Memo for Woods-Saxon form factor

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2006.11.22; rev.1 2007.7.30

1 Spherical W-S form factor

$$
\begin{aligned}
f_0(r) &= \frac{1}{1+\exp\{(r-R)/a\}} \\
g_0(r) &= \frac{\exp\{(r-R)/a\}}{[1+\exp\{(r-R)/a\}]^2} = -a\frac{d}{dr}f_0(r)
\end{aligned}
$$

1.1 Integrals of $r^n f_0(r)$ and $r^n g_0(r)$

$$
\begin{aligned}
I_n[f_0] &= \int_0^\infty dr\ r^n f_0(r) = \frac{1}{(n+1)a}\int_0^\infty dr\ r^{n+1} g_0(r) = \frac{1}{(n+1)a}\ I_{n+1}[g_0] \\
I_n[g_0] &= \int_0^\infty dr\ r^n g_0(r) = \int_{-\infty}^\infty dr\ r^n g_0(r) - \int_{-\infty}^0 dr\ r^n g_0(r) \\
&\equiv J_1(n) - J_2(n) \simeq J_1(n) \\
J_1(n) &= a\ \int_{-\infty}^\infty du\ \ (R+au)^n\ \frac{e^u}{(1+e^u)^2} = aR^n\ \sum_{m=0}^{[n/2]}\frac{n!}{(2m)!\,(n-2m)!}\left(\frac{a}{R}\right)^{2m} A_m \\
A_m &= \int_{-\infty}^\infty du\ \frac{u^{2m}e^u}{(1+e^u)^2}\ , \\
J_2(n) &= -\sum_{m=1}^\infty (-)^m m\int_{-\infty}^0 dr\ r^n\exp\left(\frac{m(r-R)}{a}\right) \\
&= (-a)^{n+1}\ n!\sum_{m=1}^\infty (-)^m\frac{\exp\left(-\frac{mR}{a}\right)}{m^n}
\end{aligned}
$$

m	0	1	2	3	4
A_m	1	$\frac{1}{3}\pi^2$	$\frac{7}{15}\pi^4$	$\frac{31}{21}\pi^6$	$\frac{127}{15}\pi^8$

n	$I_n[f_0]$	$I_n[g_0]$
1	$\frac{R^2}{2}\left[1+\frac{1}{3}\left(\frac{\pi a}{R}\right)^2\right]$	aR
2	$\frac{R^3}{3}\left[1+\left(\frac{\pi a}{R}\right)^2\right]$	$aR^2\left[1+\frac{1}{3}\left(\frac{\pi a}{R}\right)^2\right]$
3	$\frac{R^4}{4}\left[1+2\left(\frac{\pi a}{R}\right)^2+\frac{7}{15}\left(\frac{\pi a}{R}\right)^4\right]$	$aR^3\left[1+\left(\frac{\pi a}{R}\right)^2\right]$
4	$\frac{R^5}{5}\left[1+\frac{10}{3}\left(\frac{\pi a}{R}\right)^2+\frac{7}{3}\left(\frac{\pi a}{R}\right)^4\right]$	$aR^4\left[1+2\left(\frac{\pi a}{R}\right)^2+\frac{7}{15}\left(\frac{\pi a}{R}\right)^4\right]$
5	$\frac{R^6}{6}\left[1+5\left(\frac{\pi a}{R}\right)^2+7\left(\frac{\pi a}{R}\right)^4+\frac{31}{21}\left(\frac{\pi a}{R}\right)^6\right]$	$aR^5\left[1+\frac{10}{3}\left(\frac{\pi a}{R}\right)^2+\frac{7}{3}\left(\frac{\pi a}{R}\right)^4\right]$
6	$\frac{R^7}{7}\left[1+7\left(\frac{\pi a}{R}\right)^2+\frac{49}{3}\left(\frac{\pi a}{R}\right)^4+\frac{31}{3}\left(\frac{\pi a}{R}\right)^6\right]$	$aR^6\left[1+5\left(\frac{\pi a}{R}\right)^2+7\left(\frac{\pi a}{R}\right)^4+\frac{31}{21}\left(\frac{\pi a}{R}\right)^6\right]$

