July 30, 2007

# Memo relating to relativistic kinematics

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## 1 Notation

$$
\begin{array}{lcl}
m_i & : & \text{rest mass of particle } i \\
E_i & : & \text{total energy of particle } i \\
\vec{p}_i & : & \text{momentum of particle } i \\
\mathbf{p}_i & : & \text{four momentum of particle } i \ \left[ \begin{pmatrix} \vec{p}_i \\ E_i \end{pmatrix};\ \mathbf{p}_i^2 = p_i^2 - E_i^2 = -m_i^2 \right] \\
T_i & : & \text{kinetic energy of particle } i \\
\vec{\beta}_i & : & \text{velocity of particle } i \ \left[ \dfrac{\vec{p}_i}{E_i} \right] \\
\gamma_i & : & \text{Lorentz gamma of particle } i \ \left[ \dfrac{1}{\sqrt{1-\beta_i^2}} \right] \\
M & : & \text{invariant mass consisting of particle 1, 2, ..., } n \ \left[ \sqrt{\left( \sum_{i=1}^{n} E_i \right)^2 - \left( \sum_{i=1}^{n} \vec{p}_i \right)^2} \right] \\
E_{\mathrm{d}} & : & \text{decay energy (relative energy) } \left[ M - \sum_{i=1}^{n} m_i \right]
\end{array}
$$

All the calculations use the unit of $c = 1$

## 2 One-body kinematics

$$
\begin{aligned}
p^2 &= E^2 - m^2 = T\ (T + 2m) \\
T &= E - m = \frac{p^2}{\sqrt{p^2 + m^2} + m} \\
E &= m\gamma \\
\vec{p} &= m\gamma\vec{\beta} \\
T &= m(\gamma - 1) = m\frac{\beta^2\gamma^2}{1+\gamma}
\end{aligned}
$$

# 3 Invariant Mass and Decay Energy

## 3.1 Two-body kinematics

$$
\begin{aligned}
M^2 &= (E_1+E_2)^2 - (\vec{p}_1+\vec{p}_2)^2 \\
&= (m_1+m_2)^2 + 2\left(E_1E_2 - m_1m_2 - \vec{p}_1\cdot\vec{p}_2\right) \\
E_\mathrm{d} &= M-(m_1+m_2) = \frac{F}{\sqrt{(m_1+m_2)^2+F}+(m_1+m_2)} \\
F &= 2\left(E_1E_2 - m_1m_2 - \vec{p}_1\cdot\vec{p}_2\right) \\
&= \frac{T_1}{T_2}\left(T_2\left(T_2+2m_2\right)\right) + \frac{T_2}{T_1}\left(T_1\left(T_1+2m_1\right)\right) - 2\vec{p}_1\cdot\vec{p}_2 \\
&= T_1T_2\left(\frac{\vec{p}_1}{T_1}-\frac{\vec{p}_2}{T_2}\right)^2
\end{aligned}
$$

**Example 1: Non relativistic limit**

In this case, $T \to \frac{1}{2}m\beta^2$, $\vec{p} \to m\vec{\beta}$, $\vec{p}/T \to 2\vec{\beta}/\beta^2$, and $F/(m_1+m_2)^2 \to 0$, then

$$
E_\mathrm{d} = \frac{m_1m_2}{m_1+m_2}\left(\vec{\beta}_1-\vec{\beta}_2\right)^2.
$$

**Example 2: Photon emission**

If the particle 2 is photon, $T_2 = p_2 = E_2 = E_\gamma$, then

$$
\begin{aligned}
E_\mathrm{d} &= \frac{F}{\sqrt{m_1^2+F}+m_1} \\
F &= 2m_1E_\gamma\gamma_1\left(1-\beta_1\cos\theta_{12}\right),
\end{aligned}
$$

where the recoil effect by emitting photon is included. If one neglect this, i.e. $F/m_1^2 \to 0$,

$$
E_\mathrm{d} \to E_\gamma\gamma_1\left(1-\beta_1\cos\theta_{12}\right),
$$

which is well-known formula for the Doppler shift.

