流体記述の基礎付けをめざして
--- 粗視化と非線形性 ---

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Contents

The separation of scales in the relativistic heavy-ion collisions

Liouville → Boltzmann → Fluid dyn.

Hamiltonian

Navier-Stokes eq.

Slower dynamics （力学系の縮約）

Early thermalization?

Entropy production mechanism?

Toward a theory of entropy production in the little and big bang

B. Muller, A. Schaefer, A. Ohnishi and T.K.,
PTP 121(2008),555;arXiv:0809.4831(hep-ph)

Two ways of entropy production at the quantum level

1) "entanglement" with the environment

$$S_{\mathrm{rel}} = \mathrm{Tr}\,[\hat{\rho}_{\mathrm{S}} \ln \hat{\rho}_{\mathrm{S}}] \qquad \text{with} \qquad \hat{\rho}_{\mathrm{S}} = \mathrm{Tr}_{\mathrm{E}}\,\hat{\rho}.$$

Loss of information due to coupling with environment.

2) Entropy production in an isolated system,
such as in the early universe and the initial satge of H-I collisions

The time evolution $\exp[-iHt]$ is a unitary transformation;

Difficult to produce entropy!

$$|\psi(t)\rangle = \exp[-iHt]\,|\psi\rangle$$

$$\rho = |\psi\rangle\langle\psi| \longrightarrow |\psi(t)\rangle\langle\psi(t)|$$

$$S = -\mathrm{Tr}[\rho \log \rho] \longrightarrow \text{不変}$$

In classical level,
Kolmogorov-Sinai entropy λ describes the rate of entropy production.

V. Latora and M.Baranger('99)

λ= The sum of positive Lyapunov exponents : $|\delta X_i(t)| \propto e^{\lambda_i t},$

Chaos ⇨ Entropy production

Notice:

The essential role of the coarse graining (averaging of orbits)

How about in Quantum Mechanics?

How implement a coarse graining in Quantum Mechanics?

Distribution function in Quantum Mechanics

The Wigner function $$W(p,x;t) = \int du\, e^{\frac{i}{\hbar}pu} \langle x - \frac{u}{2} |\, \hat{\rho}(t)\, | x + \frac{u}{2} \rangle$$

$$i\hbar \frac{\partial}{\partial t} \hat{\rho}(t) = [\hat{\mathcal{H}}, \hat{\rho}(t)]$$

It can be negative and pure quantum mechanical object, hence no ability of describing entropy production.

The need of incorporation of coarse graining which inevitably enters through the observation process.

A choice; **Husimi function** K. Husimi (1940)

$$H_\Delta(p,x;t) \equiv \int \frac{dp'\, dx'}{\pi\hbar} \exp\left(-\frac{1}{\hbar\Delta}(p-p')^2 - \frac{\Delta}{\hbar}(x-x')^2 \right) W(p',x';t)$$

最小不確定性の分だけ粗視化された分布関数

$$\int \frac{dp\, dx}{2\pi\hbar} W(p,x;t) = \int \frac{dp\, dx}{2\pi\hbar} H_\Delta(p,x;t) = 1.$$

伏見関数の性質

$$\int \frac{dp\,dx}{2\pi\hbar} W(p,x;t) = \int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t) = 1.$$

Positive(-semi) definite ← Coherent state; minimal uncertainty **(coarse graining!)**

c.f. A. Sugita (2001, 2002)

$$H_\Delta(p,x;t) = \langle z_\Delta|\hat{\rho}(t)|z_\Delta\rangle \qquad \hat{a}_\Delta|z_\Delta\rangle = z_\Delta|z_\Delta\rangle \qquad \hat{a}_\Delta = \frac{\Delta\hat{x} + i\hat{p}}{\sqrt{2\hbar\Delta}}$$

Entropy may be defined as

$$S_{\mathrm{H},\Delta}(t) = -\int \frac{dp\,dx}{2\pi\hbar} H_\Delta(p,x;t) \ln H_\Delta(p,x;t)$$

(Husimi-Wehrl entropy)