1.2 Volume integral and moments

$$\begin{aligned}
\langle f_0 \rangle &= \int d\tau \ f_0(r) = 4\pi \int dr \ r^2 f_0(r) \simeq \frac{4\pi}{3} R^3 \left[1 + \left(\frac{\pi a}{R}\right)^2\right] \\
\left\langle r^2 f_0 \right\rangle &= 4\pi \int dr \ r^2 f_0(r) \\
&\simeq \frac{4\pi}{5} R^5 \left[1 + \frac{10}{3}\left(\frac{\pi a}{R}\right)^2 + \frac{7}{3}\left(\frac{\pi a}{R}\right)^4\right] \\
\left\langle r^2 \right\rangle_{f_0} &\simeq \frac{3}{5} R^2 \left[1 + \frac{7}{3}\left(\frac{\pi a}{R}\right)^2\right] \\
\langle g_0 \rangle &= \int d\tau \ g_0(r) = 4\pi \int dr \ r^2 g_0(r) \simeq 4\pi a R^2 \left[1 + \frac{1}{3}\left(\frac{\pi a}{R}\right)^2\right] \\
\left\langle r^2 g_0 \right\rangle &= 4\pi \int dr \ r^2 g_0(r) \\
&\simeq 4\pi a R^4 \left[1 + 2\left(\frac{\pi a}{R}\right)^2 + \frac{7}{15}\left(\frac{\pi a}{R}\right)^4\right] \\
\left\langle r^2 \right\rangle_{g_0} &\simeq R^2 \left[1 + \frac{5}{3}\left(\frac{\pi a}{R}\right)^2 - \frac{4}{45} \frac{\left(\frac{\pi a}{R}\right)^4}{1 + \frac{1}{3}\left(\frac{\pi a}{R}\right)^2}\right]
\end{aligned}$$

2 Y_2 Deformed W-S form factor

$$\begin{aligned}
f(r,\theta) &= \frac{1}{1 + \exp\{(r - R(\theta))/a\}} \\
g(r,\theta) &= \frac{\exp\{(r - R(\theta))/a\}}{\left[1 + \exp\{(r - R(\theta))/a\}\right]^2} \\
R(\theta) &= R_0 (1 + \beta_2 Y_{20}(\theta))
\end{aligned}$$

1st order or quadrupole vibration

$$\begin{aligned}
f(r,\theta) &\simeq f_0(r) + \beta_2 R_0 \frac{d}{dr} f_0(r) Y_{20}(\theta) \\
&= f_0(r) - \beta_2 R_0 \left(\frac{1}{a}\right) \frac{\exp\{(r - R_0)/a\}}{\left[1 + \exp\{(r - R_0)/a\}\right]^2} Y_{20}(\theta) \\
&= f_0(r) + \beta_2 R_0 \left(\frac{1}{a}\right) g_0(r) Y_{20}(\theta)
\end{aligned}$$

