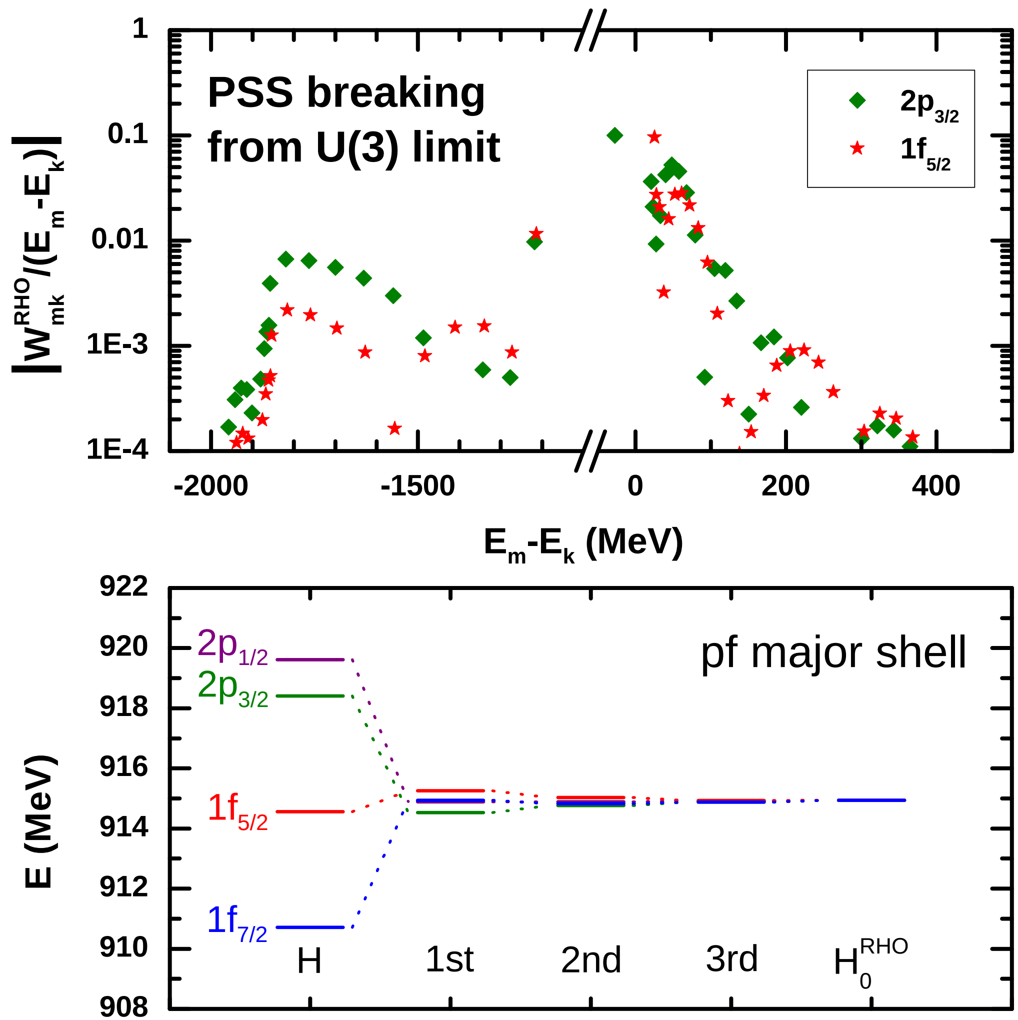
$$
\begin{aligned}
H &= H_0^{\rm RHO} + W^{\rm RHO} \\
&= \begin{pmatrix} \Sigma_{\rm HO} + M & -\frac{d}{dr} + \frac{\kappa}{r} \\ \frac{d}{dr} + \frac{\kappa}{r} & -\Delta_0 - M \end{pmatrix} + \begin{pmatrix} \Sigma - \Sigma_{\rm HO} & 0 \\ 0 & \Delta_0 - \Delta \end{pmatrix}
\end{aligned}
$$

★ single-particle potentials for neutrons in ^{132}Sn

★ solid lines: self-consistent potentials in realistic nucleus obtained by RMF with PK1

★ dashed lines: potentials $\Sigma_{\rm HO} + M$ and $-\Delta_0 - M$ in $H_0^{\rm RHO}$

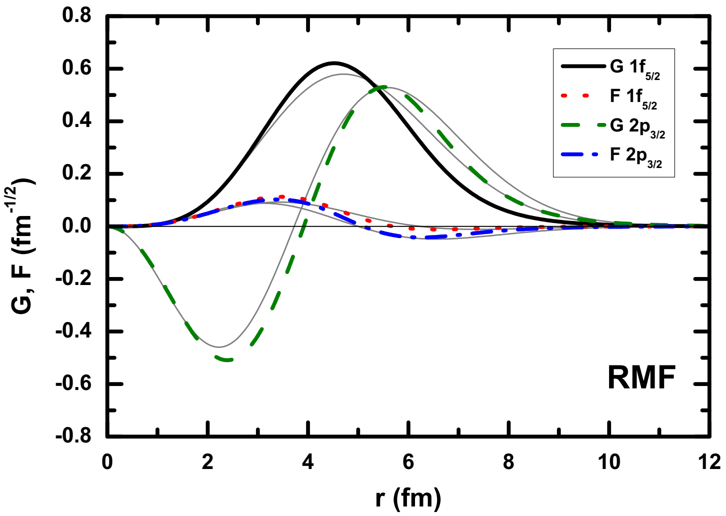
Validity of perturbation theory and restoration of symmetries

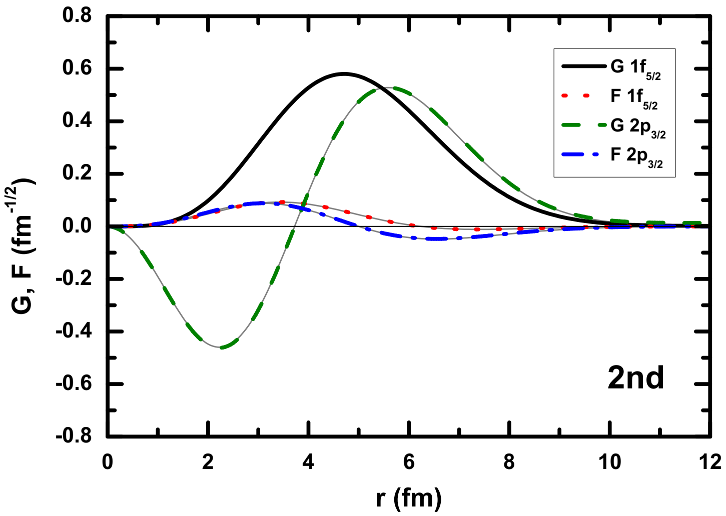


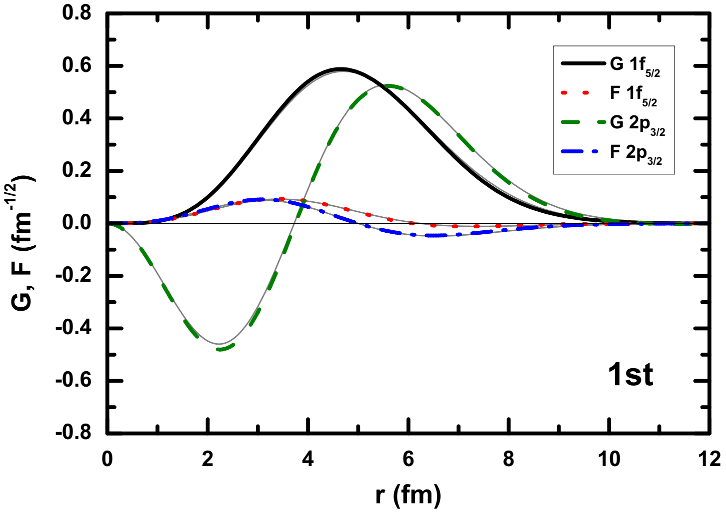
Liang, Zhao, Zhang, Meng, Giai, *PRC* **83**, 041301(R) (2011)

- The biggest perturbations ~ 0.1
- The energy degeneracy can be well restored by the 3rd order perturbation calculations.

Restoration of symmetries in single-particle wave functions







- The wave functions in the RHO symmetry limit can also be reproduced by the 2nd order perturbation calculations.