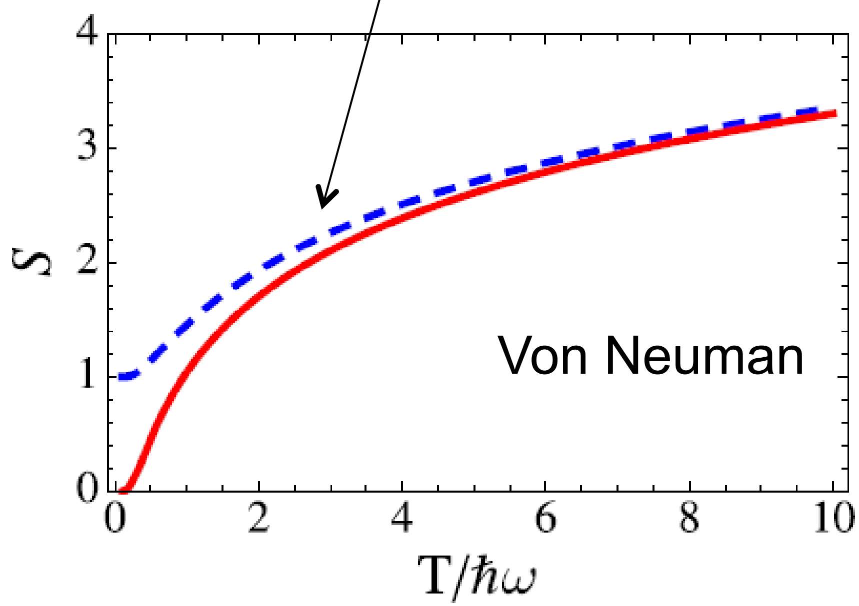
$T \gg$hv

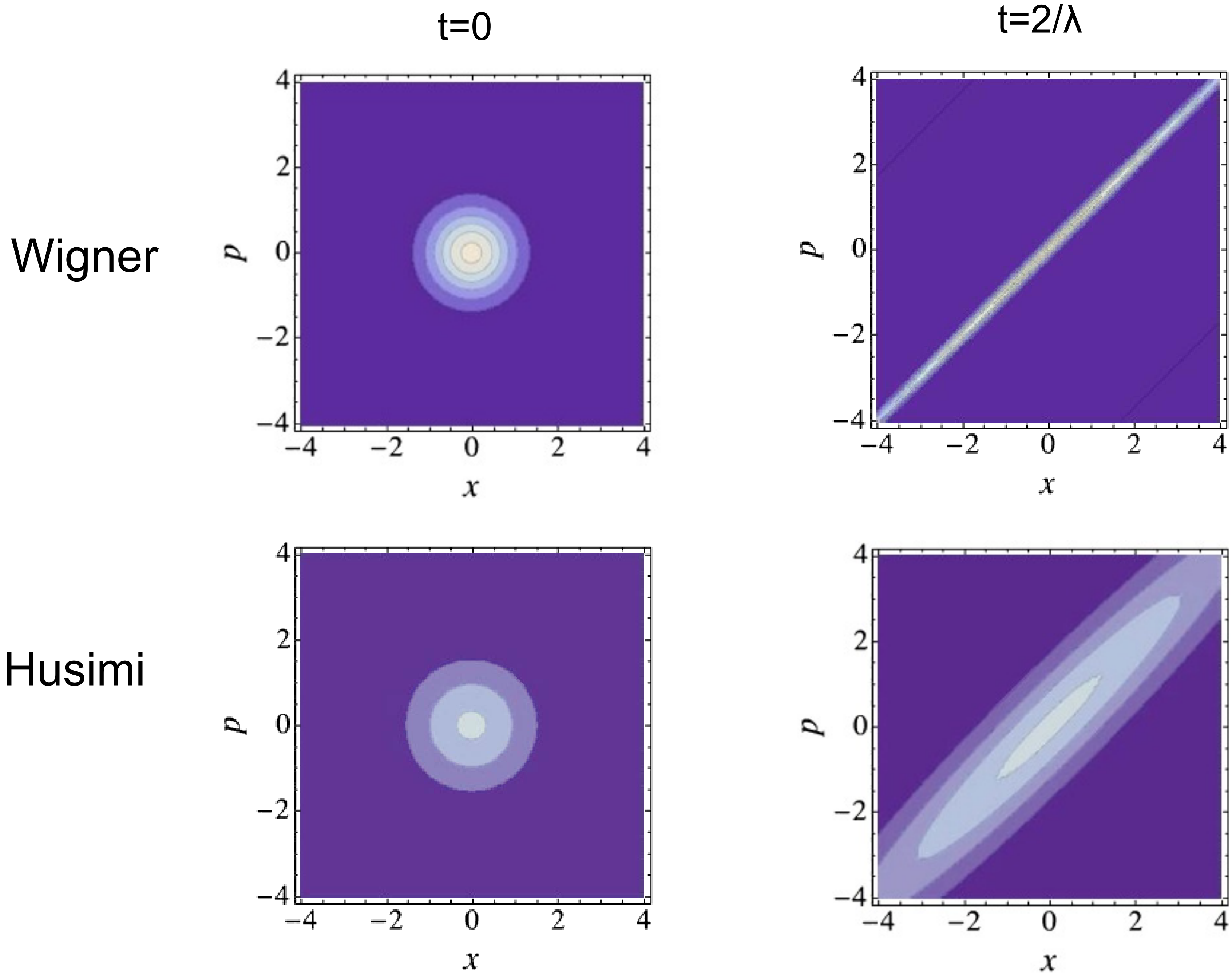
エネルギー揺らぎ小

⇩

Husimi-Wehrl = von Neuman

B. Muller, A. Schaefer,
A. Ohnishi and T.K.,
PTP 121(2008),555




t=0
t=2/λ
Wigner
Husimi
p
x
4
2
0
−2
−4

A simple example with an instability;

$$\hat{\mathcal{H}} = \frac{1}{2}\hat{p}^2 - \frac{1}{2}\lambda^2\hat{x}^2.$$

$$\langle x|\psi_0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\omega x^2/2\hbar}$$