2.1 Volume integral and moments of $f(r,\theta)$

$$
\begin{aligned}
\langle f\rangle &= \int dr\ d\Omega\ r^2\ f(r,\theta) \\
&\simeq \frac{1}{3}R_0^3\int d\Omega\ \left[(1+\beta_2 Y_{20}(\theta))^3+\left(\frac{\pi a}{R_0}\right)^2(1+\beta_2 Y_{20}(\theta))\right] \\
&= \frac{4\pi}{3}R_0^3\left[1+\left(\frac{\pi a}{R_0}\right)^2+\frac{3}{4\pi}\beta_2^2+\frac{\sqrt{5}}{28\pi^{3/2}}\beta_2^3\right] \\
\left\langle r^2 f\right\rangle &= \int dr\ d\Omega\ r^4\ f(r,\theta) \\
&\simeq \frac{1}{5}R_0^5\int d\Omega\ \left[(1+\beta_2 Y_{20}(\theta))^5+\frac{10}{3}\left(\frac{\pi a}{R_0}\right)^2(1+\beta_2 Y_{20}(\theta))^3\right. \\
&\qquad\left.+\frac{7}{3}\left(\frac{\pi a}{R_0}\right)^4(1+\beta_2 Y_{20}(\theta))\right] \\
&= \frac{4\pi}{5}R_0^5\left[\left(1+\frac{10}{3}\left(\frac{\pi a}{R_0}\right)^2+\frac{7}{3}\left(\frac{\pi a}{R_0}\right)^4\right)+\frac{10}{4\pi}\left(1+\left(\frac{\pi a}{R_0}\right)^2\right)\beta_2^2\right. \\
&\qquad\left.+\frac{10\sqrt{5}}{28\pi^{3/2}}\left(1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2\right)\beta_2^3+\frac{75}{112\pi}\beta_2^4+\frac{25\sqrt{5}}{616\pi^{5/2}}\beta_2^5\right] \\
\left\langle r^2\right\rangle_f &\simeq \frac{3}{5}R_0^2\left[\left(1+\frac{7}{3}\left(\frac{\pi a}{R_0}\right)^2\right)+\frac{7}{4\pi}\frac{1+\frac{3}{7}\left(\frac{\pi a}{R_0}\right)^2}{1+\left(\frac{\pi a}{R_0}\right)^2}\beta_2^2+\frac{9\sqrt{5}}{28\pi^{3/2}}\frac{1+\frac{1}{9}\left(\frac{\pi a}{R_0}\right)^2}{1+\left(\frac{\pi a}{R_0}\right)^2}\beta_2^3+\cdots\right] \\
\left\langle r^2 Y_2 f\right\rangle &= \int dr\ d\Omega\ r^4 Y_{20}(\theta)\ f(r,\theta) \\
&\simeq \frac{1}{5}R_0^5\int d\Omega\ Y_{20}(\theta)\left[(1+\beta_2 Y_{20}(\theta))^5+\frac{10}{3}\left(\frac{\pi a}{R_0}\right)^2(1+\beta_2 Y_{20}(\theta))^3\right. \\
&\qquad\left.+\frac{7}{3}\left(\frac{\pi a}{R_0}\right)^4(1+\beta_2 Y_{20}(\theta))\right] \\
&= \beta_2 R_0^5\left[\left(1+2\left(\frac{\pi a}{R_0}\right)^2+\frac{7}{15}\left(\frac{\pi a}{R_0}\right)^4\right)\right. \\
&\qquad\left.+\frac{2\sqrt{5}}{7\sqrt{\pi}}\left(1+\left(\frac{\pi a}{R_0}\right)^2\right)\beta_2+\frac{15}{14\pi}\left(1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2\right)\beta_2^2+\cdots\right] \\
\left\langle r^2 Y_2\right\rangle_f &\simeq \frac{3}{4\pi}\beta_2 R_0^2\left[1+\left(\frac{\pi a}{R_0}\right)^2-\frac{8}{15}\frac{\left(\frac{\pi a}{R_0}\right)^4}{1+\left(\frac{\pi a}{R_0}\right)^2}\right. \\
&\qquad\left.+\frac{2\sqrt{5}}{7\sqrt{\pi}}\beta_2+\frac{9}{28\pi}\frac{1-\frac{2}{9}\left(\frac{\pi a}{R_0}\right)^2+\frac{1}{45}\left(\frac{\pi a}{R_0}\right)^4}{\left(1+\left(\frac{\pi a}{R_0}\right)^2\right)^2}\beta_2^2\right]
\end{aligned}
$$