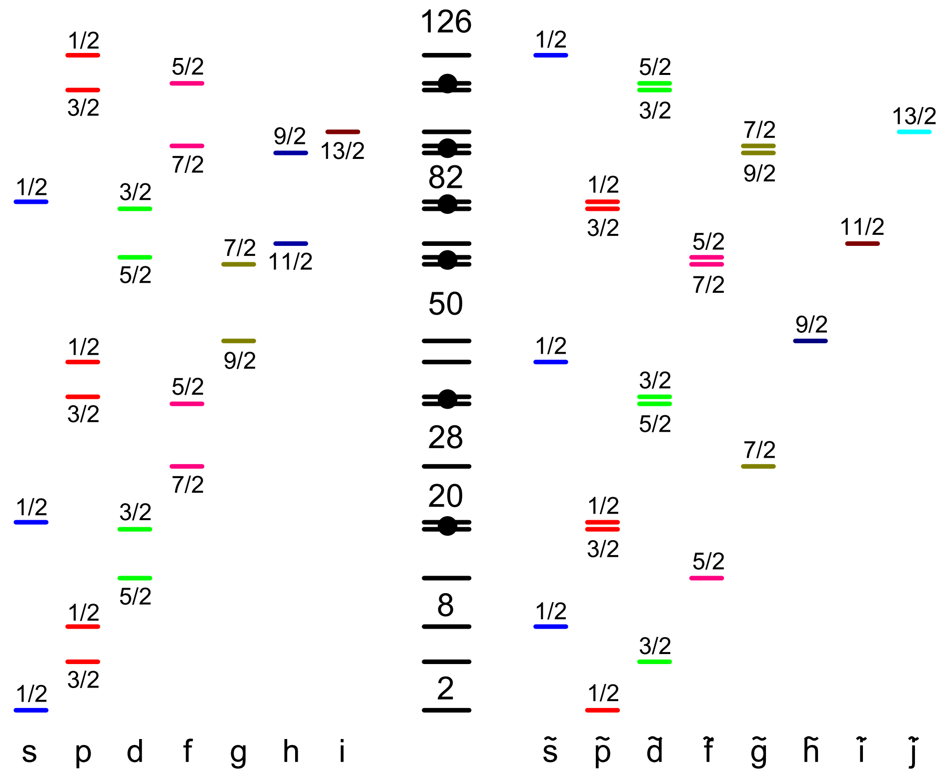
Conclusion

The nature of PSS is indeed perturbative, regarding the Dirac Hamiltonian with RHO potentials as the symmetry limit.

Outline

Intruder states in PSS

- Spin symmetry (SS) breaking, i.e., remarkable spin-orbit splitting in $(n, l, j = l \pm 1/2)$
  Haxel:1949, Mayer:1949
- Pseudospin symmetry (PSS), i.e., near degeneracy in
  $$\begin{cases} (n-1, l+2, j = l+3/2) \\ (n, l, j = l+1/2) \end{cases}$$
  by defining
  $(\tilde{n} = n-1, \tilde{l} = l+1, j = \tilde{l} \pm 1/2)$
  Arima:1969, Hecht:1969



- The intruder states do not have their own pseudospin partners.

Intruder states in PSS

- Spin symmetry (SS) breaking, i.e., remarkable spin-orbit splitting in $(n, l, j = l \pm 1/2)$

  Haxel:1949, Mayer:1949

- Pseudospin symmetry (PSS), i.e., near degeneracy in
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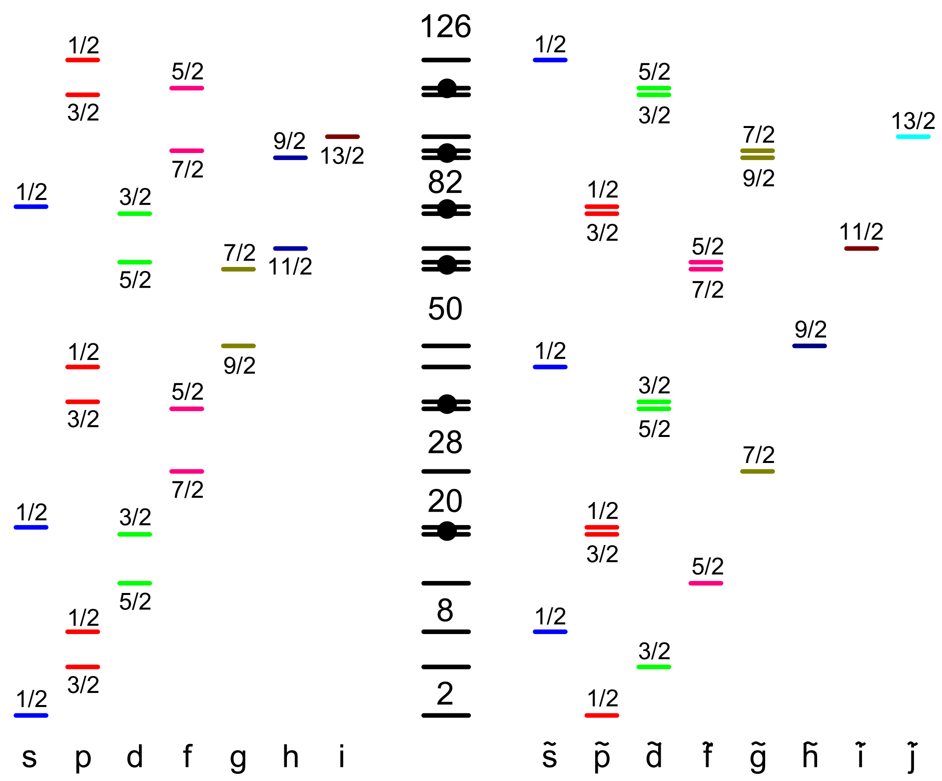


- The intruder states do not have their own pseudospin partners.

⇒ Supersymmetric (SUSY) quantum mechanics

Leviatan, *PRL* **92**, 202501 (2004); Typel, *NPA* **806**, 156 (2008)

SUSY quantum mechanics (I)

- Every second-order Hamiltonian can be factorized in a product of two Hermitian conjugate first-order operators Infeld:1951, Cooper:1995

$$H_1 = B^+ B^-.$$

- The Hermitian operators Q_1 and Q_2 called supercharges read

$$Q_1 = \begin{pmatrix} 0 & B^+ \\ B^- & 0 \end{pmatrix}, \qquad Q_2 = iQ_1\tau = \begin{pmatrix} 0 & -iB^+ \\ iB^- & 0 \end{pmatrix}.$$

- The supersymmetric Hamiltonian

$$H_S = Q_1^2 = Q_2^2 = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

is obtained with the supersymmetric partners

$$H_1 = B^+ B^- \qquad \text{and} \qquad H_2 = B^- B^+.$$

SUSY quantum mechanics (II)

- Since H_S is the square of the Hermitian operators Q_i, all eigenvalues $E_S(n)$ of the eigenvalue equation are non-negative

$$H_S \Psi_S(n) = E_S(n) \Psi_S(n)$$

with the two-component wave function

$$\Psi_S(n) = \begin{pmatrix} \psi_1(n) \\ \psi_2(n) \end{pmatrix}.$$

- H_1 and H_2 have the **same spectrum** of positive energies $E_S(n) > 0$.
- Operators B^+ and B^- connect the components of the wave function by