$$H_\Delta(p,x;t) = \frac{2}{\sqrt{A(t)}} \exp\left[-\frac{1}{\hbar A(t)}\left(K(p,x;t) + \frac{p^2}{\Delta} + \Delta x^2\right)\right]$$

$$K(p,x;t) = \frac{p^2}{\lambda}(\sigma\cosh 2\lambda t + \delta) + \lambda x^2(\sigma\cosh 2\lambda t - \delta) - 2\sigma\, p\, x \sinh 2\lambda t.$$

$$A(t) = 2(\sigma\rho\cosh 2\lambda t + 1 + \delta\delta'). \qquad \rho = \frac{\Delta^2 + \lambda^2}{2\Delta\lambda} \geq 1, \qquad \delta' = \frac{\Delta^2 - \lambda^2}{2\Delta\lambda}$$

The Wehrl entropy;

$$S_{\mathrm{H},\Delta}(t) = \frac{1}{2}\ln\frac{A(t)}{4} + 1$$

The growth rate;

$$\frac{dS_{\mathrm{H},\Delta}}{dt} = \frac{\lambda\sigma\rho\sinh 2\lambda t}{\sigma\rho\cosh 2\lambda t + 1 + \delta\delta'} \xrightarrow{t\to\infty} \lambda$$

, independent of Δ

The growth rate of the Husimi-Wehrl entropy is given by the K-S entropy (positive Lyapunov exponent) in the classical dynamics!

Extension to many-body systems:

$$\hat{\mathcal{H}} = \sum_k \frac{1}{2}\left(\hat{p}_k^2 - \lambda_k^2 \hat{x}_k^2\right)$$

$$S_{\mathrm{H},\Delta}(t) = \sum_k S_{\mathrm{H},\Delta}^{(k)}(t)$$

$$\frac{dS_{\mathrm{H},\Delta}}{dt} = \sum_k \frac{\lambda_k \sinh 2\lambda_k t}{\cosh 2\lambda_k t + (1+\delta\delta')\sigma^{-1}\rho^{-1}} \quad \xrightarrow{t\to\infty} \sum_k \lambda_k .$$

Unstable modes in the classical dynamics plays the essential role for entropy production at quantum level.

⇒ may account for entropy production in quantum level in HI collisions at RHIC,
as well as the reheating in the early universe.

古典系の不安定モードがエントロピー生成率を決める

Entropy growth rate of classical Yang-Mills fields

CYM: Mueller, Ohnishi, Schaefer, Takahashi, Yamamoto, TK, PRD82 (2010)

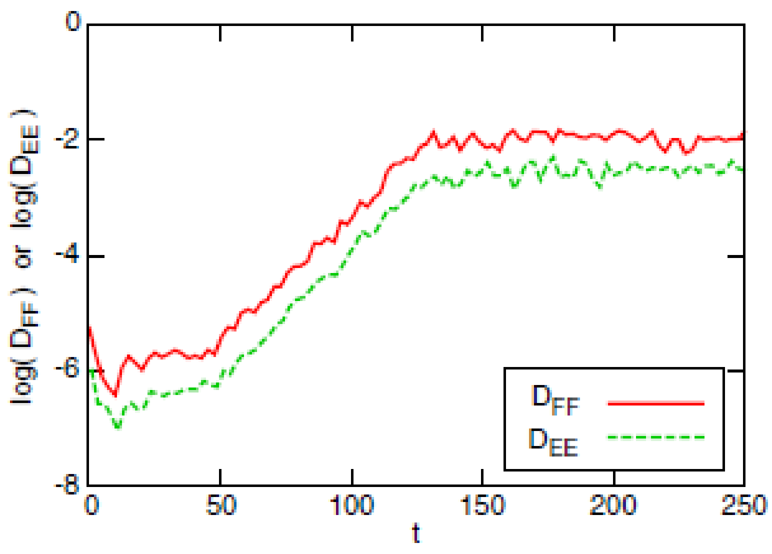
$$H = \frac{1}{2}\sum_{x,a,i} E_i^a(x)^2 + \frac{1}{4}\sum_{x,a,i,j} F_{ij}^a(x)^2\,,$$

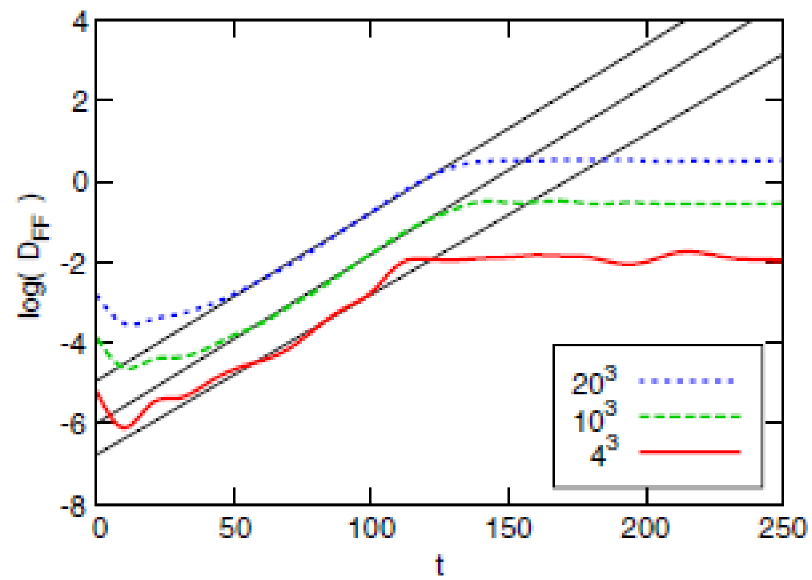
$$F_{ij}^a(x) = \partial_i A_j^a(x) - \partial_j A_i^a(x) + \sum_{b,c} f^{abc} A_i^b(x) A_j^c(x)$$

