Various Phases and Interference in Quantum Mechanics: In memory of late Dr. Akira Tonomura

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Dr. Akira Tonomura at Hitachi Laboratory passed away on May 2, 2012 at the age of 70. As a classmate at University of Tokyo, I would like to give a brief review of his contributions to basic physics and related theoretical issues.

I. Goodbye Akira!

There appeared a short column in the Asahi, a Japanese newspaper, by Mr. Tsuji several days after Tonomura's funeral ceremony in Tokyo. This is about an address of Prof. Yang at the ceremony. In brief,

1. Tonomura made fundamental contributions to quantum physics, due to the advise of Prof. Yang

2. Tonomura's contribution to basic physics, as a corporate physicist, is quite singular in Japan
3. These were the golden days of Japanese production industries
4. Prof. Yang told Tonomura that it takes 10 years to grow trees but it takes 100 years to grow (a community of) scientists
5. Japanese industry is now in a difficult situation, but we need to keep this good tradition.

At University of Tokyo, two students perform experiments by making a pair.

After graduation, we have not met each other for several years. When I was a postdoc at Enrico Fermi Institute in Chicago in 1971, Tonomura came to Chicago to see Prof. Crewe, an expert of electron microscope. Tonomura was an expert of a *needle* of the electron gun.

Again after several years of gap, in "1982", Tonomura sent me the copy of his paper together with copies of the referee reports he received from Phys. Rev. Lett. on his first experiment on Aharonov-Bohm effect.

One of the referee reports said that "There is *no* Aharonov-Bohm effect as such, thus it is meaningless to test it by experiment".

Apparently, Tonomura was in a difficult situation.

I was aware of a paper by Profs. Yang and T.T. Wu on "possible non-Abelian generalization of the Aharonov-Bohm effect", PRD12 (1975) 3845. I was thus confident that Prof. Yang believes in the Aharonov-Bohm effect. I suggested Tonomura to get contact with Prof. Yang. Apparently, Tonomura did and Tonomura had a chance to talk to Prof. Yang quite soon later, when Prof. Yang visited Tokyo in "1982".

Since then Tonomura received the advise of Prof. Yang on various aspects of the fundamental physics related to gauge fields and phases in quantum physics in general.

I was quite happy that I could be helpful to Tonomura at the very beginning of his adventure into basic physics with his technology of electron microscope. He was very successful indeed.

His major achievements include

1. Confirmation of the Aharonov-Bohm effect without any doubt. Very few people doubt the effect nowadays, in contrast to back "1982".

2. Very beautiful (one of 10 most beautiful experiments in the history of physics) experiment of the electron intereference through a double slit.

3. Observation of the movement of the magnetic vortices in the superconductor

II. Various Phases in Quantum Physics

I would like to talk on subjects related to Tonomura's experiments on two phases,

1. Double-slit experiment related to "geometric phases",

2. Aharonov-Bohm effect related to gauge fields,

3. Anomalies.

We illustrate the use of the second quantization with the action

$$S = \int dtd^3x \left[\hat{\psi}^\dagger(t.\vec{x}) \Big(i\hbar \frac{\partial}{\partial t} - \hat{H}(t) \Big) \hat{\psi}(t, \vec{x}) \right]$$

for a time-dependent Hamiltonian $\hat{H}(t)$. We then expand

$$\hat{\psi}(t, \vec{x}) = \sum_n \hat{c}_n(t) v_n(t, \vec{x})$$

$$\int d^3x v_n^\star(t,\vec{x}) v_m(t,\vec{x}) = \delta_{n,m}.$$

For the fermion, we impose anti-commutator

$$\left\{\hat{c}_l(t), \hat{c}_m^\dagger(t)\right\} = \delta_{lm}$$

The Fock states are defined by

$$|l\rangle = \hat{c}_l^\dagger(0)|0\rangle$$

By inserting the expansion into the action S,

$$
\begin{aligned}
S = & \int dt d^3x \{ \sum_n \hat{c}_n^\dagger(t) i\hbar\partial_t \hat{c}_n(t) \\
& - \sum_{n,m} [v_n^\star(t,\vec{x}) \hat{H}(t) v_m(t,\vec{x}) \\
& - v_n^\star(t,\vec{x}) i\hbar\partial_t v_m(t,\vec{x})] \hat{c}_n^\dagger(t) \hat{c}_m(t) \}
\end{aligned}
$$

Thus the appearance of "geometric phase" is automatic.