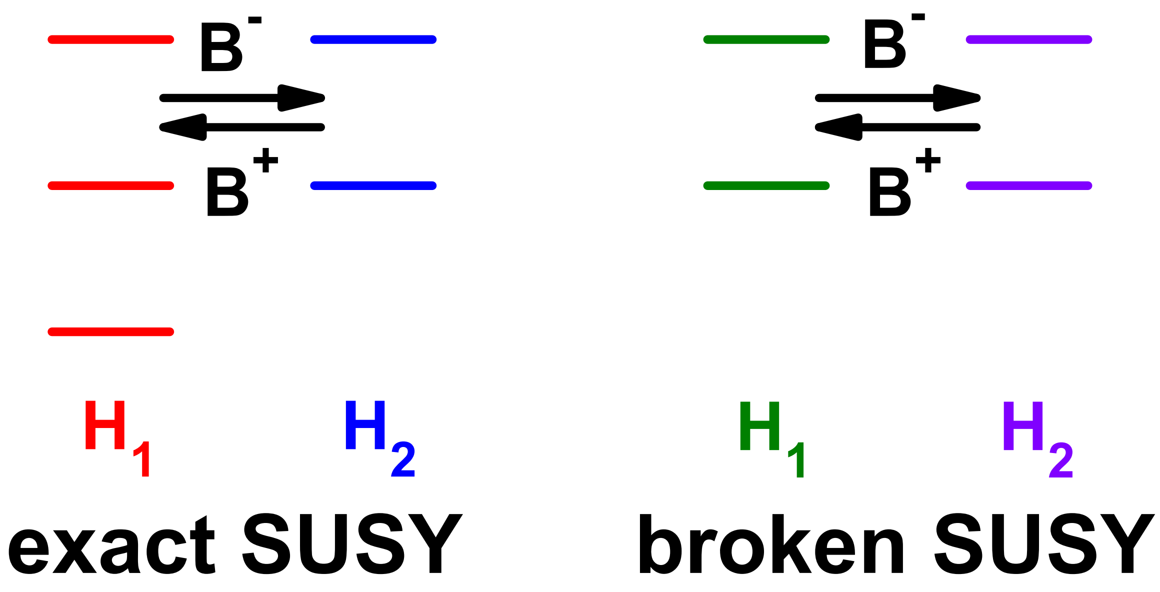
$$\psi_2(n) = \frac{B^-}{\sqrt{E_S(n)}} \psi_1(n), \qquad \psi_1(n) = \frac{B^+}{\sqrt{E_S(n)}} \psi_2(n).$$

SUSY quantum mechanics (III)

- The supersymmetry is called exact if there is an eigenstate $\Psi_S(0)$ with energy $E_S(0) = 0$.
- As usual convention, this ground-state obeys

$$B^- \psi_1(0) = 0, \quad \psi_2(0) = 0,$$

i.e., H_1 has an additional state at zero energy that is not appearing in H_2.



Schrödinger equations without spin-orbit term

- Starting point: Schrödinger equations without spin-orbit term

$$\left[-\frac{1}{2M}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

- For the spherical symmetry,

$$HR_a(r) = E_a R_a(r)$$

with the Hamiltonian and wave functions

$$H = -\frac{d^2}{2Mdr^2} + \frac{\kappa(\kappa+1)}{2Mr^2} + V(r), \qquad \psi_\alpha(\mathbf{r}) = \frac{R_a(r)}{r}\mathscr{Y}^{l}_{jm}(\hat{\mathbf{r}}),$$

where $\kappa = \mp(j+1/2)$ for $j = l \pm 1/2$ as adopted in the relativistic framework.

- H has an explicit spin symmetry (SS).
- To investigate the pseudospin symmetry (PSS) and its breaking, the critical point is to identify the $\tilde{l}(\tilde{l}+1) = \kappa(\kappa-1)$ term.
- One of the promising tricks is the SUSY quantum mechanics. Typel, *NPA* **806**, 156 (2008)

SUSY for Schrödinger equations (I)

- SUSY for Schrödinger equations without spin-orbit term

$$H = -\frac{d^2}{2Mdr^2} + \frac{\kappa(\kappa+1)}{2Mr^2} + V(r)$$

- Two Hermitian conjugate first-order operators

$$B_\kappa^+ = \left[Q_\kappa(r) - \frac{d}{dr}\right]\frac{1}{\sqrt{2M}}, \qquad B_\kappa^- = \frac{1}{\sqrt{2M}}\left[Q_\kappa(r) + \frac{d}{dr}\right],$$

- SUSY partner Hamiltonians

$$\begin{aligned} H_1 &= B_\kappa^+ B_\kappa^- = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + Q_\kappa^2 - Q_\kappa'\right], \\ H_2 &= B_\kappa^- B_\kappa^+ = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + Q_\kappa^2 + Q_\kappa'\right]. \end{aligned}$$

SUSY for Schrödinger equations (II)

- Furthermore, setting the reduced supermomenta

$$q_\kappa(r) = Q_\kappa(r) - \frac{\kappa}{r},$$

so that the SUSY partner Hamiltonians read

$$\begin{aligned} H_1 &= B_\kappa^+ B_\kappa^- = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2} + q_\kappa^2 + \frac{2\kappa}{r} q_\kappa - q_\kappa'\right], \\ H_2 &= B_\kappa^- B_\kappa^+ = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2} + q_\kappa^2 + \frac{2\kappa}{r} q_\kappa + q_\kappa'\right]. \end{aligned}$$

- The centrifugal barrier term $\kappa(\kappa+1)$ leading to SS appears in H_1.
- The pseudo-centrifugal barrier term $\kappa(\kappa-1)$ leading to PSS appears in H_2.

Energy shifts

- H and H_1 are connected by

$$H_1(\kappa) + e(\kappa) = H$$

with the **energy shifts** $e(\kappa)$ to be determined.

- It is equivalent that

$$\frac{1}{2M}\left[q_\kappa^2(r) + \frac{2\kappa}{r}q_\kappa(r) - q'_\kappa(r)\right] + e(\kappa) = V(r),$$

so that $q_\kappa(0) = 0$ and $\lim_{r\to 0} q_\kappa(r) = \frac{2M(e(\kappa)-V)}{(1-2\kappa)}r$ with regular potential $V(r)$.

- **Energy shifts** for PS doublets ($\kappa + \kappa' = 1$)
  - ⋆ For $\kappa < 0$, since the exact SUSY is achieved, it is required $E_1(\kappa) = 0$, i.e.,

$$e(\kappa) = E_{1\kappa}.$$

  - ⋆ For $\kappa > 0$, to fulfill $\lim_{r\to 0} q_\kappa(r) = \lim_{r\to 0} q_{\kappa'}(r)$, it is required Typel:2008

$$e(\kappa) = 2\left.V\right|_{r=0} - e(\kappa').$$

Exact PSS limits

- The **exact PSS limits** indicate $E_{n\kappa_1} = E_{(n-1)\kappa_2}$, it is required

$$H_2(\kappa_1) + e(\kappa_1) = H_2(\kappa_2) + e(\kappa_2),$$

i.e.,

$$\frac{1}{2M}\left[q^2_{\kappa_1}(r) + \frac{2\kappa_1}{r}q_{\kappa_1}(r) + q'_{\kappa_1}(r)\right] + e(\kappa_1) = \frac{1}{2M}\left[q^2_{\kappa_2}(r) + \frac{2\kappa_2}{r}q_{\kappa_2}(r) + q'_{\kappa_2}(r)\right] + e(\kappa_2)$$

$$q'_{\kappa_1}(r) = q'_{\kappa_2}(r)$$

- Since $q_\kappa(0) = 0$, this leads to $q_{\kappa_1}(r) = q_{\kappa_2}(r)$, and finally

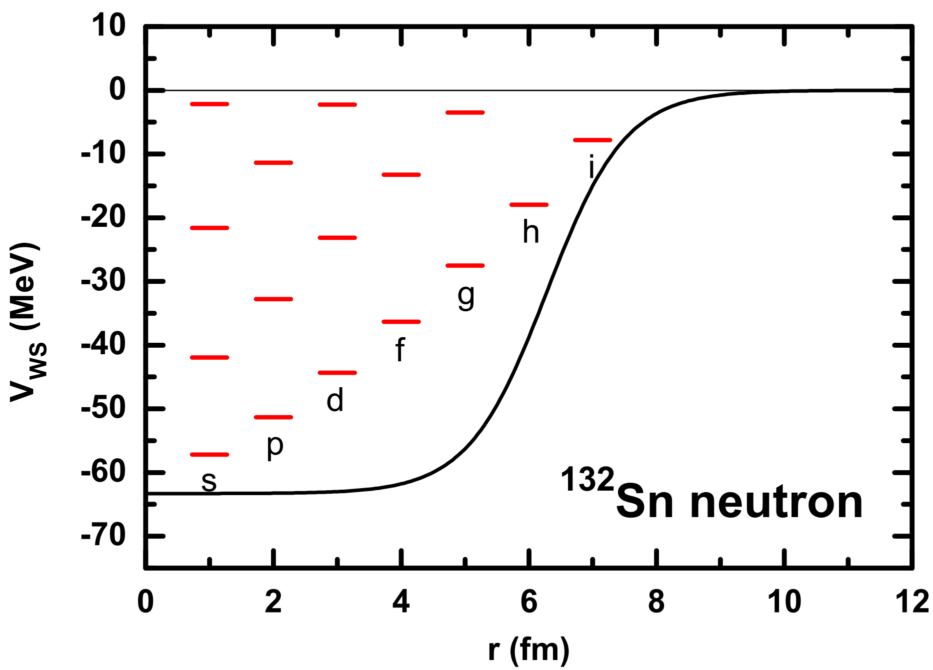
$$q_{\kappa_1}(r) = q_{\kappa_2}(r) = \frac{A}{2}\omega_{\{\kappa_1,\kappa_2\}}r \quad \text{with constants} \quad A \equiv 2M, \omega_{\{\kappa_1,\kappa_2\}} \equiv \frac{e(\kappa_1) - e(\kappa_2)}{\kappa_2 - \kappa_1}.$$