$$\dot{A}_i^a(x) = E_i^a(x)\,,$$

$$\dot{E}_i^a(x) = \sum_j \partial_j F_{ji}^a(x) + \sum_{b,c,j} f^{abc} A_j^b(x) F_{ji}^c(x)$$

$$D_{EE} = \sqrt{\sum_x \left\{ \sum_{a,i} E_i^a(x)^2 - \sum_{a,i} E_i'^a(x)^2 \right\}^2}, \quad D_{FF} = \sqrt{\sum_x \left\{ \sum_{a,i,j} F_{ij}^a(x)^2 - \sum_{a,i,j} F_{ij}'^a(x)^2 \right\}^2}.$$





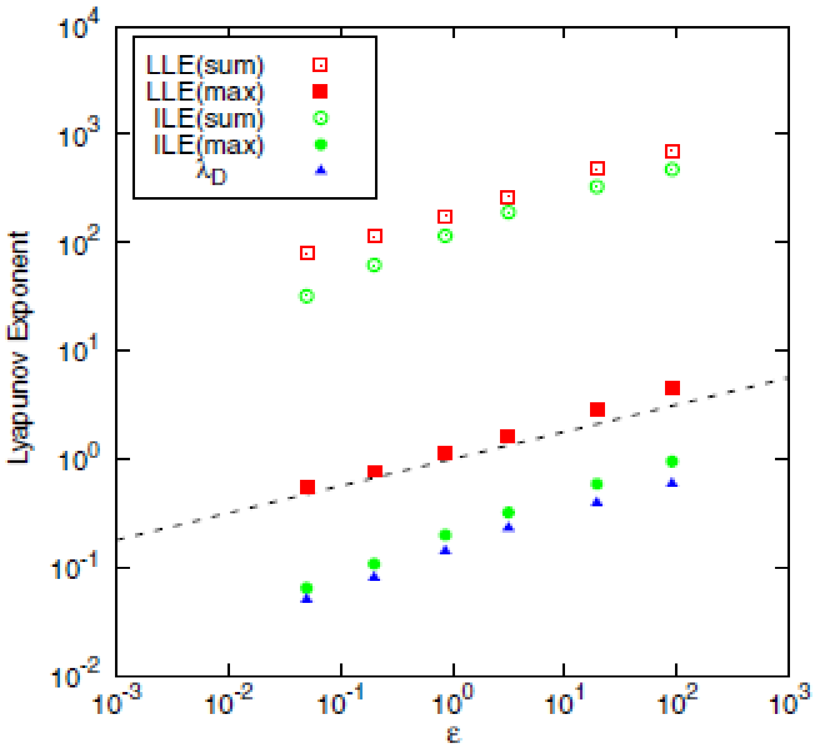


FIG. 8 (color online). The SU(3) results of the Lyapunov exponents, λ_D, $\lambda_{\max}^{\rm LLE}$, $\lambda_{\rm sum}^{\rm LLE}$, $\lambda_{\max}^{\rm ILE}$, and $\lambda_{\rm sum}^{\rm ILE}$. The broken line is $\varepsilon^{1/4}$.

