Out-of-equilibrium field theories coupled to strong external sources

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Outline

1. Preamble : classical statistical method in Quantum Mechanics
2. QFT with strong sources, Inclusive observables at LO and NLO
3. Instabilities and resummation
4. Example : Schwinger mechanism

Quantum Mechanics

Classical phase-space formulation of Quantum Mechanics

- Consider the von Neumann equation for the density operator :

$$\frac{\partial \widehat{\rho}_\tau}{\partial \tau} = i\hbar \left[\widehat{H}, \widehat{\rho}_\tau\right] \qquad (**)$$

- Introduce the Wigner transforms :

$$W_\tau(\mathbf{x},\mathbf{p}) \equiv \int d\mathbf{s}\; e^{i\mathbf{p}\cdot\mathbf{s}}\; \left\langle \mathbf{x}+\frac{\mathbf{s}}{2}\right|\widehat{\rho}_\tau\left|\mathbf{x}-\frac{\mathbf{s}}{2}\right\rangle$$

$$\mathcal{H}(\mathbf{x},\mathbf{p}) \equiv \int d\mathbf{s}\; e^{i\mathbf{p}\cdot\mathbf{s}}\; \left\langle \mathbf{x}+\frac{\mathbf{s}}{2}\right|\widehat{H}\left|\mathbf{x}-\frac{\mathbf{s}}{2}\right\rangle \quad \text{(classical Hamiltonian)}$$

(**) is equivalent to :

$$\frac{\partial W_\tau}{\partial \tau} = \mathcal{H}(\mathbf{x},\mathbf{p})\, \frac{2}{i\hbar}\, \sin\left(\frac{i\hbar}{2}\left(\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{x}} - \overleftarrow{\partial}_{\mathbf{x}}\overrightarrow{\partial}_{\mathbf{p}}\right)\right) W_\tau(\mathbf{x},\mathbf{p})$$

$$= \underbrace{\left\{\mathcal{H}, W_\tau\right\}}_{\text{Poisson bracket}} + \mathcal{O}(\hbar^2)$$

Classical statistical method in Quantum Mechanics

- Quantum effects in the time evolution are $\mathcal{O}(\hbar^2)$ corrections (i.e. they appear at NNLO and beyond)

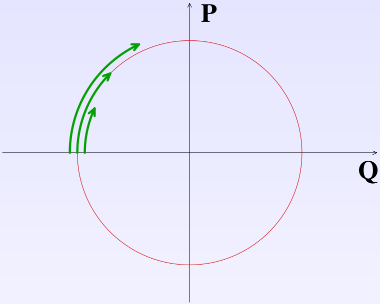
- $\mathcal{O}(\hbar)$ (NLO) contributions can only come from the initial state
  Uncertainty principle : $\Delta\mathbf{x} \cdot \Delta\mathbf{p} \geq \hbar$
  The initial Wigner distribution $W_{\tau=0}(\mathbf{x}, \mathbf{p})$ must have a support of area at least $\hbar$ (minimal area realized by coherent states)

- All the $\mathcal{O}(\hbar)$ effects can be accounted for by a Gaussian initial distribution $W_{\tau=0}(\mathbf{x}, \mathbf{p})$

Classical statistical method

- Sample by a Monte-Carlo the Gaussian distribution that approximates the initial distribution $W_{\tau=0}(\mathbf{x}, \mathbf{p})$
- For each initial $(\mathbf{x}, \mathbf{p})$, solve the classical equation of motion up to the time of interest

Decoherence and micro-canonical equilibration



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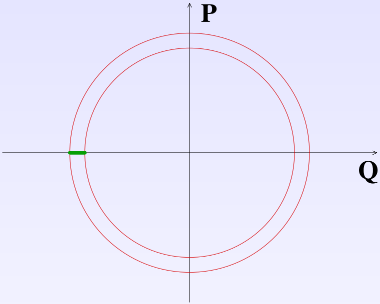

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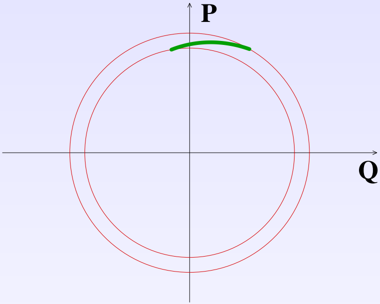

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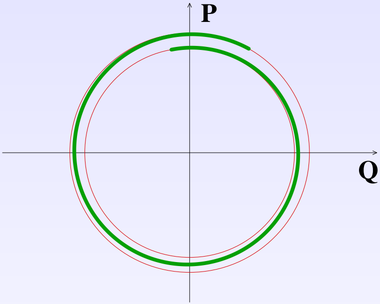