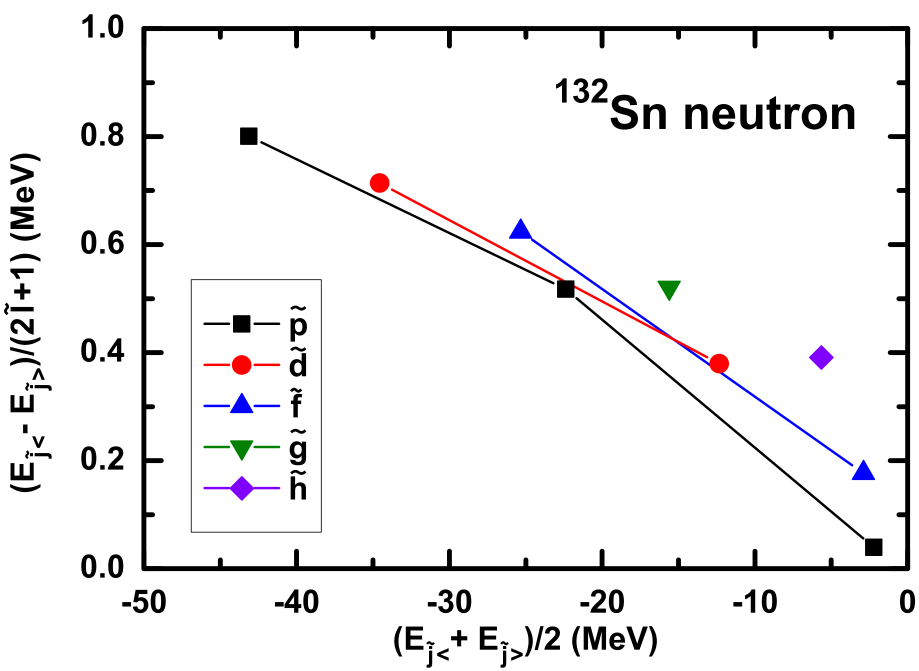
- This indicates the only possible PSS limits in the Schrödinger equations without spin-orbit term are those with harmonic oscillator (HO) potentials

$$V_{\rm HO}(r) = \frac{A}{4}\omega^2_{\{\kappa_1,\kappa_2\}}r^2 + V(0).$$

cf. $H = H_{\rm HO} + v_{ll}\mathbf{l}^2 + v_{ls}\mathbf{l}\cdot\mathbf{s}$; $\tilde{H} = \tilde{H}_{\rm HO} + v_{ll}\tilde{\mathbf{l}}^2 + (4v_{ll} - v_{ls})\tilde{\mathbf{l}}\cdot\tilde{\mathbf{s}}$ Bohr:1982

Single-particle energies and pseudospin-orbit splittings





⋆ Left: Woods-Saxon potential for ^{132}Sn and bound single-neutron energies.

Right: pseudospin-orbit splittings $(E_{\tilde{j}_<} - E_{\tilde{j}_>})/(2\tilde{l}+1)$ *vs* $(E_{\tilde{j}_<} + E_{\tilde{j}_>})/2$.

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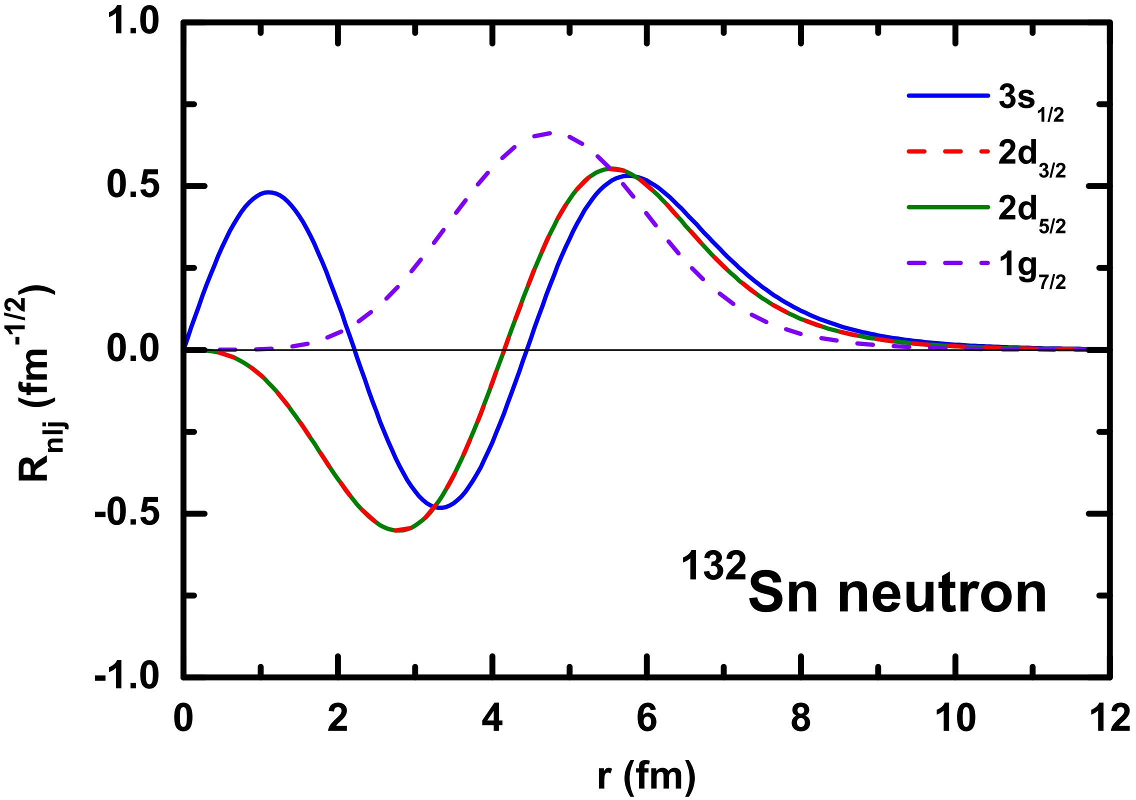
- How to understand the amplitudes of PSS splittings?
- Why do pseudospin-orbit splittings $\Delta E_{\rm PSO}$ decrease as single-particle energies $E_{\rm av}$ increase?

Single-particle wave functions

Normal representation



⋆ Single-particle wave functions of the $3s_{1/2}$, $2d_{3/2}$, $2d_{5/2}$, and $1g_{7/2}$ states.