$$\lambda_D \simeq 0.1 \times \varepsilon^{1/4},$$

$$\lambda_{\max}^{\rm LLE} \simeq 1 \times \varepsilon^{1/4},$$

$$\lambda_{\rm sum}^{\rm LLE}/L^3 \simeq 3 \times \varepsilon^{1/4},$$

$$\lambda_{\max}^{\rm ILE} \simeq 0.2 \times \varepsilon^{1/4},$$

$$\lambda_{\rm sum}^{\rm ILE}/L^3 \simeq 2 \times \varepsilon^{1/4}.$$

$\tau_{eq} \simeq$ 2-3 fm/c

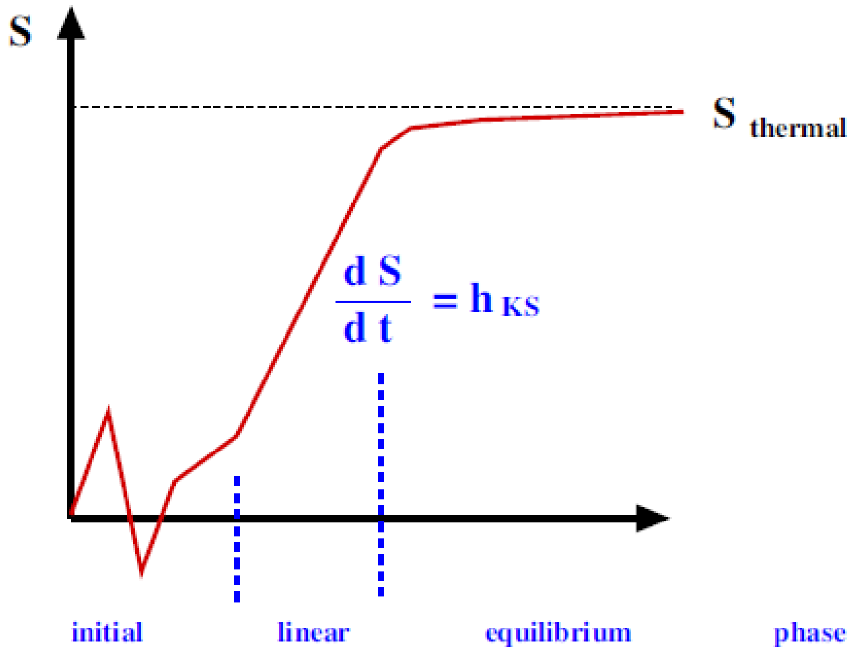
See ,Mueller, et al, PRD82 (2010)

Further development:

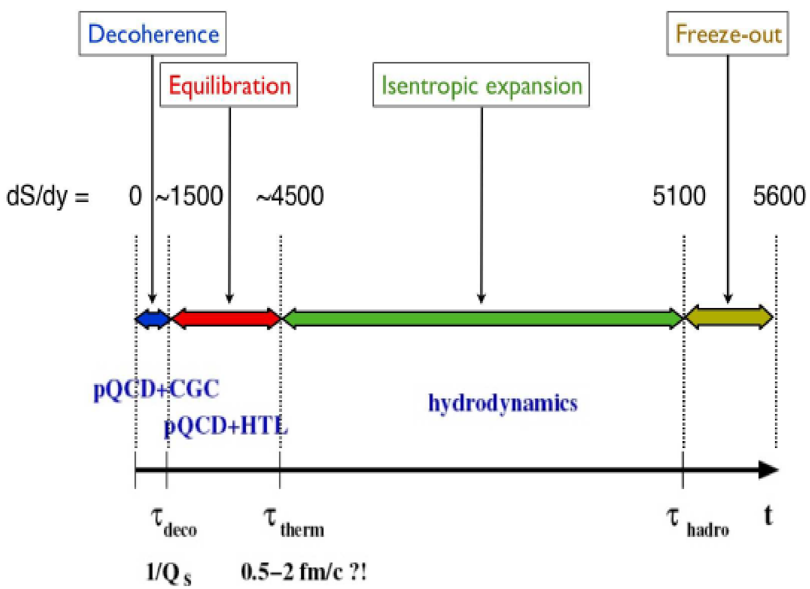
:H. Iida et al, in progress

Initial condition: CGC with randomness
Back ground: Expanding back ground

Entropy production at each stage



B.Muller and A. Schaefer,
Int. J. Mod. Phys. E20, 2235 (2011)



R.J. Fries et al,
arXiv 0906.5293

From Boltzmann to Hydrodynamic equation

K. Tsumura, K. Ohnishi and T.K., Phys. Lett. B646 (2007) 134;
K. Tsumura and T.K., Phys. Lett. B668 (2008) 425;
K. Tsumura and T.K., Prog. Theor. Phys. 126 (2011), 761.

Introduction

The separation of scales in the relativistic heavy-ion collisions

Liouville → Boltzmann → Fluid dyn.

Hamiltonian (BBGKY hierarchy)

One-body dist. fn.

T,n, u^{μ}

Slower dynamics

Fewer d.o.f

Hydrodynamics is the effective dynamics with fewer variables of the kinetic (Boltzmann) equation in the infrared refime.

Basic notions for reduction of dynamics

Averaging: Time-derivative in transport coeff. Is an average Of microscopic derivatives.

Def of coarse-grained differentiation (H. Mori, 1956, 1858, 1959)

$$\frac{\delta}{\delta t}\langle F\rangle(t) \equiv \frac{1}{\tau}\{\langle F\rangle(t+\tau) - \langle F\rangle(t)\} = \frac{1}{\tau}\int_0^\tau ds \frac{d}{ds}\langle F\rangle(t+s)$$

τ an intermidiate scale time

Construction of the invariant manifold

Set-up of Initial condition: ⇐ **invariant (or attractive) manifold**

Eg: Boltzmann: mol. chaos → Take I.C. with no two-body correl..