2.2 Volume integral and moments of $g(r,\theta)$

$$
\begin{aligned}
\langle g\rangle &= \int dr\ d\Omega\ r^2\ g(r,\theta) \\
&\simeq aR_0^2\int d\Omega\ \left[\left(1+\beta_2Y_{20}(\theta)\right)^2+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2\right] \\
&= 4\pi aR_0^3\left[1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2+\frac{1}{4\pi}\beta_2^2\right] \\
\left\langle r^2 g\right\rangle &= \int dr\ d\Omega\ r^4\ g(r,\theta) \\
&\simeq aR_0^4\int d\Omega\ \left[\left(1+\beta_2Y_{20}(\theta)\right)^4+2\left(\frac{\pi a}{R_0}\right)^2\left(1+\beta_2Y_{20}(\theta)\right)^2+\frac{7}{15}\left(\frac{\pi a}{R_0}\right)^4\right] \\
&= 4\pi aR_0^5\left[\left(1+2\left(\frac{\pi a}{R_0}\right)^2+\frac{7}{15}\left(\frac{\pi a}{R_0}\right)^4\right)+\frac{3}{2\pi}\left(1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2\right)\beta_2^2+\cdots\right] \\
\left\langle r^2\right\rangle_g &\simeq R_0^2\left[1+\frac{5}{3}\left(\frac{\pi a}{R_0}\right)^2-\frac{4}{45}\frac{\left(\frac{\pi a}{R_0}\right)^4}{1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2}+\frac{5}{4\pi}\frac{1+\frac{1}{5}\left(\frac{\pi a}{R_0}\right)^2}{1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2}\beta_2^2+\cdots\right] \\
\left\langle r^2Y_2\ g\right\rangle &= \int dr\ d\Omega\ r^4Y_{20}(\theta)\ \ g(r,\theta) \\
&\simeq aR_0^4\int d\Omega\ Y_{20}(\theta)\left[\left(1+\beta_2Y_{20}(\theta)\right)^4+2\left(\frac{\pi a}{R_0}\right)^2\left(1+\beta_2Y_{20}(\theta)\right)^2+\frac{7}{15}\left(\frac{\pi a}{R_0}\right)^4\right] \\
&= 4\beta_2aR_0^4\left[1+\left(\frac{\pi a}{R_0}\right)^2+\frac{30}{7\sqrt{\pi}}\left(1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2\right)\beta_2+\frac{5}{7\pi}\beta_2^2+\cdots\right] \\
\left\langle r^2Y_2\right\rangle_g &\simeq \frac{1}{\pi}\beta_2R_0^2\left[\frac{1+\left(\frac{\pi a}{R_0}\right)^2}{1+\frac{1}{3}\left(\frac{\pi a}{R_0}\right)^2}+\frac{30}{7\sqrt{\pi}}\beta_2+\frac{13}{28\pi}\frac{1-\frac{1}{39}\left(\frac{\pi a}{R_0}\right)^2}{\left(1+\left(\frac{\pi a}{R_0}\right)^2\right)^2}\beta_2^2+\cdots\right]
\end{aligned}
$$

3 Exact Eliptic Shape

3.1 Spheroidal coordinate

$$
\begin{aligned}
f_s(\xi) &= \frac{1}{1+\exp\{(\xi-\xi_0)/a\}} \\
\rho &= \left(\xi - \frac{\alpha\xi_0^2}{\xi}\right)\sin\eta \ ; \ \ \rho(\xi_0) = \rho_0 = \xi_0(1-\alpha)\sin\eta \\
z &= \left(\xi + \frac{\alpha\xi_0^2}{\xi}\right)\cos\eta \ ; \ \ z(\xi_0) = z_0 = \xi_0(1+\alpha)\cos\eta \\
\xi &\geq \sqrt{|\alpha|}\xi_0 \ ; \ |\alpha| < 1 \\
d\tau &= \xi^2\left(1-\frac{\alpha\xi_0^2}{\xi^2}\right)\left[\left(1+\frac{\alpha\xi_0^2}{\xi^2}\right)^2 - 4\frac{\alpha\xi_0^2}{\xi^2}\cos^2\eta\right]\sin\eta \ d\xi d\eta d\phi \\
r^2 &= \rho^2 + z^2 = \left(\xi - \frac{\alpha\xi_0^2}{\xi}\right)^2\sin^2\eta + \left(\xi + \frac{\alpha\xi_0^2}{\xi}\right)^2\cos^2\eta \\
&= \xi^2\left[\left(1-\frac{\alpha\xi_0^2}{\xi^2}\right)^2 + 4\ \frac{\alpha\xi_0^2}{\xi^2}\cos^2\eta\right] \\
r^2 Y_{20} &= \sqrt{\frac{5}{4\pi}}\ \xi^2\ \frac{1}{2}\ \left[\left\{3 + 2\ \frac{\alpha\xi_0^2}{\xi^2} + 3\ \left(\frac{\alpha\xi_0^2}{\xi^2}\right)^2\right\}\cos^2\eta - \left(1-\frac{\alpha\xi_0^2}{\xi^2}\right)^2\right]
\end{aligned}
$$