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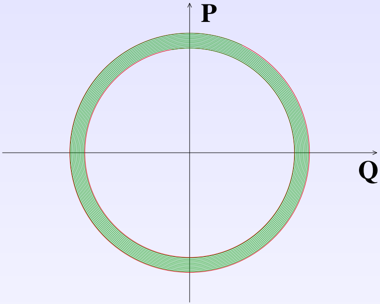

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- Because of QM, the initial ensemble is a set of width $\gtrsim \hbar$
- This ensemble of initial configurations spreads in time
- At large times, the ensemble fills densely all the region allowed by energy conservation $\Rightarrow$ microcanonical equilibrium

Quantum Field Theory w/ Strong Sources

Typical situation

$$\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - V(\phi) + J\phi$$

- In general, the source J is space and time dependent
- The system starts at $t = -\infty$ from a known initial state (example: vacuum state)
- The source may be turned off at some point, and the system evolves by itself afterwards

Strong source : $J \sim$ inverse coupling

- Non-perturbative
  $\Longrightarrow$ can we expand in powers of the coupling?
- Spectrum of produced particles?
- After the sources are switched off : how does the system equilibrate?

Example I : Schwinger mechanism

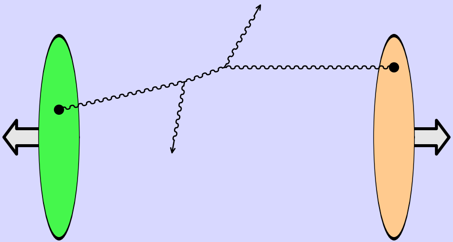
- Consider a constant and uniform electrical field $\vec{E}_{ext}$
- Perturbatively, energy conservation prevents the production of e^+e^- pairs
- Pairs can be produced via a vacuum instability
- Rate : $\exp(-\pi m^2/eE)$ (non analytic in the coupling e)
- In Quantum Field Theory, can be obtained at one loop (but one must use the propagator dressed by the external field)

Example II : Nucleus-Nucleus collisions at high energy

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \underbrace{(J_1^\mu + J_2^\mu)}_{J^\mu} A_\mu$$

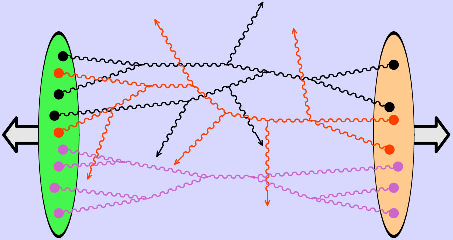
- Given the sources $J_{1,2}$ in each projectile, how do we calculate observables? Is there some kind of perturbative expansion?
- Loop corrections, factorization?
- Thermalization?

Strong source regime



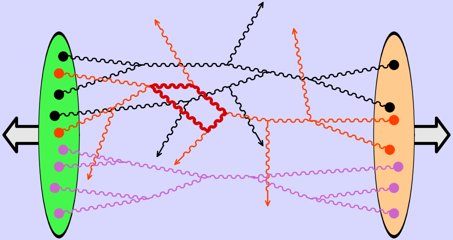
- Weak sources : perturbative treatment

Strong source regime

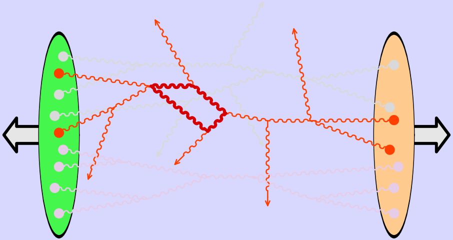


- Weak sources : perturbative treatment
- Strong sources : non-perturbative (when $J \sim 1/g$)

Power counting



Power counting



Order of connected subdiagram when $J \sim g^{-1}$:

$$\frac{1}{g^2} \quad g^{\#\ \text{produced gluons}} \quad g^{2(\#\ \text{loops})}$$

Power counting

- Example : single particle spectrum :

$$\frac{dN_1}{d^3\vec{p}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

- The coefficients $c_0, c_1, \cdots$ are themselves series that resum all orders in $(gJ)^n$. For instance,

$$c_0 = \sum_{n=0}^{\infty} c_{0,n}\,(gJ)^n$$

- We want to calculate at least the entire c_0/g^2 contribution, and a subset of the higher order terms

Inclusive observables

- Inclusive observables do not veto any final state
  Example: moments of the transition probabilities :

$$\frac{dN_1}{d^3\vec{p}} \sim \sum_{n=0}^{\infty} (n+1) \int \frac{1}{(n+1)!} \Big[\underbrace{d\Phi_1 \cdots d\Phi_n}_{n \text{ part. phase-space}} \Big] \Big| \big\langle \mathbf{p}\,\mathbf{p}_1 \cdots \mathbf{p}_{n\,\mathrm{out}} \big| 0_{\mathrm{in}} \big\rangle \Big|^2$$