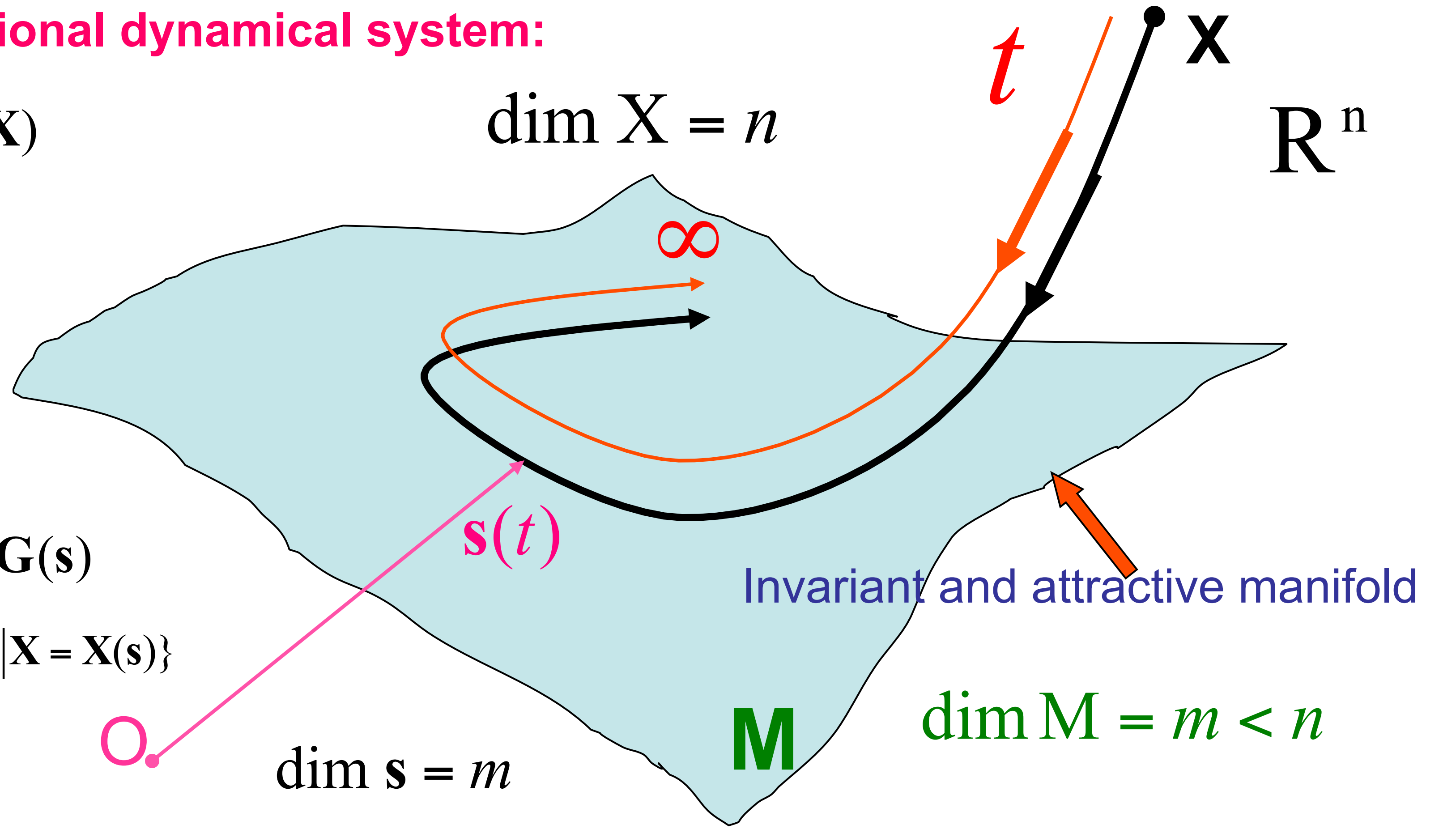
Bogoliubov (1946), Kubo et al (Iwanami, Springer)
J.L. Lebowitz, Physica A 194 (1993),1.
K. Kawasaki (Asakura, 2000), chap. 7.

Geometrical image of reduction of dynamics

n-dimensional dynamical system:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$

$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M} = \{\mathbf{X} \mid \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$



eg.

In Field theory, $\mathbf{X} = (g_1, g_2, \ldots, g_n) \equiv \boldsymbol{g} \rightarrow \mathbf{s} = (s_1, s_2, \ldots, s_m)$

renormalizable ⟺ $\exists$ **Invariant manifold M** dim M $= m < n \leq \infty$

Relativistic Boltzmann equation

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x),$$

Collision integral: $C[f]_p(x) \equiv \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p\,,\,p_1|p_2\,,\,p_3) \left(f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x) \right),$

Symm. property of the transition probability:

$$\omega(p\,,\,p_1|p_2\,,\,p_3) = \omega(p_2\,,\,p_3|p\,,\,p_1) = \omega(p_1\,,\,p|p_3\,,\,p_2) = \omega(p_3\,,\,p_2|p_1\,,\,p) \qquad \text{--- (1)}$$

Energy-mom. conservation; $\omega(p\,,\,p_1|p_2\,,\,p_3) \propto \delta^4(p + p_1 - p_2 - p_3)$ --- (2)

