Geometric variational problems involving competition between line and surface energy

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Outline

1 Euler–Plateau problem
2 Planar specialization
3 Recasting of the Euler–Plateau problem in parametric form
4 Stability of flat, circular solutions
5 Bifurcation from flat, circular solutions
6 Extensions of the Euler–Plateau problem
7 Synopsis and discussion
In an inventive generalization of experiments conducted by Plateau (Mém. Acad. Sci. Belgique 23 (1849), 1–151), Giomi & Mahadevan (Proc. R. Soc. A 468 (2012), 1851–1864) explored what happens when closed loops of fishing line of various lengths are dipped into and extracted from soapy water.
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Experiments and photos by Aisa Biria

- For a loop of given length, a flat circular disk has maximal area and, thus, maximum surface energy.
- A circular loop has minimum bending energy.
- If the length of the loop exceeds a certain threshold, it becomes energetically favorable to reduce the area of the film in favor of bending the loop away from circular.
Formulation of Giomi & Mahadevan

- Following the conventional approach to the Plateau problem, the soap film is modeled as a surface $S$ with uniform tension $\sigma > 0$. 
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- Following the conventional approach to the Plateau problem, the soap film is modeled as a surface $S$ with uniform tension $\sigma > 0$.
- Motivated by Euler’s work on the elastica, the fishing line is modeled as an inextensible ring with uniform flexural rigidity $a > 0$ and rectilinear rest configuration, the midline of which coincides with the boundary $C = \partial S$ of $S$. 

Overlooked contribution: Granted that $C$ is free of self-contact, Bernatzki & Ye (Ann. Glob. Anal. Geom. 19 (2001), 1–9) established the existence of minimizers of the functional $\int_C |\kappa - \kappa_0|^2 + \int_S \sigma$, where $\kappa$ denotes the vector curvature of $C$ and $\kappa_0$ is an intrinsic vector curvature.
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If gravitational effects are negligible, then the net potential-energy $E$ of the system comprised by the loop and the film is given by

$$E := \int_C \frac{1}{2} a \kappa^2 + \int_S \sigma,$$

where $\kappa$ denotes the curvature of $C$. 
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Using an energy comparison argument based on a particular trial solution, Giomi & Mahadevan find that for a ring of length \( L = 2\pi R \) the system prefers a flat configuration with circular boundary if

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\nu := \frac{R^3 \sigma}{a} < 3.
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On this basis, they reasoned that a bifurcation to a flat, oval configuration should occur at $\nu = R^3 \sigma / a = 3$. 

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Analytical and numerical results of Giomi & Mahadevan

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- On this basis, they reasoned that a bifurcation to a flat, oval configuration should occur at $\nu = R^3 \sigma/a = 3$. They also performed numerical experiments that seem to support this assertion.

Giomi & Mahadevan (Proc. R. Soc. A 468 (2012), 1851–1864)
The first variation condition $\delta F = 0$ requires that:

- At all points on the surface $S$, $H = 0$.
- At all points on the boundary $C = \partial S$, $\beta' a t - (\kappa' + \frac{1}{2} \kappa^3 - (\tau^2 + \beta a) \kappa - \sigma \sin \vartheta a) p - (2 \kappa' \tau + \kappa \tau' + \sigma \cos \vartheta a) b = 0$.

$\beta$ is a Lagrange multiplier needed to ensure the inextensibility of $C$, a prime signifies differentiation with respect to arclength along $C$, $\{t, p, b\}$ is the Frenet frame of $C$, $\kappa$ and $\tau$ are the curvature and torsion of $C$, and $\cos \vartheta = \frac{p \cdot n}{|C|}$, where $n$ is a unit normal to $S$.

For $\sigma = 0$, the condition on $C$ is classical. See, for example, Langer & Singer (J. Diff. Geom. 20 (1984), 10–22).