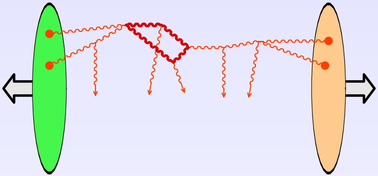
(single inclusive particle distribution)

Equivalent definition :

$$\frac{dN_1}{d^3\vec{p}} \sim \big\langle 0_{\mathrm{in}} \big| a^\dagger_{\mathrm{out}}(\mathbf{p})\, a_{\mathrm{out}}(\mathbf{p}) \big| 0_{\mathrm{in}} \big\rangle$$

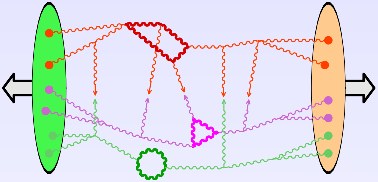
(completeness of the out-states)

Bookkeeping



- Start with transition amplitudes : sources $\rightarrow$ particles

Bookkeeping



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- Consider squared amplitudes (including interferences)

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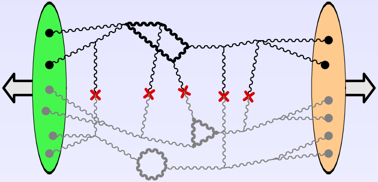

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- See them as cuts through vacuum diagrams
  Cut propagator $\sim \delta(p^2)$

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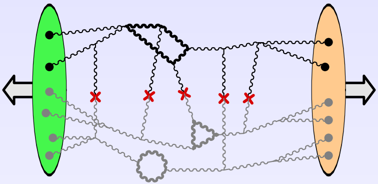

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Weight each cut by $z(p)$ $\longrightarrow$ generating functional

$$F[z] \equiv \sum_n \frac{1}{n!} \int \Big[d\Phi_1 \cdots d\Phi_n \Big] \, z(p_1) \cdots z(p_n) \, \Big| \langle p_1 \cdots p_{n\,\mathrm{out}} | 0_{\mathrm{in}} \rangle \Big|^2$$

Generating functional

- Observables are given by derivatives of $F[z]$, e.g.

$$\frac{dN_1}{d^3\vec{p}} = \left.\frac{\delta F[z]}{\delta z(\mathbf{p})}\right|_{z=1}$$

(inclusive observables are derivatives at the point $z = 1$)

unitarity implies $F[1] = 1$

Exact formula for the first derivative :

$$\frac{\delta \log F[z]}{\delta z(\mathbf{p})} = \int d^4x d^4y \; e^{ip\cdot(x-y)} \; \square_x \square_y \left[\mathcal{A}_+(x)\mathcal{A}_-(y) + \mathcal{G}_{+-}(x,y)\right]$$

where $\mathcal{A}_\pm$ and $\mathcal{G}_{+-}$ are connected 1- and 2-point functions in the Schwinger-Keldysh formalism, with cut propagators weighted by $z(\mathbf{p})$

Schwinger-Keldysh formalism

- Set of Feynman rules to compute directly transition probabilities (i.e. $\mathcal{A}\mathcal{A}^*$)
- This can be achieved as follows :
  - A vertex is $-ig$ on one side of the cut, and $+ig$ on the other side
  - There are four propagators, depending on the location w.r.t. the cut of the vertices they connect :

$$G^0_{++}(p) = i/(p^2 - m^2 + i\epsilon) \qquad \text{(standard Feynman propagator)}$$
$$G^0_{--}(p) = -i/(p^2 - m^2 - i\epsilon) \qquad \text{(complex conjugate of } G^0_{++}(p)\text{)}$$
$$G^0_{+-}(p) = 2\pi\, z(\mathbf{p})\, \theta(-p^0)\delta(p^2 - m^2)$$

  - At each vertex of a given diagram, sum over the types $+$ and $-$ (2^n terms for a diagram with n vertices)

Inclusive Observables at Leading Order

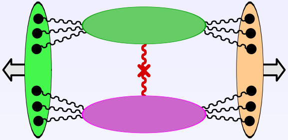
Single inclusive spectrum

- The single inclusive spectrum is given by :

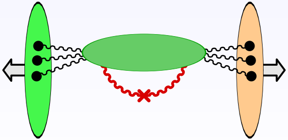
$$\frac{dN_1}{d^3\mathbf{p}} = \frac{\delta F[z]}{\delta(\mathbf{p})}\bigg|_{z=1} = \int d^4x d^4y \; e^{ip\cdot(x-y)} \; \square_x \square_y \Big[A_+(x)A_-(y) + G_{+-}(x,y)\Big]_{z=1}$$

- Two types of terms :