Owing to (1),

$$\sum_p \frac{1}{p^0} \varphi_p(x) C[f]_p(x) = \frac{1}{2!} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \frac{1}{4} \Big[\omega(p\,,\,p_1|p_2\,,\,p_3) \left(\varphi_p(x) + \varphi_{p_1}(x) - \varphi_{p_2}(x) - \varphi_{p_3}(x) \right) \times \left(f_{p_2}(x) f_{p_3}(x) - f_p(x) f_{p_1}(x) \right) \Big]. \qquad (3)$$

Collision Invariant $\varphi_p(x)$ **:** $\sum_p \frac{1}{p^0} \varphi_p(x) C[f]_p(x) = 0,$

Eq.'s (3) and (2) tell us that

the general form of a collision invariant; $\varphi_p(x) = \alpha(x) + p^\mu \beta_\mu(x),$

which can be x-dependent!

Ambiguities of the definition of the flow and the LRF

In the kinetic approach, one needs conditions of fit or matching conditions., irrespective of Chapman-Enskog or Maxwell-Grad moment methods:

In the literature, the following plausible ansatz are taken;

$$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu = \epsilon_0 \equiv u_\mu T_0^{\mu\nu} u_\nu$$
$$n \equiv u \cdot N = n_0 \equiv u \cdot N_0$$

de Groot et al (1980),
Cercignani and Kremer (2002)

Is this always correct, irrespective of the frames?

Particle frame is the same local equilibrium state as the energy frame?
Note that the distribution function in non-eq. state should be quite different from that in eq. state.

Eg. $\exists$ the bulk viscosity

Local equilibrium $\longrightarrow$ No dissipation!

D. H. Rischke, nucl-th/9809044

Previous attempts to derive the dissipative hydrodynamics as a reduction of the dynamics

N.G. van Kampen, J. Stat. Phys. 46(1987), 709

unique but non-covariant form and hence not Landau either Eckart!

Cf. Chapman-Enskog method to derive Landau and Eckart eq.'s; see, eg, de Groot et al ('80)

Here,

In the covariant formalism,
in a unified way and systematically
derive dissipative rel. hydrodynamics at once!

Derivation of the relativistic hydrodynamic equation from the rel. Boltzmann eq. --- an RG-reduction of the dynamics

K. Tsumura, T.K. K. Ohnishi;   Phys. Lett. B646 (2007) 134-140

c.f. Non-rel. Y.Hatta and T.K., Ann. Phys. 298 ('02), 24; T.K. and K. Tsumura, J.Phys. A:39 (2006), 8089

Ansatz of the origin of the dissipation= the spatial inhomogeneity,
leading to Navier-Stokes in the non-rel. case .

$\boldsymbol{a}_p^\mu$ would become a macro flow-velocity ⟵ **Coarse graining of space-time**

$\boldsymbol{a}_p^\mu$ may not be u^μ

$$\tau \equiv \boldsymbol{a}_p^\mu x_\mu, \quad \sigma^\mu \equiv \left(g^{\mu\nu} - \frac{\boldsymbol{a}_p^\mu \boldsymbol{a}_p^\nu}{\boldsymbol{a}_p^2} \right) x_\nu \equiv \boldsymbol{\Delta}_p^{\mu\nu} x_\nu \qquad x^\mu \Longrightarrow \tau \quad \sigma^\mu$$

$$\frac{\partial}{\partial \tau} = \frac{1}{\boldsymbol{a}_p^2} \boldsymbol{a}_p^\mu \partial_\mu \equiv \boldsymbol{D},$$ time-like derivative

$$\boldsymbol{\Delta}_p^{\mu\nu} \frac{\partial}{\partial \sigma^\nu} = \boldsymbol{\Delta}_p^{\mu\nu} \partial_\nu \equiv \boldsymbol{\nabla}^\mu$$ space-like derivative

Rewrite the Boltzmann equation as,

$$\Longrightarrow \quad \frac{\partial}{\partial \tau} f_p(\tau\,,\,\sigma) = \frac{1}{p \cdot \boldsymbol{a}_p} C[f]_p(\tau\,,\,\sigma) - \frac{1}{p \cdot \boldsymbol{a}_p} p \cdot \boldsymbol{\nabla} f_p(\tau\,,\,\sigma)$$

↑ perturbation

Only spatial inhomogeneity leads to dissipation.

RG gives a resummed distribution function, from which $T^{\mu\nu}$ and N^μ are obtained.

Chen-Goldenfeld-Oono('95), T.K.('95), **S.-I. Ei, K. Fujii and T.K. (2000)**

Solution by the perturbation theory

0th

$$\frac{\partial}{\partial \tau}\tilde{f}_p^{(0)} = \frac{1}{p\cdot \boldsymbol{a}_p} C[f]_p \Big|_{f=\tilde{f}^{(0)}}$$

"slow" $\frac{\partial}{\partial \tau}\tilde{f}_p^{(0)} = 0$ → $\frac{1}{p\cdot \boldsymbol{a}_p} C[f]_p \Big|_{f=\tilde{f}^{(0)}} = 0$

→ $\tilde{f}_p^{(0)}(\tau\,,\,\sigma\,;\,\tau_0) = (2\pi)^{-3}\exp\left[\frac{\mu(\sigma\,;\,\tau_0) - p^\mu\, u_\mu(\sigma\,;\,\tau_0)}{T(\sigma\,;\,\tau_0)}\right] \equiv f_p^{\rm eq}(\sigma\,;\,\tau_0)$

→ $\tilde{f}^{(0)}(\tau) \;=\; f^{\rm eq}$

written in terms of the hydrodynamic variables.
Asymptotically, the solution can be written solely in terms of the hydrodynamic variables.