Requiring $\kappa$ and $\tau$ to be smooth and periodic does not generally suffice to determine a closed space curve: Efimov (Usp. Mat. Nauk 2 (1947), 193–194), Fenchel (Bull. Amer. Math. Soc. 57 (1951), 44–54).
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- \left( 2\kappa' \tau + \kappa \tau' + \frac{\sigma \cos \vartheta}{a} \right) b = 0.
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Planar specialization: Pressurized cylindrical tube

Considered by:

- Lévy (J. Math. Pures Appl. 10 (1884), 5–42)
- Halphen (Fonctions elliptiques, Gauthier-Villars et fils, Paris, 1888)
- Greenhill (Math. Ann. 52 (1889), 465–500)
- Carrier (J. Math. Phys. 26 (1947), 94–103)
- Flaherty, Keller & Rubinow (SIAM J. Appl. Math. 23 (1972), 446–455)
- Giomi (Soft Matter 9 (2013), 8121–8139)
Recast Euler–Plateau problem (with Yi-chao Chen)

Suppose that $\mathcal{S}$ admits a (sufficiently) smooth parametrization

$$\mathcal{S} = \{ \mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = \mathbf{x}(r, \theta), \ 0 \leq r \leq R, 0 \leq \theta \leq 2\pi \}.$$
Suppose that $S$ admits a (sufficiently) smooth parametrization
\[ S = \{ x \in \mathbb{R}^3 : x = x(r, \theta), 0 \leq r \leq R, 0 \leq \theta \leq 2\pi \}. \]

Then $C = \partial S$ is parametrized according to
\[ C = \{ x \in \mathbb{R}^3 : x = x(R, \theta), 0 \leq \theta \leq 2\pi \}, \]
where, to ensure that the boundary is inextensible, $x$ must satisfy
\[ |x_\theta(R, \theta)| = R, \quad 0 \leq \theta \leq 2\pi. \]
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Periodicity requires that $x(r, 0) = x(r, 2\pi)$ for $0 < r \leq R$ and so on...
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$E$ can then be represented as a functional of $x$:

$$E[x] = \int_0^{2\pi} \frac{a |x_{\theta\theta}(R, \theta)|^2}{2R^3} \, d\theta + \int_0^{2\pi} \int_0^R \sigma |x_r(r, \theta) \times x_\theta(r, \theta)| \, dr \, d\theta.$$
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Notice that the highest derivatives of $x$ appear in the boundary term.
Invariance and scaling of the recast energy functional

Given an orthogonal linear transformation $Q$ and a vector $c$, it follows that

$$E[Qx + c] = E[x]$$

and, thus, that $E$ is invariant under rigid transformations. Any minimizer of $E$ is, at best, determined up to such a transformation.
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- For the simple choice $x_g(r, \theta) = r \hat{r}(\theta)$, which represents a circular disc of radius $R$ and can thus be thought of as a base state, $E$ specializes to yield a reference value of the energy

$$E[x_g] = \frac{\pi(1 + \nu)a}{R},$$

$$\nu = \frac{R^3 \sigma}{a} = \frac{\pi R^2 \sigma}{\pi a/R} > 0.$$
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- For a circular disc of radius $R$, $\nu$ represents the ratio of the surface energy $\pi R^2\sigma$ of the film to the bending energy $\pi a/R$ of the loop and...
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For a circular disc of radius $R$, $\nu$ represents the ratio of the surface energy $\pi R^2\sigma$ of the film to the bending energy $\pi a/R$ of the loop and is the only dimensionless parameter entering the problem.

There are three conceivable ways to adjust $\nu$, the simplest of which is to alter $R$ while holding $\sigma$ and $a$ fixed.
Equilibrium conditions for the recast problem

The first variation condition $\delta E = 0$ yields two equilibrium conditions, a scalar second-order partial-differential equation

$$n \cdot (x_\theta \times n_r + n_\theta \times x_r) = 0,$$

and a vector fourth-order ordinary-differential equation

$$\nu x_\theta \times n + (x_\theta \theta \theta - \lambda x_\theta \theta) \theta = 0,$$

where $n = \frac{x_r \times x_\theta}{|x_r \times x_\theta|}$, to be satisfied on the interior of the disc of radius $R$, and

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to be satisfied on the interior of the disc of radius $R$, and a vector fourth-order ordinary-differential equation

$$[\nu \mathbf{x}_\theta \times \mathbf{n} + (\mathbf{x}_{\theta\theta\theta} - \lambda \mathbf{x}_\theta)_{\theta}]_{r=R} = 0,$$

to be satisfied on the boundary of the disc of radius $R$. 