$A_+(x)A_-(y) \quad \longrightarrow$



$G_{+-}(x,y) \quad \longrightarrow$

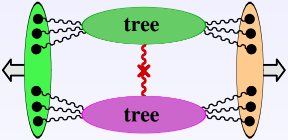


Leading Order

- LO $\equiv$ tree diagrams
  - ▷ the second terms can be ignored

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\left.\frac{dN_1}{d^3\mathbf{p}}\right|_{\rm LO} = \sum_{\rm trees}\sum_{\rm cuts}$$



- Note : at this point, we set $z(\mathbf{p}) = 1$

Expression in terms of classical fields

- When summing over the cuts in a tree diagram, we only get the following combinations of propagators :

$$G^0_{++}(p) - G^0_{+-}(p)$$
$$G^0_{-+}(p) - G^0_{--}(p)$$

Retarded propagator

$$G^0_{++}(p) - G^0_{+-}(p) = G^0_{-+}(p) - G^0_{--}(p) = G^0_{R}(p)$$ (retarded propagator)

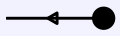
- For any tree diagram contributing to the 1-point functions $A_\pm$, the sum over the $\pm$ indices at the vertices simply transforms all the propagators into retarded propagators

Retarded classical fields

Sum of trees with retarded propagators :

$$\square\mathcal{A} + U'(\mathcal{A}) = j \qquad , \qquad \lim_{x_0\to-\infty}\mathcal{A}(x) = 0$$

- Expansion in powers of J (for cubic interactions) :



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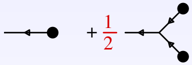

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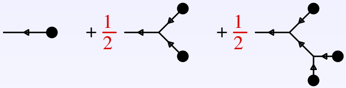

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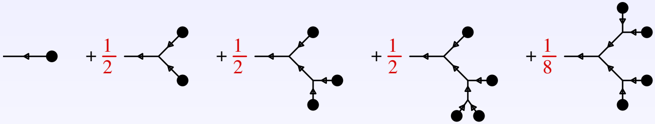

- Built with retarded propagators

- Classical solutions resum the full series of tree diagrams

Inclusive particle spectrum at LO

- The particle spectrum at LO is given by :

$$\left.\frac{dN_1}{d^3\vec{\mathbf{p}}}\right|_{\text{LO}} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y\,\mathcal{A}(x)\mathcal{A}(y)$$

where $\mathcal{A}(x)$ is the classical solution such that $\lim_{x^0\to-\infty}\mathcal{A}(x)=0$

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- NOTE : if the source J is time independent, no particles can be produced at LO (because $\mathcal{A}(x)$ has no time-like Fourier modes)

Next–to–Leading Order

Why the LO may be insufficient ?

- Naive perturbative expansion :

$$\frac{dN}{d^3\vec{p}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

  Note : so far, we have seen how to compute c_0

- The source is time dependent, and particle production is impossible at LO

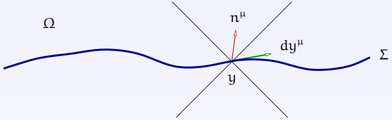
- The description of the projectiles as external sources is valid for modes with large longitudinal momenta $k^\pm > \Lambda$. Loop corrections produce logs of this unphysical cutoff. The logs must be computed and resummed

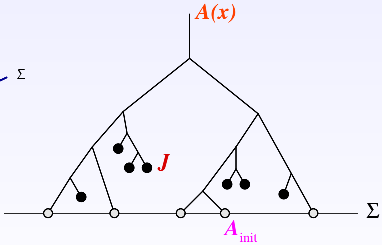
- In QCD and other theories, there are instabilities that cause the coefficients c_n to grow indefinitely with time. These secular terms must be resummed

Cauchy problem for classical fields

Green's formula for classical solutions

$$\mathcal{A}(x) = i \int_{y\in\Omega} G_R^0(x,y)\,\Big[J(y) - V'(\mathcal{A}(y))\Big] + i \int_{y\in\Sigma} G_R^0(x,y)\,(n\cdot \overleftrightarrow{\partial}_y)\,\mathcal{A}_{\rm init}(y)$$





Small perturbations of a classical field

Linearized equation of motion around a classical background

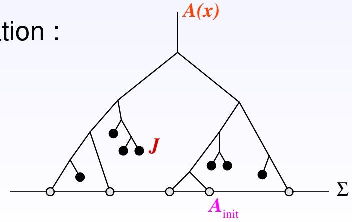
$$\Big[\square_x + V''(\mathcal{A}(x))\Big]\, a(x) = 0 \qquad , \qquad a(x) = \alpha(x) \ \text{ on } \ \Sigma$$