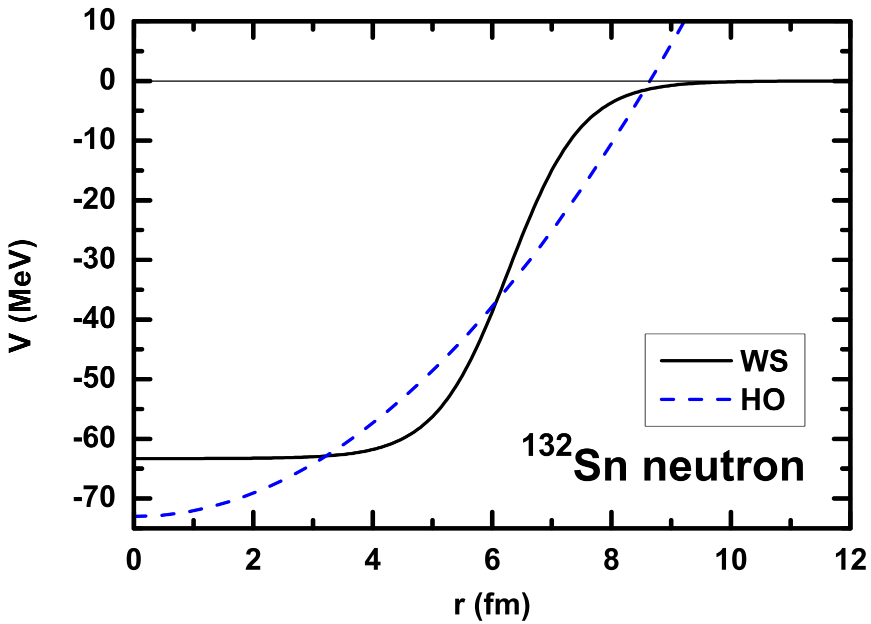
- Wave functions of spin doublets are exact the same since there is no spin-orbit term.
- However, wave functions of the PS doublets are very different to each other, so it is difficult to analyze the origin of PSS and its breaking.

Implicit PSS limit: Schrödinger equations with HO potentials

- Hamiltonian can be divided as

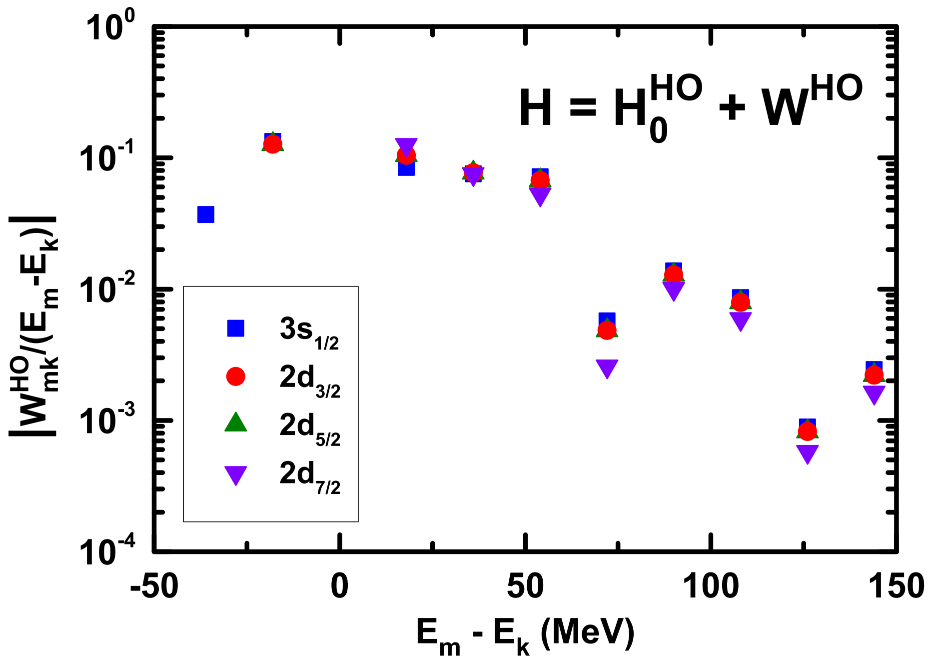
$$H = H_0^{\mathrm{HO}} + W^{\mathrm{HO}}$$

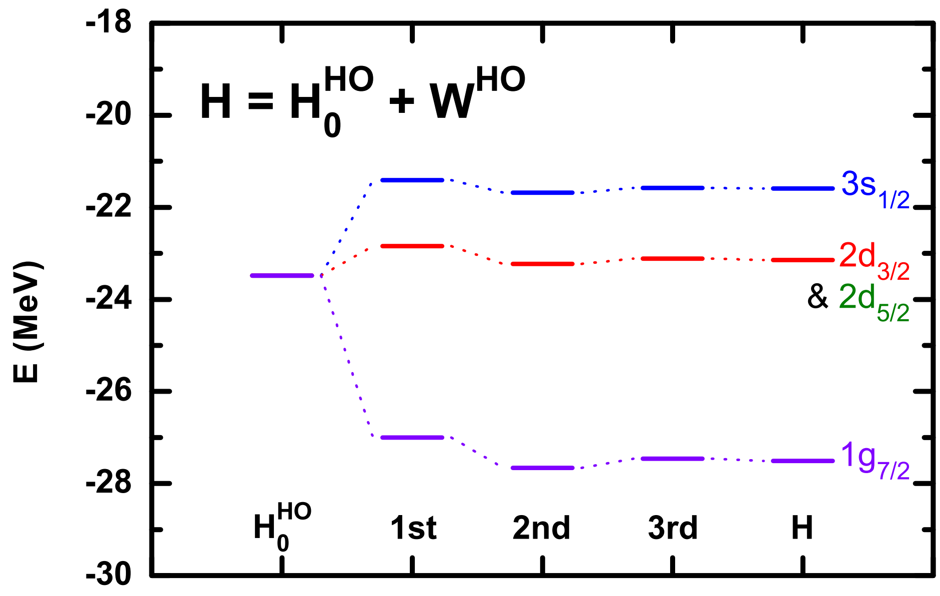
  - ⋆ H_0^{HO} leading to PSS
  - ⋆ W^{HO} symmetry breaking potential



$$H_0^{\mathrm{HO}} = -\frac{1}{2M}\left[\frac{d^2}{dr^2} + \frac{\kappa(\kappa+1)}{r^2}\right] + \frac{A}{4}\omega^2 r^2 + V(0)$$

Validity of perturbation theory and perturbation corrections





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- The biggest perturbations ~ 0.13.
- Pseudospin-orbit splittings are reproduced by the 3rd-order perturbation calculations.

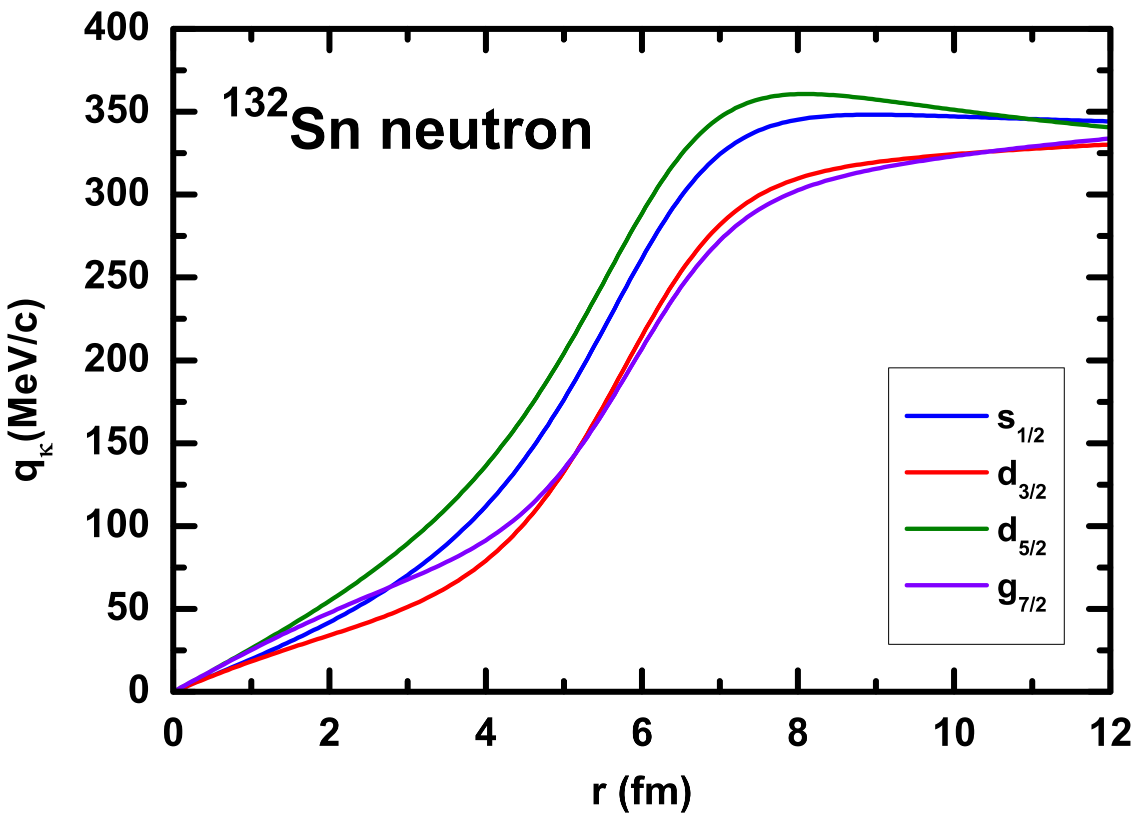
Conclusion

The nature of PSS is perturbative, and its breaking can be understood in such implicit way.