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RIKEN–Oaska–OIST Joint Workshop 2016

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to be satisfied on the interior and boundary of the disc of radius \( R \), respectively.

The unknowns are the parametrization \( \mathbf{x} \) and a Lagrange multiplier \( \lambda \).
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$$[\nu x_\theta \times n + (x_{\theta\theta\theta} - \lambda x_\theta)_{\theta}]_{r=R} = 0,$$

to be satisfied on the boundary of the disc of radius $R$.

- The unknowns are the parametrization $x$ and a Lagrange multiplier $\lambda$.

- The ratio $a\lambda/R^2$ is the reactive force density needed to ensure satisfaction of the constraint

$$|x_\theta|_{r=R} = R,$$

which must be imposed along with the equilibrium conditions.
The problem for $x$ and $\lambda$ involves a second-order partial differential equation subject to a fourth-order boundary condition. Notwithstanding the important contributions of Agmon, Douglis & Nirenberg (Comm. Pure Appl. Math. 12 (1959), 623–727; 18 (1964), 35–92), this problem provides involves many new mathematical challenges, as do related problems that we will mention.

The partial-differential equation is equivalent to $H = 0$. 

A smoothly periodic $x$ satisfying the equilibrium conditions determines $\kappa$, $\tau$, and $\vartheta$. Since the closed-curve problem remains unresolved, the converse assertion is not generally true.
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The ordinary-differential equation is equivalent to

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- \left( 2 \kappa' \tau + \kappa \tau' + \frac{\sigma \cos \vartheta}{a} \right) \mathbf{b} = 0,
\]

where \( \beta \) is a constant force, per unit length, related to \( \lambda \) and \( \kappa \) via

\[
\beta = \frac{a}{R^2} \left( \lambda + \frac{3R^2 \kappa^2}{2} \right).
\]
The problem for \( x \) and \( \lambda \) involves a second-order partial differential equation subject to a fourth-order boundary condition. Notwithstanding the important contributions of Agmon, Douglis & Nirenberg (Comm. Pure Appl. Math. 12 (1959), 623–727; 18 (1964), 35–92), this problem provides involves many new mathematical challenges, as do related problems that we will mention.

The partial-differential equation is equivalent to \( H = 0 \).

The ordinary-differential equation is equivalent to

\[
\frac{\beta'}{a} t - \left( \kappa'' + \frac{1}{2} \kappa^3 - \left( \tau^2 + \frac{\beta}{a} \right) \kappa - \frac{\sigma \sin \vartheta}{a} \right) p \\
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$$

A smoothly periodic $x$ satisfying the equilibrium conditions determines $\kappa$, $\tau$, and $\vartheta$. Since the closed-curve problem remains unresolved, the converse assertion is not generally true.
Second variation condition for the recast problem

- A parametrization $\mathbf{x}$ satisfying the equilibrium conditions is stable if the second-variation condition

$$\delta^2 F = \int_0^{2\pi} \left( |\mathbf{u}_{\theta\theta}|^2 + \lambda |\mathbf{u}_\theta|^2 \right) |r=R| \, d\theta$$

$$+ \int_0^{2\pi} \int_0^R \nu \left( \frac{|\mathbf{P}(\mathbf{u}_r \times \mathbf{x}_\theta + \mathbf{x}_r \times \mathbf{u}_\theta)|^2}{|\mathbf{x}_r \times \mathbf{x}_\theta|} + 2 \mathbf{m} \cdot (\mathbf{u}_r \times \mathbf{u}_\theta) \right) \, dr \, d\theta \geq 0$$

holds for all admissible variations $\mathbf{u} = \delta \mathbf{x}$, where

$$\mathbf{P} = \mathbf{I} - \mathbf{m} \otimes \mathbf{m}$$

is the perpendicular projector onto the tangent space of $\mathcal{S}$.