Formal solution

$$\big[\alpha\,\mathbb{T}\big]_y \equiv \alpha(y)\,\frac{\delta}{\delta\mathcal{A}_{\rm init}(y)} + (n\cdot\partial\alpha(y))\,\frac{\delta}{\delta(n\cdot\partial\mathcal{A}_{\rm init}(y))}$$

$$a(x) \equiv \int\limits_{y\in\Sigma} \big[\alpha\,\mathbb{T}\big]_y \ \ \mathcal{A}(x)$$

- Diagrammatic interpretation :



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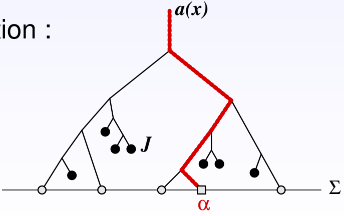


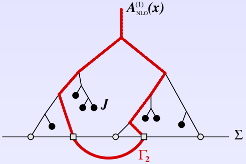
Small perturbations of a classical field

Translation operator for the initial field

$$\mathcal{F}[\mathcal{A}_{\text{initial}} + \alpha] \equiv \exp\left[\int_{\vec{\mathbf{u}}\in\Sigma} \left[\alpha \cdot \mathbb{T}\right]_{\mathbf{u}}\right] \mathcal{F}[\mathcal{A}_{\text{initial}}]$$

- This formula means that $\mathbb{T}_{\mathbf{u}}$ is the generator of shifts of the initial value of the classical field $\mathcal{A}$

Reconstructing the NLO from the LO



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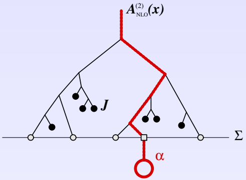
Reconstructing the NLO from the LO



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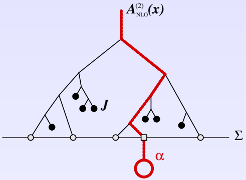

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Single field at NLO

$$A_{\rm NLO} = \left[\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\,\mathbb{T}_u\mathbb{T}_v + \int_u \alpha(u)\,\mathbb{T}_u\right] A_{\rm LO}$$

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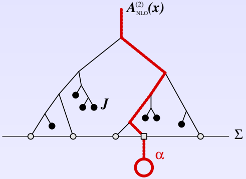

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Single field at NLO

$$\mathcal{A}_{\rm NLO} = \left[\frac{1}{2}\int_{\mathbf{u},\mathbf{v}} \Gamma_2(\mathbf{u},\mathbf{v})\,\mathbb{T}_\mathbf{u}\mathbb{T}_\mathbf{v} + \int_\mathbf{u} \alpha(\mathbf{u})\,\mathbb{T}_\mathbf{u}\right]\mathcal{A}_{\rm LO}$$

Works also for any observable expressible in terms of the field at LO

$$\mathcal{O}_{\rm NLO} = \left[\frac{1}{2}\int_{\mathbf{u},\mathbf{v}} \Gamma_2(\mathbf{u},\mathbf{v})\,\mathbb{T}_\mathbf{u}\mathbb{T}_\mathbf{v} + \int_\mathbf{u} \alpha(\mathbf{u})\,\mathbb{T}_\mathbf{u}\right]\mathcal{O}_{\rm LO}$$

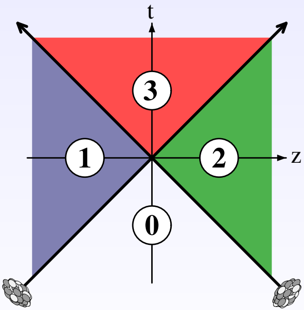
But there is no free lunch...

- For this formula to be true, the functions Γ_2 and α must be determined consistently :
  - Γ_2 = dressed propagator with endpoints on Σ
  - α = 1-point function at 1-loop with endpoint on Σ
- This is almost as hard as doing the NLO calculation !
  (law of conservation of difficulty...)
- In some cases, there is an advantage :
  - If Σ is at $t = -\infty$, then Γ_2 is trivial and $\alpha = 0$
  - When the classical field is simple below Σ and complicated above, then Γ_2 and α are simpler to calculate than the NLO observable above Σ

Example : nucleus-nucleus collisions

- Sources located on the light-cone:

$$J^\mu = \delta^{\mu +} \underbrace{\rho_1(x^-, \boldsymbol{x}_\perp)}_{\sim\delta(x^-)} + \delta^{\mu -} \underbrace{\rho_2(x^+, \boldsymbol{x}_\perp)}_{\sim\delta(x^+)}$$



- Region 0 : $\mathcal{A}^\mu = 0$
- Regions 1,2 : $\mathcal{A}^\mu$ depends only on ρ_1 or ρ_2 (known analytically)
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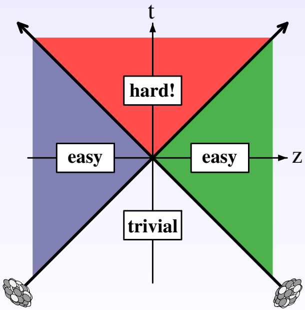