The solution of the conventional Schrödinger equation with the initial condition $\psi(0,\vec{x}) = v_n(0,\vec{x})$ is given by

$$\psi_n(t,\vec{x}) = \langle 0|\hat{\psi}(t,\vec{x})\hat{c}_n^{\dagger}(0)|0\rangle$$

This second quantized formulation contains the following *gauge (or redundant) freedom*

$$\hat{c}_n(t) \to e^{-i\alpha_n(t)}\hat{c}_n(t), \quad v_n(t) \to e^{i\alpha_n(t)}v_n(t)$$

where the phase freedom $\{\alpha_n(t)\}$ are arbitrary functions of time.

Under this *hidden gauge transformation*

$$\psi_n(t,\vec{x}) = \langle 0|\hat{\psi}(t,\vec{x})\hat{c}_n^\dagger(0)|0\rangle \to e^{i\alpha_n(0)}\psi_n(t,\vec{x}).$$

Ray representation of the state vector.

What is the physical implication of this hidden gauge symmetry?

The answer is *"it controls all the geometric phases, either adiabatic or non-adiabatic"*.

In the analysis of geometric phases, it is crucial that the combination

$$\psi_n^\star(0, \vec{x})\psi_n(t, \vec{x})$$

is manifestly gauge invariant.

Exactly solvable example:

Motion of a spin inside the rotating magnetic field

$$\mathbf{B}(t) = B\Big(\sin\theta\cos\varphi(t), \sin\theta\sin\varphi(t), \cos\theta\Big)$$

and $\varphi(t) = \omega_0 t$ with a constant ω_0.
Action is written as

$$S = \int dt \left[\hat{\psi}^\dagger(t)\Big(i\hbar\frac{\partial}{\partial t} + \mathbf{B}\cdot\sigma/2\Big)\hat{\psi}(t)\right],$$

with σ Pauli matrix.

Field operator

$$\hat{\psi}(t, \vec{x}) = \sum_{l=\pm} \hat{c}_l(t) w_l(t)$$

with the anti-commutation relation, $\left\{\hat{c}_l(t), \hat{c}_m^\dagger(t)\right\} = \delta_{lm}$.

The effective Hamiltonian for the isolated spin system is exactly diagonalized.

$$w_+(t) = \begin{pmatrix} e^{-i\varphi(t)} \cos\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2} \end{pmatrix}, \; w_-(t) = \begin{pmatrix} e^{-i\varphi(t)} \sin\frac{\vartheta}{2} \\ -\cos\frac{\vartheta}{2} \end{pmatrix}$$

with $\vartheta = \theta - \theta_0$ and the constant parameter θ_0 defined by

$$\tan\theta_0 = \frac{\hbar\omega_0 \sin\theta}{B + \hbar\omega_0 \cos\theta}. \qquad (1)$$

Effective Hamiltonian

$$\hat{\mathcal{H}}_{\text{eff}}(t) \equiv \sum_l E_l \hat{c}_l^\dagger(0)\hat{c}_l(0)$$

with time-independent effective energy eigenvalues

$$\begin{aligned} E_\pm &= w_\pm^\dagger(t')\big(\hat{H} - i\hbar\partial_{t'}\big)w_\pm(t') \\ &= \mp\frac{1}{2}B\cos\theta_0 - \frac{1}{2}\hbar\omega_0\big[1 \pm \cos(\theta - \theta_0)\big]. \end{aligned} \tag{2}$$

The *exact* solution is given

$$
\begin{aligned}
& \psi_\pm(t) \\
= \ & \langle 0|\hat{\psi}(t)\hat{c}^\dagger_\pm(0)|0\rangle \\
= \ & w_\pm(t)\exp\left[-\frac{i}{\hbar}\int_0^t dt' w^\dagger_\pm(t')\big(\hat{H} - i\hbar\partial_{t'}\big)w_\pm(t')\right], \\
= \ & w_\pm(t)\exp\left[-\frac{i}{\hbar}E_\pm t\right], \qquad (3)
\end{aligned}
$$

where the exponent has been calculated in Eq.(2).

$w_\pm(T) = w_\pm(0)$ with the period $T = 2\pi/\omega_0$, and the solution is *cyclic* (namely, periodic up to a phase freedom) and, as an exact solution, it is applicable to the *non-adiabatic* case also.

For an arbitrary time-dependent $\mathbf{B}(t)$, *any exact* solution of the Schrödinger equation can be written in the form of Eq.(3) , if one chooses basis vectors $w_\pm(t)$ suitably.