Volume integral, Moments

$$
\begin{aligned}
V &= \int d\tau f_s(\xi) \\
&\simeq \frac{4\pi}{3}\xi_0^3\left\{(1-\alpha)^2(1+\alpha) + \left(\frac{\pi a}{\xi_0}\right)^2\left(1-\frac{1}{3}\alpha^2+2\alpha^3\right)\right. \\
&\qquad\left. -\frac{7}{15}\left(\frac{\pi a}{\xi_0}\right)^4\alpha^2\left(1-15\alpha^2\right)+\cdots\right\} \\
\left\langle r^2\right\rangle &\simeq \frac{3}{5}\ \xi_0^2\ \left\{1-\frac{2}{3}\alpha+\alpha^2+\frac{7}{3}\left(\frac{\pi a}{\xi_0}\right)^2\left(1+\frac{4}{7}\alpha+\frac{9}{7}\alpha^2\right)\cdots\right\} \\
\left\langle r^2 Y_{20}\right\rangle &= \sqrt{\frac{4}{5\pi}}\ \xi_0^2\ \alpha \simeq \frac{3}{4\pi}\ \beta_2\ \xi_0^2\ \left(1+\frac{1}{28}\sqrt{\frac{5}{\pi}}\beta_2\right)\ \ ; \text{ see below}
\end{aligned}
$$

Expansion with spherical harmonics

$$
\begin{aligned}
R_0^3 &= \xi_0^3 \left(1+\alpha\right)\left(1-\alpha\right)^2 \ ; \ \delta = \frac{2\alpha}{1+\frac{1}{3}\alpha} \ ; \ \alpha = \frac{\delta}{2-\frac{1}{3}\delta} \\
\tan\theta &= \frac{1-\alpha}{1+\alpha}\tan\eta \\
R &= \xi_0 \sqrt{\left(1-\alpha\right)^2 \sin^2\eta + \left(1+\alpha\right)^2 \cos^2\eta} \\
&= \xi_0 \ \frac{\left(1-\alpha\right)\left(1+\alpha\right)}{\sqrt{\left(1+\alpha\right)^2 - 4\alpha\cos^2\theta}} \\
&= \tilde{R}_0 \left[1 + \sum_{\ell\geq 2} \beta_\ell Y_{\ell 0}\left(\theta\right)\right] \\
\tilde{R}_0 &= \xi_0 \left(1 - \frac{1}{3}\alpha - \frac{4}{5}\alpha^2 - \frac{4}{21}\alpha^3 - \ \cdots\right) \\
&= R_0 \left(1 - \frac{16}{45}\alpha^2 - \frac{64}{2835}\alpha^3 - \ \cdots\right) \\
\beta_2 &= \frac{8}{3}\sqrt{\frac{\pi}{5}} \left(\alpha - \frac{2}{21}\alpha^2 + \frac{17}{315}\alpha^3 + \cdots\right) \\
\beta_4 &= \frac{32\sqrt{\pi}}{35} \left(\alpha^2 - \frac{4}{35}\alpha^3 + \frac{478}{6435}\delta^4 + \cdots\right) \\
&\simeq \frac{9}{14\sqrt{\pi}}\beta_2^2 = 0.36269 \ \beta_2^2 \\
\beta_6 &= \frac{640}{231}\sqrt{\frac{\pi}{13}} \left(\alpha^3 - \frac{2}{15}\delta^4 + \cdots\right) \\
&\simeq \frac{225}{308\pi}\sqrt{\frac{5}{13}}\beta_2^3 = 0.14421 \ \beta_2^3 \\
\alpha &\simeq \alpha_0 + \frac{2}{21}\alpha_0^2 - \frac{79}{2205}\alpha_0^3 \ \ ; \ \ \alpha_0 \equiv \frac{3}{8}\sqrt{\frac{5}{\pi}}\beta_2
\end{aligned}
$$