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$\Longrightarrow$ choose Σ just above the forward light-cone

Instabilities and Resummation

Mode decomposition of Γ_2

$$\Gamma_2(x,y) = \int_{\text{modes } \mathbf{k}} \alpha_{\mathbf{k}}(x)\alpha^*_{\mathbf{k}}(y)$$

with

$$\left[\square_x + V''(\mathcal{A}(x))\right] \alpha_{\mathbf{k}}(x) = 0 \qquad , \qquad \lim_{t\to-\infty} \alpha_{\mathbf{k}}(x) = e^{i k\cdot x}$$

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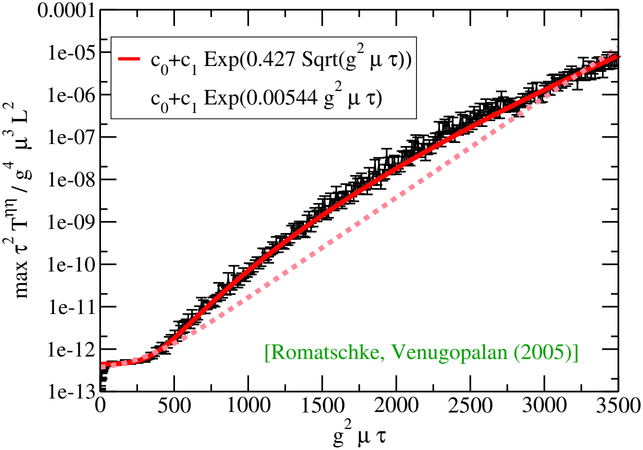
- The equation of motion of the of the mode functions α_k is linear
- Some of the modes can be unstable
- What happens to the NLO observables?

Yang-Mills theory : Weibel instabilities for small perturbations

[Mrowczynski (1988), Romatschke, Strickland (2003), Arnold, Lenaghan, Moore (2003), Rebhan, Romatschke, Strickland (2005), Arnold, Lenaghan, Moore, Yaffe (2005), Romatschke, Rebhan (2006), Bodeker, Rummukainen (2007),...]



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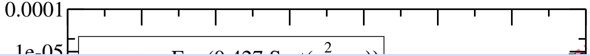


- For some k's, the field fluctuations a_k diverge like $\exp\sqrt{\mu\tau}$ when $\tau \to +\infty$
- Some components of $T^{\mu\nu}$ have secular divergences when evaluated at fixed loop order
- When $a_k \sim \mathcal{A} \sim g^{-1}$, the power counting breaks down and additional contributions must be resummed :

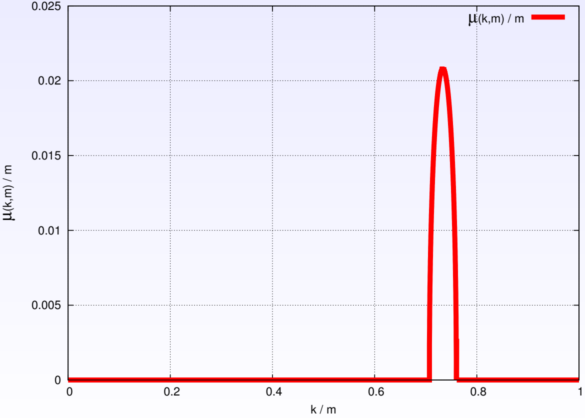
$$g\, e^{\sqrt{\mu\tau}} \sim 1 \quad \text{at} \quad \tau_{max} \sim \mu^{-1} \log^2(g^{-1})$$

$g^2\mu\tau$

ϕ^4 scalar field theory

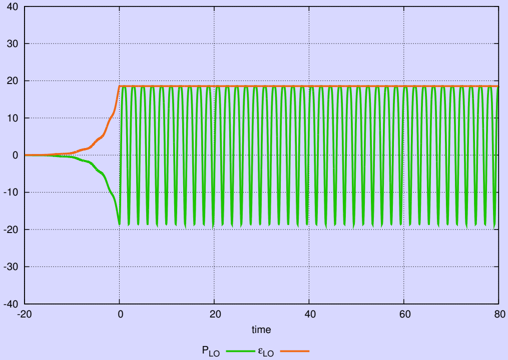
- Lyapunov exponent for the mode k :

$$\mu_k \equiv \frac{1}{T} \ln \left(\frac{a_k(t+T)}{a_k(t)} \right)$$



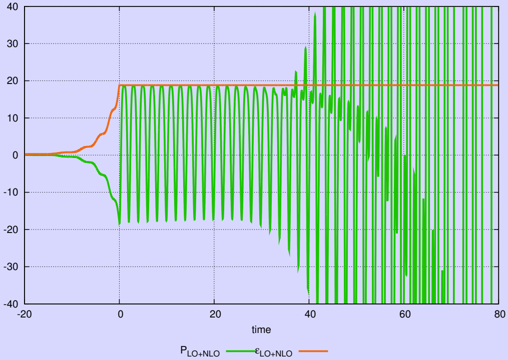
ϕ^4 scalar field theory : pathologies in fixed order calculations