### 3.2 Many-body case

$$
\begin{aligned}
E_{\mathrm{d}} &= \frac{F}{\sqrt{\left(\sum_{i=1}^{n} m_i\right)^2 + F} + \sum_{i=1}^{n} m_i} \\
F &= \sum_{i<j} F_{ij} \\
F_{ij} &= T_i T_j \left(\frac{\vec{p}_i}{T_i} - \frac{\vec{p}_j}{T_j}\right)^2
\end{aligned}
$$

**Example: Non relativistic limit**

In this case, $T \to \frac{1}{2} m \beta^2$, $\vec{p} \to m\vec{\beta}$, $\vec{p}/T \to 2\vec{\beta}/\beta^2$, and $F/\left(\sum_{i=1}^{n} m_i\right)^2 \to 0$, then

$$
\begin{aligned}
E_{\mathrm{d}} &= \frac{\sum_{i<j}\left(m_i + m_j\right) E_{ij}}{\sum_{i=1}^{n} m_i} \\
E_{ij} &= \frac{m_i m_j}{m_i + m_j}\left(\vec{\beta}_i - \vec{\beta}_j\right)^2 .
\end{aligned}
$$

# 4 Lorentz Transformation

Suppose that the system $X'Y'Z'$ moves relative to the system $XYZ$ with the velocity of $\vec{\beta} = \beta\vec{n}$ ($|\vec{n}| = 1$) under the condition that $\vec{n}$ and $\vec{n'}$ are "quasi-parallel"[1]. The transform matrix $A$ (${}^T(p'_x, p'_y, p'_z, E') = A\ {}^T(p_x, p_y, p_z, E)$) is given by:

$$
\begin{aligned}
A &= \begin{pmatrix}
1+(\gamma-1)n_x^2 & (\gamma-1)n_x n_y & (\gamma-1)n_x n_z & -\beta\gamma n_x \\
(\gamma-1)n_x n_y & 1+(\gamma-1)n_y^2 & (\gamma-1)n_y n_z & -\beta\gamma n_y \\
(\gamma-1)n_x n_z & (\gamma-1)n_y n_z & 1+(\gamma-1)n_z^2 & -\beta\gamma n_z \\
-\beta\gamma n_x & -\beta\gamma n_y & -\beta\gamma n_z & \gamma
\end{pmatrix} \\
&= \begin{pmatrix}
1 & 0 & 0 & -\beta\gamma n_x \\
0 & 1 & 0 & -\beta\gamma n_y \\
0 & 0 & 1 & -\beta\gamma n_z \\
-\beta\gamma n_x & -\beta\gamma n_y & -\beta\gamma n_z & 1
\end{pmatrix} + (\gamma-1)\begin{pmatrix}
n_x^2 & n_x n_y & n_x n_z & 0 \\
n_x n_y & n_y^2 & n_y n_z & 0 \\
n_x n_z & n_y n_z & n_z^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{aligned}
$$

The first term in the second line denotes the boost in the space relating to the Galilei transformation. However, the second term corresponds to the rotation in the space, which is Relativistic effect relating to the Thomas precession.

[1]Consider the space component $\vec{a} = {}^T(a_x, a_y, a_z)$ of a four vector a in the $XYZ$ system. If the space component $\vec{a'} = {}^T(a'_x, a'_y, a'_z)$ in the $X'Y'Z'$ system has the relation of $a'_x/a_x = a'_y/a_y = a'_z/a_z$, $\vec{a}$ and $\vec{a'}$ are called as "quasi-parallel".
Proof that $\vec{n}$ and $\vec{n'}$ are quasi-parallel but $|\vec{n'}| \neq 1$, when using the above transformation matrix $A$.

# 5 Binary Reaction

Consider a binary reaction $A(a,b)B$ with a incident momentum $\vec{p}$ of $a$ on a fixed target $A$. The rest masses are denoted as $M$, $m$, $m'$, and $M'$ for $A$, $a$, $b$, and $B$, respectively. The momenta of $b$ and $B$ after the reaction are denoted as $\vec{p'}$ (scattering angle of $\theta$) and $\vec{P'} = \vec{q}$, respectively (Fig. 1).

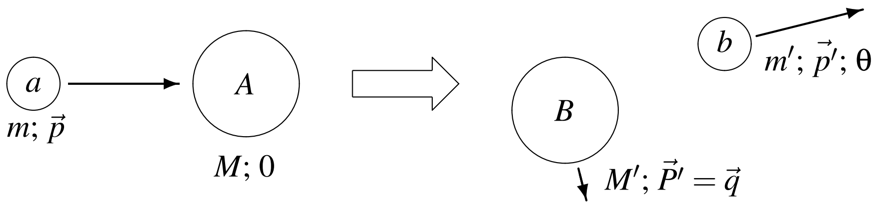


Figure 1: A binary reaction in the laboratory frame (before and after).