Adiabatic limit $|\hbar\omega_0/B| \ll 1$: $\theta_0 \to 0$.

$$\begin{aligned}
\psi_\pm(t) &= w_\pm(t)\exp\left[-\frac{i}{\hbar}[-\frac{1}{2}\hbar\omega_0\big(1\pm\cos(\theta-\theta_0)\big)t\right] \\
&\quad \times \exp\left[-\frac{i}{\hbar}[\mp\frac{1}{2}B\cos\theta_0]t\right] \qquad (4)\\
&\Rightarrow w_\pm(t)\exp\left[\frac{i}{2}[\omega_0\big(1\pm\cos\theta\big)t\right] \\
&\quad \times \exp\left[-\frac{i}{\hbar}[\mp\frac{1}{2}B]t\right]
\end{aligned}$$

where the first phase factor is called *geometric phase*, and the second phase factor as *dynamical phase.*

The conventional "Berry's phase"

$$\exp\left[i\pi(1 \pm \cos\theta)\right] \tag{5}$$

is recovered after one cycle $t = T = 2\pi/\omega_0$ of motion.

This Berry's phase is known to have an *approximately topological meaning* as the phase generated by a magnetic monopole located at the origin of the parameter space $\mathbf{B}$.

Note that the dynamical phase in (4) vanishes at $\mathbf{B} = 0$, namely, the *level crossing* appears in the conventional adiabatic approximation.

In the generic case with period $T = \frac{2\pi}{\omega_0}$, one can measure $\psi_+^\dagger(0)\psi_+(T)$ by the interference

$$
\begin{aligned}
|\psi_+(T) + \psi_+(0)|^2 &= 2|\psi_+(0)|^2 + 2\mathrm{Re}\psi_+^\dagger(0)\psi_+(T) \\
&= 2 + 2\cos[(\mu B \cos\theta_0)T - \frac{1}{2}\Omega_+]
\end{aligned}
$$

where

$$\Omega_+ = 2\pi[1 - \cos(\theta - \theta_0)]$$

stands for the solid angle drawn by $w_+^\dagger(t)\vec{\sigma}w_+(t)$.

Non-adiabatic limit $|\hbar\omega_0/B| \gg 1$:

$\theta_0 \to \theta$ in Eq.(1) so that the geometric phase vanishes
$\exp\left[-\frac{i}{\hbar}[-\frac{1}{2}\hbar\omega_0\big(1 \pm \cos(\theta - \theta_0)\big)\frac{2\pi}{\omega_0}\right] = 1.$

Namely, the *adiabatic Berry's phase is smoothly connected to the trivial phase inside the exact solution* and thus the topology of Berry's phase is trivial.

In our *unified formulation* of adiabatic and non-adiabatic phases, we can analyze a transitional region from the adiabatic limit to the non-adiabatic region in a reliable way, which was not possible in the past formulation.

Cf. Majorana in 1930s.

Gauge invariance of geometric phases.

$$\psi_l^\dagger(0)\psi_l(t)$$

is manifestly gauge invariant. Its phase becomes

$$\beta_l = \arg\left\{ w_l^\dagger(0) w_l(T) \exp\left[i \int_0^T dt w_l^\dagger(t) i\partial_t w_l(t) \right] \right\},$$

(

which is also manifestly gauge invariant.

This β_l is shown to be the *holonomy* of the basis vector associated with the exact hidden local symmetry, $e^{i\alpha(t)} w_n$, for all geometric phases, either adiabatic or non-adiabatic.

This is based on parallel transport condition

$$\int d^3x w_n^\dagger(\vec{x},t)\partial_t w_n(\vec{x},t) = 0.$$

Comparison with the *conventional formulation* in the *projective Hilbert space* with the equivalence class (Aharonov-Anandan)

$$\{e^{i\alpha(t)}\psi(t)\}$$

where $\psi(t)$ stands for Schrödinger amplitude. This equivalence class or gauge symmetry $\psi(t) \to e^{i\alpha(t)}\psi(t)$ is not a symmetry of the Schrödinger equation $i\hbar\partial_t\psi(t) = \hat{H}(t)\psi(t)$.

As a consequence, the gauge invariant non-adiabatic phase on the basis of the projective Hilbert space

$$\beta = \arg\{\psi^\dagger(0)\psi(T)\exp[i\int_0^T dt\psi^\dagger(t)i\partial_t\psi(t)]\} \tag{7}$$

is *non-local and non-linear* in the Schrödinger amplitude $\psi(t)$, and thus consistency with the superposition principle is not obvious.

In contrast, our β_l in Eq. (6), which numerically agrees with Aharonov-Anandan's β in (7) when one uses the exact solution Eq. (8) in the definition of β, is *bi-linear* in the Schrödinger amplitude and thus consistency with the superposition principle is manifest.

Comparison with Aharonov-Bohm effect

Some similarities between the Aharonov-Bohm effect and the adiabatic Berry's phase. However, there is a fundamental difference. The topology of Berry's phase is valid only in the ideal adiabatic limit and it is lost once one moves away from ideal adiabaticity.