LO



ϕ^4 scalar field theory : pathologies in fixed order calculations

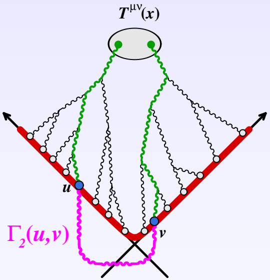
LO + NLO



- Small correction to the energy density
  (protected by energy conservation)
- Secular divergence in the pressure

Improved power counting and resummation

$$\text{Loop} \sim g^2 \quad , \quad \mathbb{T} \sim e^{\sqrt{\mu\tau}}$$



- 1 loop : $(g e^{\sqrt{\mu\tau}})^2$

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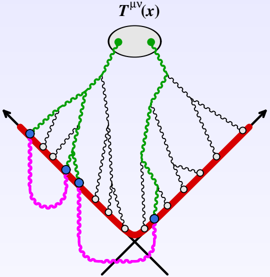

- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$

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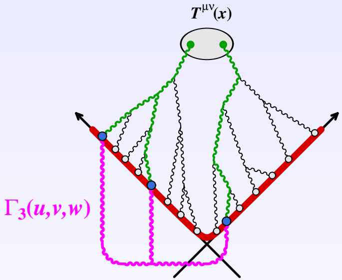


- 1 loop : $(g e^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(g e^{\sqrt{\mu\tau}})^4$
- 2 entangled loops : $g (g e^{\sqrt{\mu\tau}})^3$ $\triangleright$ subleading

Leading terms

- All disconnected loops to all orders
  $\triangleright$ exponentiation of the 1-loop result

Interlude...

$$e^{\frac{\alpha}{2}\partial_x^2}\,f(x) = \int_{-\infty}^{+\infty} dz\,\frac{e^{-z^2/2\alpha}}{\sqrt{2\pi\alpha}}\,f(x+z)$$

Resummation of the leading secular terms

$$T^{\mu\nu}_{\text{resummed}} = \exp\left[\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\,\mathbb{T}_u\mathbb{T}_v\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}]$$

$$= \underbrace{T^{\mu\nu}_{\text{LO}} + T^{\mu\nu}_{\text{NLO}}}_{\text{in full}} + \underbrace{T^{\mu\nu}_{\text{NNLO}} + \cdots}_{\text{partially}}$$

- The exponentiation of the 1-loop result collects all the terms with the worst time behavior

Resummation of the leading secular terms

$$
\begin{aligned}
T^{\mu\nu}_{\text{resummed}} &= \exp\left[\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\mathbb{T}_u\mathbb{T}_v\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}] \\
&= \int [Da]\;\exp\left[-\frac{1}{2}\int_{u,v} a(u)\Gamma_2^{-1}(u,v)a(v)\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}+a]
\end{aligned}
$$

- The exponentiation of the 1-loop result collects all the terms with the worst time behavior
- Equivalent to Gaussian fluctuations of the initial field + classical time evolution

Note : Classical field + Fluctuations = Coherent state

- This Gaussian distribution of initial fields is the Wigner distribution of a **coherent state** $|\mathcal{A}\rangle$

  Coherent states are the "most classical quantum states"

  Their Wigner distribution has the minimal support permitted by the uncertainty principle ($\mathcal{O}(\hbar)$ for each mode)

- $|\mathcal{A}\rangle$ is not an eigenstate of the full Hamiltonian
  $\triangleright$ decoherence via interactions

Example :
Schwinger mechanism

[FG, N. Tanji (2013)]