## 5.1 Momentum transfer ($q$) and Energy transfer ($\omega$) in laboratory frame

We define the momentum and energy transfer to the target system and the mass difference between $b$ and $a$ as:

$$
\begin{aligned}
\vec{q} &\equiv \vec{P'} = \vec{p} - \vec{p'} \\
\omega &\equiv M' - M \\
\Delta m &\equiv m - m' \ .
\end{aligned}
$$

From the momentum and energy conservation law, one obtain

$$
\begin{aligned}
q^2 &= \left(p - p'\cos\theta\right)^2 + p'^2 \sin^2\theta \\
\sqrt{m^2 + p^2} + M &= \sqrt{(M+\omega)^2 + q^2} + \sqrt{(m - \Delta m)^2 + (\vec{p} - \vec{q})^2}
\end{aligned}
$$

In the following small momentum transfer comparing to the rest masses and a small scattering angle:

$$
q \ll M \ , \ q \ll \sqrt{m^2 + p^2} \ , \ \theta \ll 1 \ ,
$$

the above relation is deduced to be

$$\sqrt{m^2+p^2}+M \cong M+\omega+\sqrt{m^2+p^2}\left(1-\frac{m\Delta m\,(1-\Delta m/\,(2m))\pm p\sqrt{q^2-(p\theta)^2}}{m^2+p^2}\right)$$

One obtain a simple relation between $q$ and $\omega$ as

$$\begin{aligned}(\beta q)^2-\left[\omega-\frac{\Delta m}{\gamma}\left(1-\frac{\Delta m}{2m}\right)\right]^2 &\cong (\beta p\theta)^2\\ q\cong\frac{1}{\beta}\left[\omega-\frac{\Delta m}{\gamma}\left(1-\frac{\Delta m}{2m}\right)\right] &\qquad \text{at } \theta=0.\end{aligned}$$

## 5.2 Center-of-mass frame

We consider the momentum transfer $\vec{q}$ in the center-of-mass (CM) frame. The momenta of $A$, $a$, $b$, and $B$ in the CM system are denoted as $\vec{p}_{\rm c}$, $-\vec{p}_{\rm c}$, $\vec{p}'_{\rm c}$, $-\vec{p}'_{\rm c}$, respectively. It is noted that $q$ is not invariant, *i.e.* $\vec{q}$ is not always equal to $\vec{p}_{\rm c}-\vec{p}'_{\rm c}$. The relative velocity of the CM frame with respect to the laboratory frame, $\beta_{\rm CM}$, and $\gamma_{\rm CM}=1/\sqrt{1-\beta_{\rm CM}^2}$ are expressed as

$$\begin{aligned}\beta_{\rm CM} &= \frac{p}{\sqrt{p^2+m^2}+M}=\frac{p_{\rm c}}{\sqrt{p_{\rm c}^2+M^2}}\\ \gamma_{\rm CM} &= \frac{\sqrt{p^2+m^2}+M}{\sqrt{M^2+m^2+2M\sqrt{p^2+m^2}}}=\frac{\sqrt{p_{\rm c}+M^2}}{M}\quad .\end{aligned}$$

Using these quantities, the momenta in the laboratory frame are represented by those in the CM frame:

$$\begin{aligned}p &= \gamma_{\rm CM}\left(p_{\rm c}+\beta_{\rm CM}\sqrt{p_{\rm c}^2+m^2}\right)\\ p'\cos\theta &= \gamma_{\rm CM}\left(p'_{\rm c}\cos\theta_{\rm c}+\beta_{\rm CM}\sqrt{p'^2_{\rm c}+m^2}\right)\\ p'\sin\theta &= p'_{\rm c}\sin\theta_{\rm c}\quad .\end{aligned}$$

Then, the momentum transfer $\vec{q}$ is expressed by the CM momenta as

$$\begin{aligned}q\cos\theta &= \frac{\sqrt{p'^2_{\rm c}+M'^2}}{M}p_{\rm c}-\frac{\sqrt{p_{\rm c}^2+M^2}}{M}p'_{\rm c}\cos\theta\\ &\rightarrow \frac{M'}{M}p_{\rm c}-p'_{\rm c}\cos\theta \quad \text{(non-relativistic limit)}\\ q\sin\theta &= -p'_{\rm c}\sin\theta_{\rm c}\\ \vec{q} &\rightarrow \frac{M'}{M}\vec{p}_{\rm c}-\vec{p}'_{\rm c} \quad \text{(non-relativistic limit)}\end{aligned}$$