Reduced supermomenta

SUSY representation

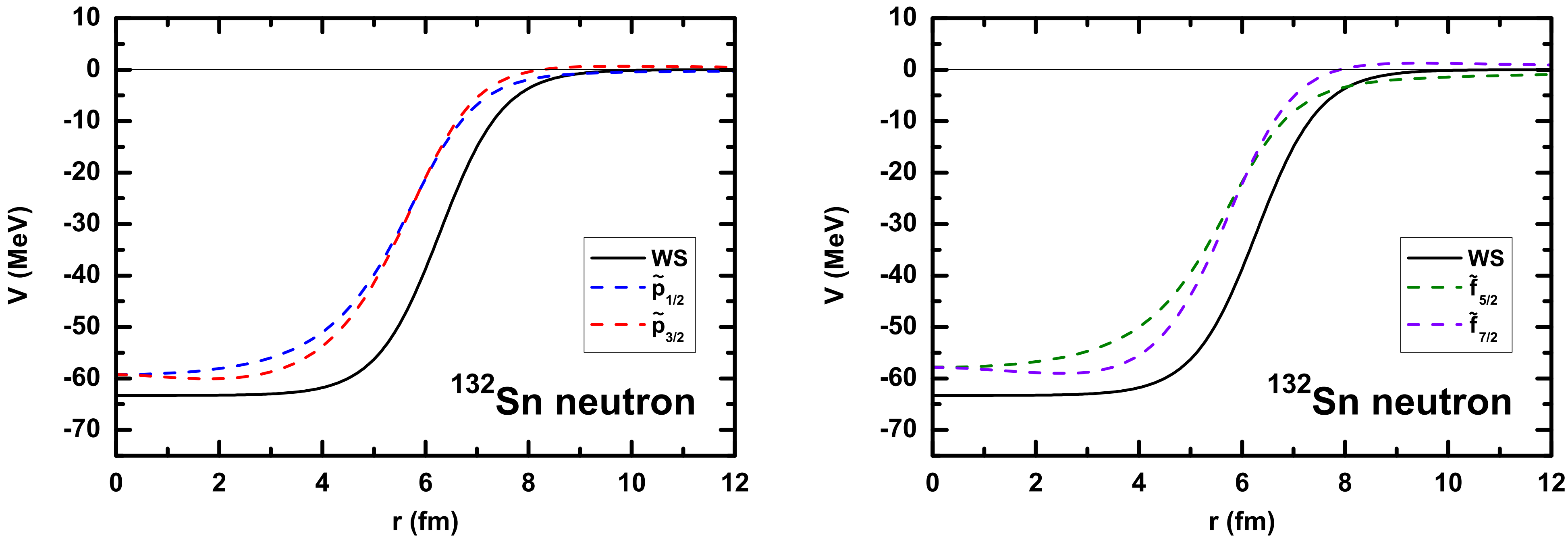
$$\frac{1}{2M}\left[q_\kappa^2(r) + \frac{2\kappa}{r}q_\kappa(r) - q_\kappa'(r)\right] + e(\kappa) = V(r)$$



⋆ Reduced supermomenta $q_\kappa(r)$ for the $s_{1/2}$, $d_{3/2}$, $d_{5/2}$, and $g_{7/2}$ blocks.

- $q_\kappa(r)$ are block-dependent.
- Asymptotic behaviors: $\lim_{r\to 0} q_\kappa(r) = \frac{2M(e(\kappa)-V)}{(1-2\kappa)} r$ and $\lim_{r\to\infty} q_\kappa(r) = \sqrt{-2Me(\kappa)}$.

Central potentials in SUSY partner Hamiltonians



⋆ Central potentials $\tilde{V}_\kappa(r)$ in $\tilde{H} = H_2 + e(\kappa)$ for the $\tilde{p}_{1/2}$, $\tilde{p}_{3/2}$, $\tilde{f}_{5/2}$, and $\tilde{f}_{7/2}$ blocks.

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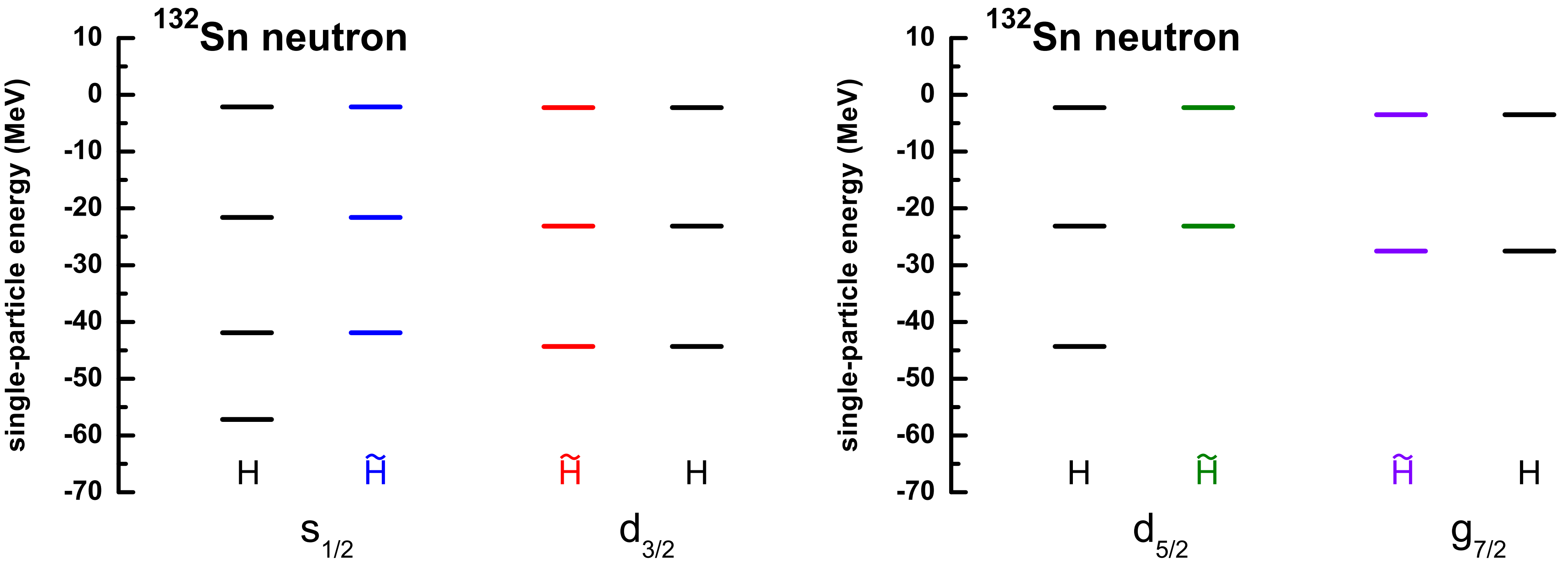
- $\tilde{V}_\kappa(r) = V(r) + q'_\kappa(r)/M$ are regular and block-dependent.
- Asymptotic behaviors: $\lim_{r\to 0} \tilde{V}_\kappa(r) = V + \frac{2(e(\kappa)-V)}{(1-2\kappa)}$ and $\lim_{r\to\infty} \tilde{V}_\kappa(r) = 0$.