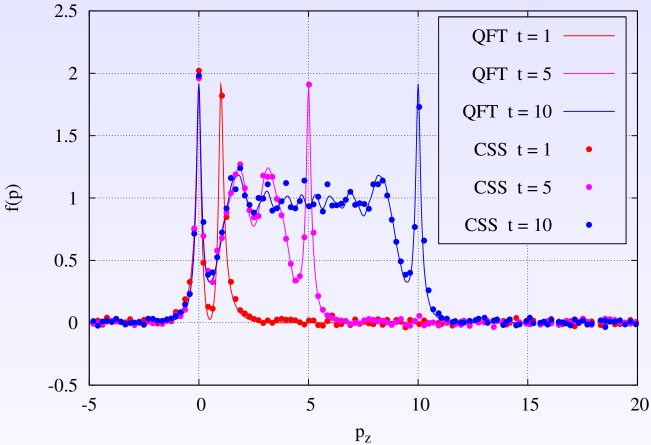
Scalar QED model

$$\mathcal{L} \equiv \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{photons}} + \underbrace{(D_\mu\phi)(D^\mu\phi)^* - m^2\phi^*\phi - \frac{\lambda}{4}(\phi\phi^*)^2}_{\text{charged scalars}} + J^\mu_{ext}A_\mu$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad , \quad D^\mu \equiv \partial^\mu - ieA^\mu \, ,$$

- Two coupling constants :
  e (electrical charge) and λ (self-coupling)
- When the external field is static, no perturbative production
- Non perturbative production $\sim \exp(-\pi m^2/eE_{ext})$

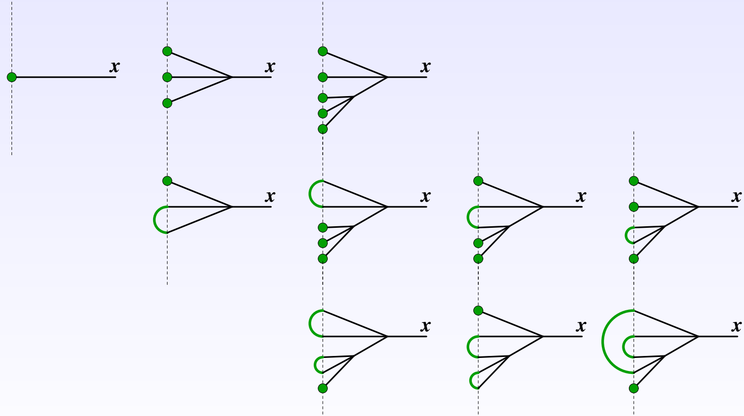
- Comparison with the 1-loop QFT result (for $\lambda = 0$) :



- QFT = 1-loop quantum field theory
- CSS = classical statistical simulation

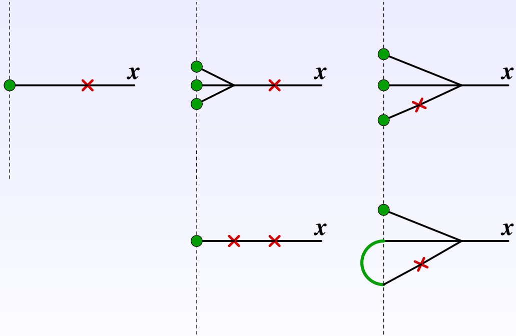
Mass renormalization

- When $\lambda \neq 0$, tadpoles give a quadratic cutoff dependence $\sim \Lambda^2$

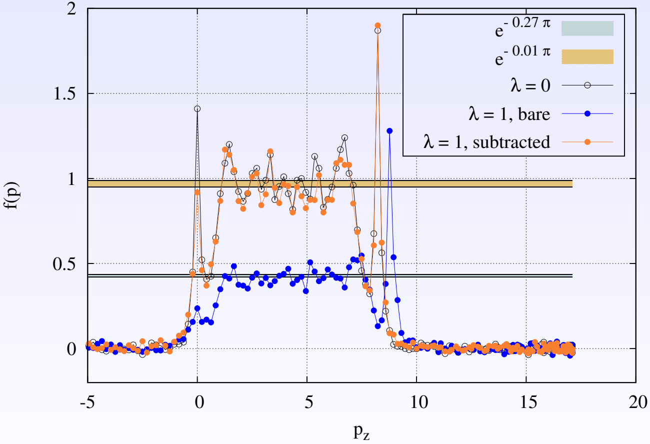


- This can be compensated by a counterterm in the equation of motion :

$$\Big(D_0^2 - \sum_i D_i D_i + m_0^2 + \delta m^2\Big)\,\varphi + \frac{\lambda}{2}(\varphi^*\varphi)\,\varphi = 0\;,$$



- Comparison of bare and mass-renormalized results, for $\lambda = 1$ (at very short time, so that we know that the scalar self-interactions should not have affected the system yet) :



Summary

Summary

- In Quantum Field Theories coupled to strong sources :
  - The LO is expressible in terms of classical fields
  - The NLO can be related to the LO by an operator acting on the initial fields
- When the classical fields are unstable (i.e. in theories where the classical Hamiltonian has chaotic dynamics), the loop expansion is ill behaved
- The terms that have the fastest growth can be summed to all loop orders

  The result of the resummation can be obtained by averaging the LO result over a Gaussian ensemble of initial conditions
- Accuracy : LO + NLO + leading secular terms