4 Other useful formula

Gaussian, sine, cosine, and spherical Bessel

$$\begin{aligned}
g(r,R,\sigma) &= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(r-R)^2}{2\sigma^2}\right] \\
f_0(r) &\cong \int_r^\infty g\left(t,R,\sqrt{2}a\right)\,dt \quad : \text{approximation of W-S form factor} \\
g_0(r) &\cong -a\; g\left(r,R,\sqrt{2}a\right) \quad : \text{approximation so as to fit the curvature at } r=R \\
j_0(x) &= \frac{\sin x}{x} \\
j_1(x) &= \frac{\sin x}{x^2}-\frac{\cos x}{x} \\
j_2(x) &= \left(\frac{3}{x^3}-\frac{1}{x}\right)\sin x-\frac{3}{x^2}\cos x \\
j_3(x) &= \left(\frac{15}{x^4}-\frac{6}{x^2}\right)\sin x-\left(\frac{15}{x^3}-\frac{1}{x}\right)\cos x
\end{aligned}$$

Integrals

$$\begin{aligned}
\int_{-\infty}^{\infty} g(r,R,\sigma)\sin qr\;dr &= e^{-\frac{\sigma^2q^2}{2}}\sin qR \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\cos qr\;dr &= e^{-\frac{\sigma^2q^2}{2}}\cos qR \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\frac{\sin qr}{r}\;dr &= q\;e^{-\frac{\sigma^2q^2}{2}}\sum_{n=0}^{\infty}\left(\frac{\sigma}{R}\right)^{2n}(qR)^n j_n(qR) \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\,r\sin qr\;dr &= R\;e^{-\frac{\sigma^2q^2}{2}}\left[\sin qR+\left(\frac{\sigma}{R}\right)^2(qR)\cos qR\right] \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\,r^2\sin qr\;dr &= R^2\;e^{-\frac{\sigma^2q^2}{2}}\left[\sin qR+\left(\frac{\sigma}{R}\right)^2(\sin qR+2\;qR\cos qR)\right. \\
&\qquad \left.-\left(\frac{\sigma}{R}\right)^4(qR)^2\sin qR\right] \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\,r\cos qr\;dr &= R\;e^{-\frac{\sigma^2q^2}{2}}\left[\cos qR-\left(\frac{\sigma}{R}\right)^2(qR)\sin qR\right] \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\,r^3 j_2(qr)\;dr &= R^3\;e^{-\frac{\sigma^2q^2}{2}}\left[j_2(qR)+2\left(\frac{\sigma}{R}\right)^2(qR)\,j_1(qR)\right. \\
&\qquad \left.+\left(\frac{\sigma}{R}\right)^4(qR)^2 j_0(qR)\right] \\
\int_{-\infty}^{\infty} g(r,R,\sigma)\,r^2 j_2(qr)\;dr &= R^2\;e^{-\frac{\sigma^2q^2}{2}}\left[\left\{1+\left(\frac{\sigma}{R}\right)^2+3\left(\frac{\sigma}{R}\right)^4\right\}j_2(qR)\right. \\
&\qquad \left.+\left(\frac{\sigma}{R}\right)^2(qR)\,j_1(qR)+\frac{3}{(qR)^2}\sum_{n=3}^{\infty}\left(\frac{\sigma}{R}\right)^{2n}(qR)^n j_n(qR)\right]
\end{aligned}$$