On the other hand, the topology of the Aharonov-Bohm effect is provided by the external boundary condition for the gauge field,

$$\langle \vec{x}_f, T | \vec{x}_i, 0 \rangle$$
$$= \int \mathcal{D}\vec{x} \exp\{\frac{i}{\hbar} \int_0^T [\frac{m\dot{\vec{x}}^2}{2} - e\vec{A}(\vec{x})\frac{d\vec{x}}{dt}]dt\}$$

The Aharonov-Bohm phase is precise for the non-adiabatic as well as adiabatic motion of the electron.

Geometric phase,

$$
\begin{aligned}
& \psi_\pm(t) \\
= \; & w_\pm(t) \exp\left[-\frac{i}{\hbar} \int_0^t dt' w_\pm^\dagger(t') \big(\hat{H} - i\hbar \partial_{t'} \big) w_\pm(t') \right] \\
= \; & w_\pm(t) \exp\left[-\frac{i}{\hbar} \int_0^t dt' w_\pm^\dagger(t') \hat{H} w_\pm(t') \right] \\
\times \; & \exp\left[-\frac{i}{\hbar} \int_0^t dt' [w_\pm^\dagger(t')(-i\hbar \frac{\partial}{\partial \vec{B}}) w_\pm(t')] \frac{d\vec{B}}{dt'} dt' \right],
\end{aligned}
$$

where the last term gives an analogue of gauge potential.

Tetative conclusion so far:

Second quantization

$$\hat{\psi}(t, \vec{x}) = \sum_n \hat{c}_n(t) v_n(t, \vec{x})$$

induces a "hidden local gauge symmetry"

$$\hat{c}_n(t) \to e^{-i\alpha_n(t)} \hat{c}_n(t), \quad v_n(t) \to e^{i\alpha_n(t)} v_n(t)$$

and that this hidden symmetry defines the parallel transport and holonomy associated with geometric phases.

This approach allows a unified treatment of all the known geometric phases, either adiabatic or non-adiabatic, and thus one can analyze the transitional region from adiabatic to non-adiabatic phases in a reliable way.

One then recognizes that the topology of the adiabatic Berry's phase is actually trivial.

We have also emphasized the basic difference between Berry's phase (topology is approximate) and Aharonov-Bohm phase (topology is exact).

Anomaly (quantum breaking of Noether theorem): simplest example

$$\mathcal{L} = \bar{\psi}(x)[i\gamma^\mu(\partial_\mu - ieQA_\mu) - mU(\pi)]\psi(x)$$
$$+ \frac{f_\pi^2}{16}\mathrm{Tr}\partial_\mu U(\pi)\partial^\mu U(\pi)^\dagger$$

where

$$U(\pi) = e^{2i(1/f_\pi)\gamma_5\pi^a(x)T^a},$$

$$\psi(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix},$$

We now perform a field-dependent unitary transformation

$$\psi(x) = V(\pi)\psi'(x), \qquad \bar{\psi}(x) = \bar{\psi}'(x)V(\pi)$$
$$V(\pi) = e^{-i(1/f_\pi)\gamma_5\pi^a(x)T^a}.$$

One then obtains the result

$$\int \mathcal{D}U(\pi)\mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\{i\int d^4x\mathcal{L}\}$$
$$= \int \mathcal{D}U(\pi)\mathcal{D}\bar{\psi}'\mathcal{D}\psi' \exp\{i\int d^4x[\mathcal{L}' + \mathcal{L}_{\mathrm{W-Z}}]\}$$

where

$$\mathcal{L}' = \bar{\psi}'(x)[i\gamma^\mu(\partial_\mu - ieQA_\mu + V^\dagger(\pi)D_\mu V(\pi)) \\ -m]\psi'(x) \\ +\frac{f_\pi^2}{16}\mathrm{Tr}\partial_\mu U(\pi)\partial^\mu U(\pi)^\dagger$$

with

$$D_\mu V(\pi) = \partial_\mu V(\pi) - ie[QA_\mu, V(\pi)]$$

The Jacobian (anomaly)

$$\begin{aligned}
\mathcal{D}\bar{\psi}\mathcal{D}\psi &= J\mathcal{D}\bar{\psi}'\mathcal{D}\psi', \\
\ln J &= i\int d^4x \mathcal{L}_{\mathrm{Wess-Zumino}} \\
&= i\int d^4x \frac{1}{f_\pi}\pi^0(x)\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} + \ldots
\end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

which correctly describes the decay $\pi^0 \to \gamma + \gamma$.

This gives the simplest example of chiral anomaly.

Review with references:

K. Fujikawa," Geometric phases and hidden gauge symmetry", Bulletin of Asia-Pacific Center for Theoretical Physics (APCTP),23-24 (2009) 29. arXiv:0910.0396 [quant-ph]