- To be admissible, $\mathbf{u}$ must satisfy

$$\mathbf{x}_\theta \cdot \mathbf{u}_\theta |_{r=R} = 0.$$
Let \( \rho = \frac{r}{R} \) and \( \xi(\rho, \theta) = \frac{x(r, \theta)}{R} \).
Stability of flat, circular solutions

Let $\rho = r/R$ and $\xi(\rho, \theta) = x(r, \theta)/R$. Then the flat, circular solution $\xi(\rho, \theta) = \rho \hat{r}(\theta)$ is a disk of (dimensionless) radius unity and the corresponding value of the Lagrange multiplier $\lambda$ is

$$\lambda = -(1 + \nu).$$
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- Express the variation \( u \) in terms of radial and transverse perturbations \( v \) and \( w \) of the flat, circular solution:

  \[
  u = v \hat{r} + w \hat{r} \times \hat{\theta}.
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Express the variation $u$ in terms of radial and transverse perturbations $v$ and $w$ of the flat, circular solution:

$$u = v \hat{r} + w \hat{r} \times \hat{\theta}.$$  

The second-variation condition then yields decoupled inequalities for the radial and transverse perturbations $v$ and $w$:

$$\int_0^{2\pi} [(v_{\theta \theta} + \nu)^2 - \nu(v_{\theta}^2 - v^2)]_{\rho=1} \, d\theta \geq 0,$$

$$\int_0^{2\pi} [w_{\theta \theta}^2 - (1 + \nu)w_{\theta}^2]_{\rho=1} \, d\theta + \int_0^{2\pi} \int_0^R \nu \left( \rho w_{\rho}^2 + \frac{1}{\rho} w_{\theta}^2 \right) \, d\rho \, d\theta \geq 0.$$

The left-hand side of the first inequality admits a minimum if and only if \( \nu \geq 0 \) for every solution of

\[
[v_{\theta\theta\theta\theta} + (2 + \nu)v_{\theta\theta} + (1 + \nu - \nu)v]_{\rho=1} = 0,
\]

which is the case if \( \nu \leq 3 \). Evaluating the inequality for the particular choice \( \nu(1, \theta) = \sin 2\theta \) shows that the foregoing condition is also necessary.

The left-hand side of the second inequality admits a minimum if and only if \( \gamma \geq 0 \) for every solution of

\[
\nu \left( w_{\rho\rho} + \frac{1}{\rho} w_{\rho} + \frac{1}{\rho^2} w_{\theta\theta} \right) + \gamma w = 0,
\]

\[
[w_{\theta\theta\theta\theta} + (1 + \nu)w_{\theta\theta} + \nu w_{\rho}]_{\rho=1} = 0,
\]

which is the case if \( \nu \leq 6 \). Evaluating the inequality for the particular choice \( w(\rho, \theta) = \rho^2 \sin 2\theta \) shows that the foregoing condition is also necessary.
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**Conclusion**

The trivial solution is stable if and only if $\nu \leq 3$. 
Bifurcation from flat, circular solutions

By the implicit function theorem, the boundary-value problem for $\xi$ and $\lambda$ possesses a nontrivial solution branch that bifurcates from the flat, circular solution branch only if the linearized equations have a nontrivial solution.

To linearize about the flat, circular solution, consider

$$\xi = \rho \hat{r} + \eta + w \hat{r} \times \hat{\theta}, \quad \lambda = -(1 + \nu) + \epsilon,$$

where $\eta$ obeys $\eta \cdot (\hat{r} \times \hat{\theta}) = 0$ and is, thus, planar.
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$$\begin{align*}
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\end{align*}$$

where $\eta$ obeys $\eta \cdot (\hat{r} \times \hat{\theta}) = 0$ and is, thus, planar.

The linearized problem for $\eta$ and $\epsilon$ is

$$\begin{align*}
[\eta_{\theta\theta\theta\theta} + (1 + \nu)\eta_{\theta\theta} - \nu(\hat{r} \cdot \eta_{\theta})\hat{\theta}]_{\rho=1} + \epsilon \hat{r} - \epsilon_{\theta} \hat{\theta} &= 0, \\
\hat{\theta} \cdot \eta_{\theta}|_{\rho=1} &= 0,
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\]

\[
\hat{\theta} \cdot \eta_{\theta}\big|_{\rho=1} = 0,
\]

while that for \( w \) is

\[
\rho(\rho w_\rho)_\rho + w_{\theta\theta} = 0,
\]

\[
[w_{\theta\theta\theta\theta} + (1 + \nu)w_{\theta\theta} + \nu w_\rho]_{\rho=1} = 0.
\]
Without loss of generality, we neglect rigid transformations. Then:

- The in-plane problem has nontrivial solutions

\[
\eta(1, \theta) = m(D_1 \sin m\theta + D_2 \cos m\theta)\hat{r} + m(D_1 \cos m\theta - D_2 \sin m\theta)\hat{\theta},
\]
\[
\epsilon(\theta) = -3\nu m(D_1 \sin m\theta + D_2 \cos m\theta),
\]
where \(D_1\) and \(D_2\) are constants and \(\nu\) must obey

\[
\nu = m^2 - 1 \geq 3.
\]

- The out-of-plane problem has nontrivial solutions

\[
w(\rho, \theta) = \rho^n(C_1 \cos n\theta + C_2 \sin n\theta),
\]
where \(C_1\) and \(C_2\) are constants and \(\nu\) must obey

\[
\nu = n(n+1) \geq 6.
\]

**Conclusion**

- The mode \(m = 2\) describes a stable bifurcation to a flat, noncircular solution branch. All remaining modes \(m \geq 3\) describe unstable bifurcations.

- All choices of the mode \(n \geq 2\) describe unstable out-of-plane bifurcations.
Observations

- As $\nu$ increases monotonically from some value $\nu_0 < 3$, a stable bifurcation to a flat, noncircular shape occurs at $\nu = 3$.
- Any other bifurcation solution branch that emanates from the flat, circular solution branch is unstable.
- Any stable nonplanar solution branch must emanate from the stable branch of flat but noncircular solutions.
Obervations

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Numerical results (with Abdul Majid)

- $\nu = 4.30$
- $\nu = 4.42$
- $\nu = 4.50$
- $\nu = 4.65$
- $\nu = 4.77$
- $\nu = 4.92$
Questions

(with Giulio Giusteri)

Can the theory be modified to suppress in-plane or out-of-plane bifurcations?

\( \nu = 3 \quad \text{in-plane bifurcation suppressed} \)

\( \nu \approx 4.42 \quad \text{out-of-plane bifurcation suppressed} \)
Questions
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\[ \nu = 3 \quad \Rightarrow \quad \text{in-plane bifurcation suppressed} \]

\[ \nu \approx 4.42 \quad \Rightarrow \quad \text{out-of-plane bifurcation suppressed} \]

Is there a dissipative dynamical generalization of the theory that is ‘nice’ in the sense that it is both physically sound and mathematically useful?
Suppressing in-plane or out-of-plane bifurcations

In-plane bifurcations can be suppressed by making the loop from a filament with:

- a circular cross-section having intrinsic curvature or twist density;

- an elliptical cross-section having major axis in the plane of the loop.
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Out-of-plane bifurcations can be suppressed by making the loop from a filament with an elliptical cross-section having minor axis in the plane of the loop.
To gain control over bifurcation pathways, it suffices to replace \( \int_C \frac{1}{2} a \kappa^2 \) by

\[
\int_C \frac{1}{2} (a_1 (\kappa_2 - \bar{\kappa}_2)^2 + a_2 (\kappa_1 - \bar{\kappa}_1)^2 + b (\omega - \bar{\omega})^2).
\]

Here:

- \( \kappa_1 \) and \( \kappa_2 \) are measures of curvature given by \( \kappa_1 = (t \times d) \cdot \kappa \) and \( \kappa_2 = d \cdot \kappa \), where \( t \) is the unit tangent of \( C \), \( d \) is a unit vector field orthogonal to \( t \) and oriented along the minor axis of the cross-section of the filament, and \( \kappa \) is the previously encountered vector curvature of \( C \).
- \( \omega \) is a measure of twist density given by \( \omega = t \cdot (d \times d') \).
- \( a_1 > 0 \) and \( a_2 \geq 0 \) are flexural rigidities and \( b \geq 0 \) is the twisting rigidity.
- \( \bar{\kappa}_1 \geq 0 \) and \( \bar{\kappa}_2 \geq 0 \) are intrinsic curvatures and \( \bar{\omega} \) is the intrinsic twist density.