4.1 Expantion of Eliptic Shape (obsolate)

$$
\begin{aligned}
R^2(\theta) &= \left(\frac{\cos^2\theta}{R_3^2}+\frac{\sin^2\theta}{R_\perp}\right)^{-1} \quad ; \quad \frac{R^2\cos^2\theta}{R_3^2}+\frac{R^2\sin^2\theta}{R_\perp^2}=1 \\
R_1 &= R_2 = R_\perp \quad ; \quad \frac{R_3}{R_\perp}=\frac{1+\frac{1}{3}\delta}{1-\frac{2}{3}\delta} \quad ; \quad R_3R_\perp^2=R_0^3 \\
\frac{1}{R_3} &= \left(\frac{1-\frac{2}{3}\delta}{1+\frac{1}{3}\delta}\right)^{2/3}\frac{1}{R_0} \quad ; \quad \frac{1}{R_\perp}=\left(\frac{1+\frac{1}{3}\delta}{1-\frac{2}{3}\delta}\right)^{1/3}\frac{1}{R_0} \\
R(\theta) &= R_0\frac{\left(1+\frac{1}{3}\delta\right)^{2/3}\left(1-\frac{2}{3}\delta\right)^{1/3}}{\sqrt{\left(1+\frac{1}{3}\delta\right)^2-\delta\left(2-\frac{1}{3}\delta\right)\cos^2\theta}} \\
&= \tilde{R}_0\left[1+\sum_{\ell\geq 2}\beta_\ell Y_{\ell 0}(\theta)\right] \\
\tilde{R}_0 &= R_0\left(1-\frac{4}{45}\delta^2-\frac{76}{2835}\delta^3-\frac{13}{945}\delta^4-\cdots\right) \\
\beta_2 &= \frac{4}{3}\sqrt{\frac{\pi}{5}}\left(\delta+\frac{5}{21}\delta^2+\frac{8}{315}\delta^3+\frac{59}{20790}\delta^4+\cdots\right) \\
&= 1.0569\ \delta\left(1+\frac{5}{21}\delta+\frac{8}{315}\delta^2+\frac{59}{20790}\delta^3+\cdots\right) \\
\beta_4 &= \frac{8\sqrt{\pi}}{35}\left(\delta^2+\frac{3}{11}\delta^3+\frac{1843}{25740}\delta^4+\cdots\right) \\
&= 0.40513\ \delta^2\left(1+\frac{3}{11}\delta+\frac{1843}{25740}\delta^2+\cdots\right) \\
&\simeq \frac{9}{14\sqrt{\pi}}\beta_2^2 = 0.36269\ \beta_2^2 \\
\beta_6 &= \frac{80}{231}\sqrt{\frac{\pi}{13}}\left(\delta^3+\frac{13}{30}\delta^4+\cdots\right) \\
&= 0.17025\ \delta^3\left(1+\frac{13}{30}\delta+\cdots\right) \\
&\simeq \frac{225}{308\pi}\sqrt{\frac{5}{13}}\beta_2^3 = 0.14421\ \beta_2^3 \\
\beta_8 &= \frac{224}{1287}\sqrt{\frac{\pi}{17}}\left(\delta^4+\cdots\right) \\
&= 0.074820\ \delta^4+\cdots \\
&\simeq \frac{1575}{1184\sqrt{17}\pi^{3/2}}\beta_2^4 = 0.057940\ \beta_2^4
\end{aligned}
$$

$$
\delta \simeq \delta_0-\frac{5}{21}\delta_0^2+\frac{194}{2205}\delta_0^3-\frac{40841}{1018710}\delta_0^4 \quad ; \quad \delta_0\equiv\frac{3}{4}\sqrt{\frac{5}{\pi}}\beta_2
$$