Single-particle energies of SUSY Hamiltonians

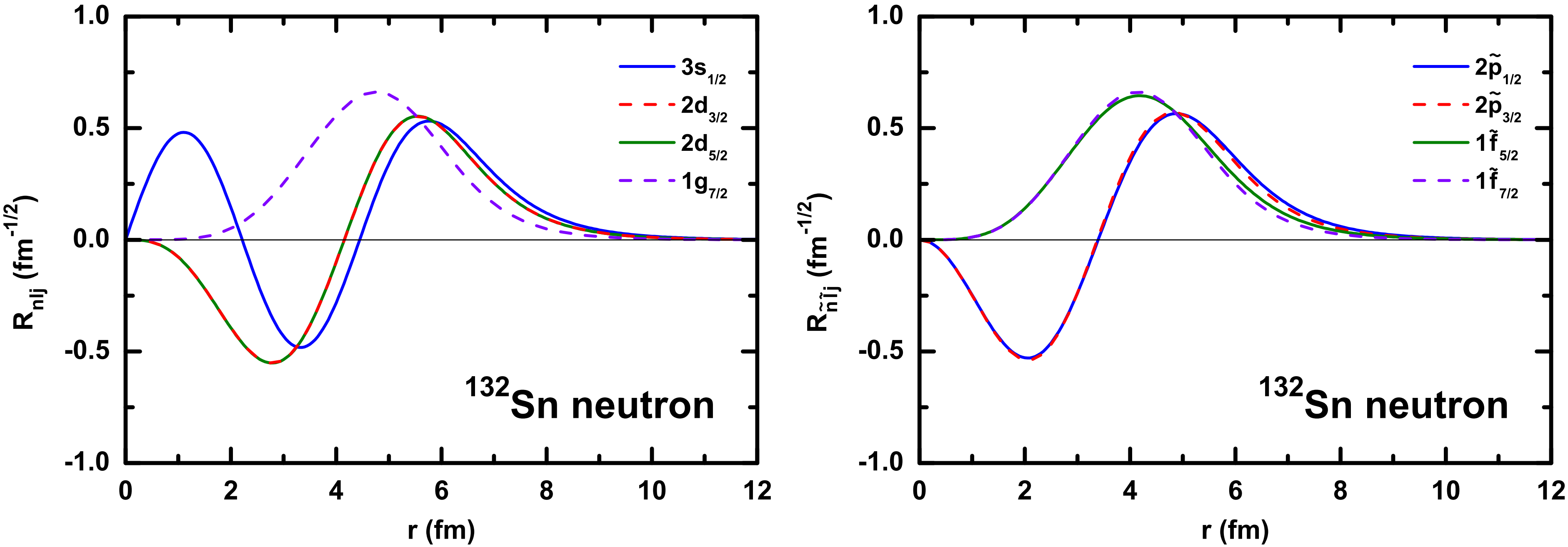


⋆ Single-particle energies of both H and $\tilde{H}$ for the $s_{1/2}$, $d_{3/2}$, $d_{5/2}$, and $g_{7/2}$ blocks.

- H and $\tilde{H}$ have **identical spectra**, expect **an additional eigenstate** with $E_1 = 0$, corresponding to the states without pseudospin partners.
- The pseudospin-orbit splittings $\Delta E_{\rm PSO}$ can be explicitly understood as the splitting appearing in $\tilde{H}$ with the SUSY representation.

Single-particle wave functions in $\tilde{H}$

- Wave function transformation: $\psi_2(n) = \frac{B^-}{\sqrt{E_S(n)}}\psi_1(n)$



⋆ Single-particle wave functions in H and $\tilde{H}$ of the $2\tilde{p}_{1/2}$, $2\tilde{p}_{3/2}$, $1\tilde{f}_{5/2}$, and $1\tilde{f}_{7/2}$ states.

- Single-particle wave functions of PS doublets are almost identical to each other.
- It is a natural result as they are quasi-degenerate.

Explicit PSS limit: symmetry conserving and breaking terms

- SUSY Hamiltonian can be divided as

$$\tilde{H} = \tilde{H}_0^{\rm PSS} + \tilde{W}^{\rm PSS}$$

where

- ⋆ symmetry conserving term

$$\tilde{H}_0^{\rm PSS} = \frac{1}{2M}\left[-\frac{d^2}{dr^2} + \frac{\kappa(\kappa-1)}{r^2}\right] + \tilde{V}_{\rm PSS}$$

- ⋆ symmetry breaking term

$$\tilde{W}^{\rm PSS} = \kappa \tilde{V}_{\rm PSO}$$

- $\tilde{V}_{\rm PSO}(r)$ with amplitudes of ~ 1 MeV are negative inside and positive outside.
- This is why $\Delta E_{\rm PSO}$ decrease as main quantum numbers n increase.

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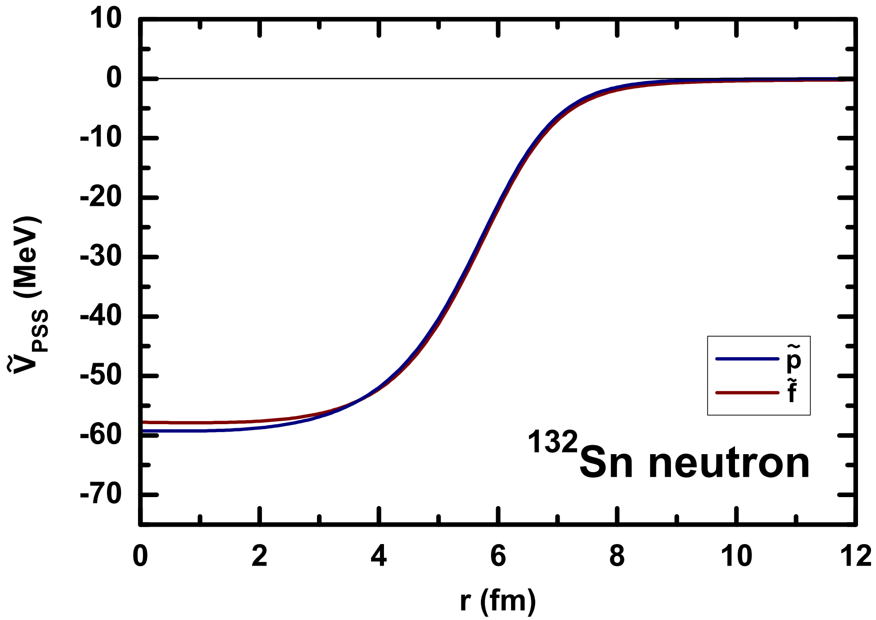
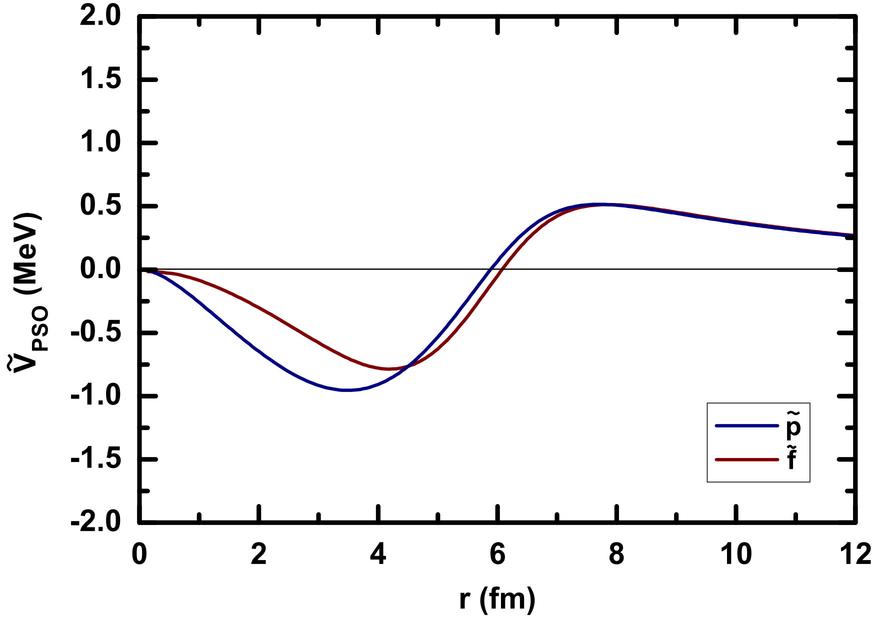
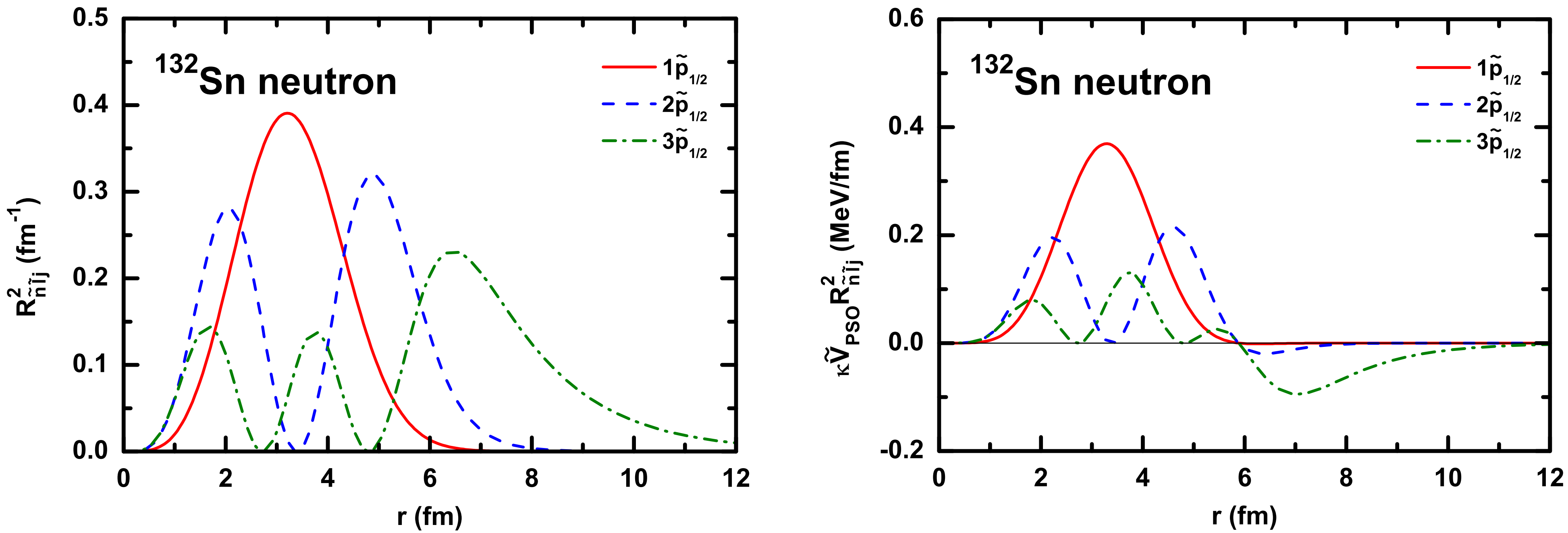