This is Kirchhoff’s (J. reine angew. Math. 56 (1859), 285–313) energy for an inextensible, unshearable rod.
To gain control over bifurcation pathways, it suffices to replace $\int_C \frac{1}{2} a \kappa^2$ by

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This is Kirchhoff’s (J. reine angew. Math. 56 (1859), 285–313) energy for an inextensible, unshearable rod. Taking $a_1 = a_2 = a > 0$, $b = 0$, and $\bar{\kappa}_1 = \bar{\kappa}_2 = \bar{\omega} = 0$ reduces it to $\int_{C} \frac{1}{2}a\kappa^2$. 

Sample results: Moderate curvature mismatch regime

Set $\bar{\kappa}_1 = \bar{\omega} = 0$ and introduce the curvature mismatch $\zeta = 1 - R \bar{\kappa}_2$. 

For $\zeta < \frac{1}{2}$, the flat, circular solution branch becomes unstable at $\nu = \frac{R^2}{\sigma_a}$

$$\alpha_2 = \frac{a_2}{a_1}, \quad \alpha_3 = \frac{b}{a_1}.$$
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For $\zeta < \frac{1}{2}$, the flat, circular solution branch becomes unstable at

$$\nu = \frac{R^3\sigma}{a_1} = \min \left\{ 3, \frac{6\zeta(\alpha_2 + \alpha_3 - \zeta) + 18\alpha_2\alpha_3}{\alpha_2 + 4\alpha_3 - \zeta} \right\},$$

$$\alpha_2 = \frac{a_2}{a_1}, \quad \alpha_3 = \frac{b}{a_1}.$$
Dissipative dynamics

Instead of gradient flow versions of the equilibrium conditions, consider

\[ x_\theta \times n_r + n_\theta \times x_r = \eta_0 (n \cdot \Delta \dot{x}) n \]

on \([0, R) \times [0, 2\pi]\) in conjunction with

\[
\begin{align*}
(a_1 (k_2 - \bar{k}_2) x_\theta \times d + a_2 (k_1 - \bar{k}_1) d)_{\theta \theta} \\
- (a_1 (k_2 - \bar{k}_2) d \times x_{\theta \theta} + b (\omega - \bar{\omega}) d \times d_{\theta})_{\theta} \\
+ \sigma x_\theta \times n - (\lambda_1 x_\theta + \lambda_3 d)_{\theta} = (\eta_1 \dot{k}_1 x_\theta \times d - \eta_2 \dot{k}_2 d)_{\theta}
\end{align*}
\]

and

\[
\begin{align*}
a_1 (k_2 - \bar{k}_2) x_{\theta \theta} \times x_\theta + a_2 (k_1 - \bar{k}_1) x_{\theta \theta} \\
- b ((\omega - \bar{\omega}) x_{\theta \theta} \times d + (\omega - \bar{\omega})_\theta x_\theta \times d + 2 (\omega - \bar{\omega}) x_\theta \times d_{\theta}) \\
+ \lambda_2 d + \lambda_3 x_\theta = (\eta_3 \dot{\omega} x_\theta)_{\theta}
\end{align*}
\]

on \([0, 2\pi]\), where \(\eta_i \geq 0, i = 0, 1, 2, 3\) are viscosities and \(\lambda_1, \lambda_2,\) and \(\lambda_3\) are multipliers associated with the constraints on \(x\) and \(d\).
The Euler–Plateau problem has been recast to avoid the closed-curve problem.
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A bifurcation analysis reveals that:

- A stable bifurcation from the trivial solution branch to the flat noncircular solution branch occurs at $\nu = 3$.
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- Any stable nonplanar solution branch must emanate from the flat, noncircular solution branch.

Numerical studies indicate that:
- A stable bifurcation from the flat, noncircular solution branch to a nonplanar solution branch occurs at $\nu \approx 4.42$.
- A stable bifurcation from the noncircular solution branch to a planar figure-eight like configuration occurs at $\nu \approx 4.92$. 
Replacing the simple bending energy $\int_C \frac{1}{2} a\kappa^2$ by the energy for an inextensible, unshearable Kirchhoff rod allows for different bifurcation pathways.
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A framework for dissipative dynamics has been provided as a physically sound alternative to more conventional gradient flow approaches.
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The Plateau problem inspired significant advances in differential geometry, variational calculus, analysis, and various other areas of mathematics.

The class of geometrical variational problems including the Euler–Plateau problem and its various generalizations provides a new set of challenges that might lead to further progress.