General pattern of PSO splittings

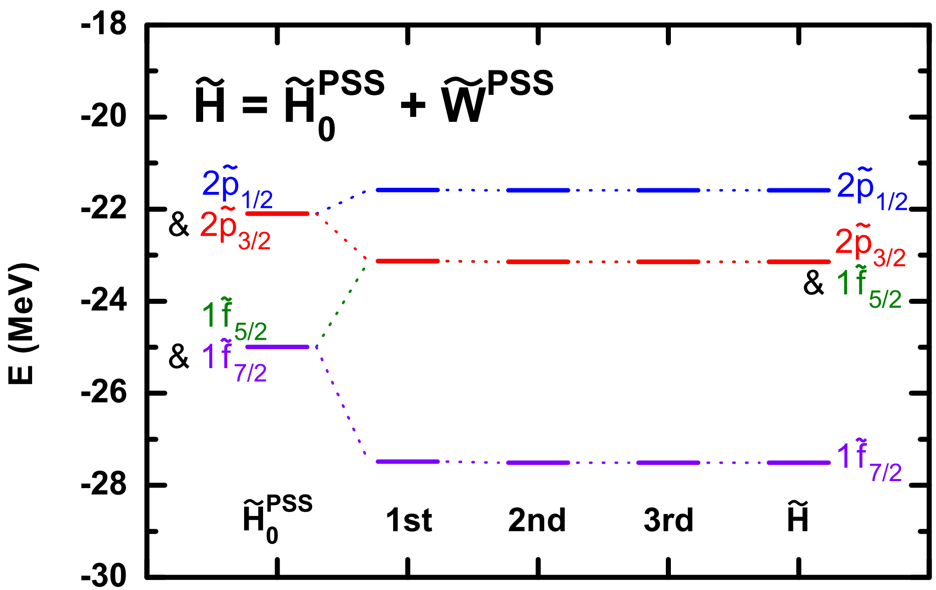


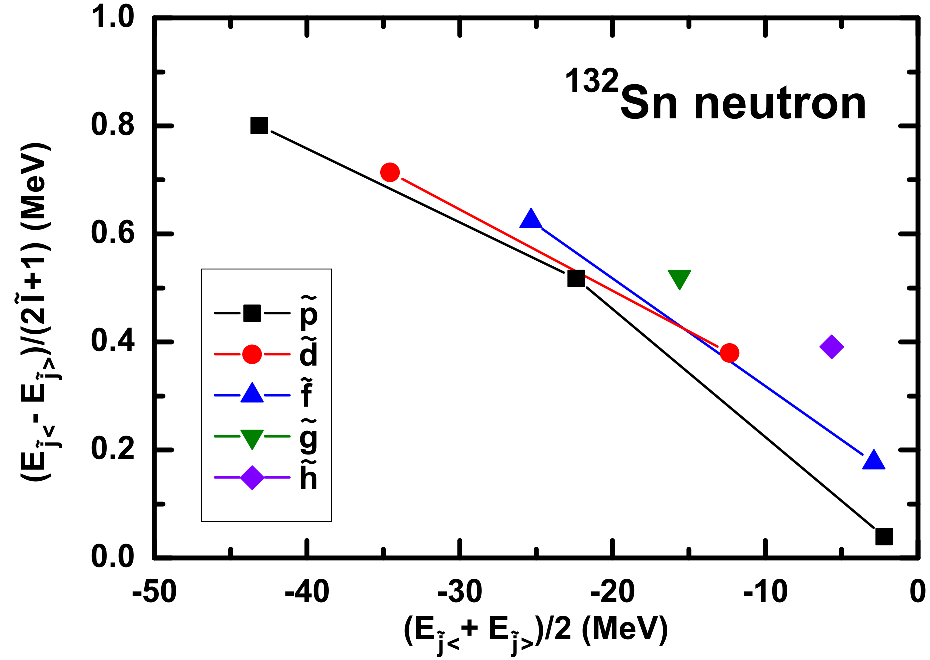
⋆ Left: $\tilde{R}^2(r)$ for the $1\tilde{p}_{1/2}$, $2\tilde{p}_{1/2}$, and $3\tilde{p}_{1/2}$ states.

Right: $\kappa \tilde{V}_{\rm PSO}(r)\tilde{R}^2(r)$ for the $1\tilde{p}_{1/2}$, $2\tilde{p}_{1/2}$, and $3\tilde{p}_{1/2}$ states.

- main quantum numbers n increase $\Rightarrow$ wave functions $\tilde{R}(r)$ move outward $\Rightarrow E_{\rm PSO} = \int \kappa \tilde{V}_{\rm PSO}(r)\tilde{R}^2(r)dr$ decrease

Validity of perturbation theory and perturbation corrections





- Pseudospin-orbit splittings are reproduced by the 1st-order perturbation calculations.

Conclusions

The nature of PSS is perturbative. In the SUSY representation $\tilde{H}$:

- ⋆ Both single-particle energies and wave functions of PS doublets are quasi-degenerate.
- ⋆ $\tilde{W}^{\rm PSS}$ can be explicitly identified.
- ⋆ Shape of $\tilde{W}^{\rm PSS} \Rightarrow \Delta E_{\rm PSO}$ decrease as main quantum numbers n increase

Outline

Summary and Perspectives

Summary

★ We deem it promising to understand PSS and its breaking mechanism in a fully quantitative way by combining the similarity renormalization group (SRG) technique, SUSY quantum mechanics, and perturbation theory.

✓ **SRG**: to transform the Dirac Hamiltonian into a Schrödinger-like form yet keeping all operators Hermitian.

✓ **SUSY**: to identify the PSS conserving and breaking terms naturally; to clarify the reason why the intruder states have no pseudospin partners.

✓ **Perturbation theory**: to understand the behavior of pseudospin-orbit splitting in a quantitative way.

Perspectives

?′ Schrödinger equations with spin-orbit term

?′ Dirac equations and/or Schrödinger-like equations

?′ Why $\Delta E_{\rm PSO} \lesssim \Delta E_{\rm SO}$ in realistic nuclei?

?′ ……

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