- Five conserved quantities $T(\sigma\,;\,\tau_0),\;\; \mu(\sigma\,;\,\tau_0),\;\; u_\mu(\sigma\,;\,\tau_0)$

reduced degrees of freedom m = 5 $u^\mu(\sigma\,;\,\tau_0)\, u_\mu(\sigma\,;\,\tau_0) = 1$

- 0th invariant manifold $f_p^{(0)}(\tau_0\,,\,\sigma) = f_p^{\rm eq}(\sigma\,;\,\tau_0)$

→ $f^{(0)}(\tau_0) \;=\; f^{\rm eq}$

Local equilibrium

1st

$$\frac{\partial}{\partial \tau} \tilde{f}_p^{(1)} = \sum_q A_{pq} \tilde{f}_q^{(1)} + F_p$$

Evolution op.：
$$A_{pq} \equiv \frac{1}{p \cdot \boldsymbol{a}_p} \frac{\partial}{\partial f_q} C[f]_p \Big|_{f=f^{\mathrm{eq}}}$$

inhomogeneous：
$$F_p \equiv -\frac{1}{p \cdot \boldsymbol{a}_p} p \cdot \boldsymbol{\nabla} f_p^{\mathrm{eq}}$$

Collision operator
$$L_{pq} \equiv f_p^{\mathrm{eq}\text{-}1} A_{pq} f_q^{\mathrm{eq}}$$

$$L_{pq} = -\frac{1}{p \cdot \boldsymbol{a}_p} \frac{1}{2!} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p\,,\, p_1 | p_2\,,\, p_3)\, f_{p_1}^{\mathrm{eq}} \left(\delta_{pq} + \delta_{p_1 q} - \delta_{p_2 q} - \delta_{p_3 q} \right)$$

The lin. op. L has good properties:

Def. **inner product:** $\langle \varphi\,,\, \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot \boldsymbol{a}_p)\, f_p^{\mathrm{eq}}\, \varphi_p\, \psi_p$

1. $\langle \varphi\,,\, L\,\psi \rangle = \langle L\,\varphi\,,\, \psi \rangle$ — Self-adjoint

2. $\langle \varphi\,,\, L\,\varphi \rangle \leq 0$ for all φ — Semi-negative definite

3. $L\,\varphi_0^\alpha = 0$ ⇒ $\varphi_{0p}^\alpha = \begin{cases} p^\mu & \alpha = \mu, \\ m & \alpha = 4 \end{cases}$

L has 5 zero modes、other eigenvalues are negative.

1. Proof of self-adjointness

$$\begin{aligned}\langle \varphi , L\psi \rangle &= \sum_{pq} \frac{1}{p^0} (p \cdot a_p) f_p^{eq} \varphi_p L_{pq} \psi_q \\ &= -\frac{1}{4}\frac{1}{2!} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p , p_1 | p_2 , p_3) \\ &\qquad f_p^{eq} f_{p_1}^{eq} \left(\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}\right) \left(\psi_p + \psi_{p_1} - \psi_{p_2} - \psi_{p_3}\right) \\ &= \langle L\varphi , \psi \rangle.\end{aligned}$$

2. Semi-negativeness of the L

$$\begin{aligned}\langle \varphi , L\varphi \rangle &= -\frac{1}{4}\frac{1}{2!} \sum_p \frac{1}{p^0} \sum_{p_1} \frac{1}{p_1^0} \sum_{p_2} \frac{1}{p_2^0} \sum_{p_3} \frac{1}{p_3^0} \omega(p , p_1 | p_2 , p_3) f_p^{eq} f_{p_1}^{eq} \left(\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3}\right)^2 \\ &\leq 0 \text{ for all } \varphi\end{aligned}$$

3.Zero modes

$$\varphi_{0p}^{\alpha} = \begin{cases} p^{\mu} & \alpha = \mu \\ m & \alpha = 4 \end{cases}$$

en-mom.

Particle #

$$\varphi_p + \varphi_{p_1} = \varphi_{p_2} + \varphi_{p_3}$$

Collision invariants!
or conserved quantities.

Def. Projection operators:

$$\begin{cases} \left[P\,\psi\right]_p \equiv \sum_{\alpha\beta} \varphi_{0p}^{\alpha}\,\eta^{-1}_{\alpha\beta}\,\langle\,\varphi_0^{\beta}\,,\,\psi\,\rangle, \\ Q \equiv 1 - P. \end{cases}$$

metric

$$\eta^{\alpha\beta} \equiv \langle\,\varphi_0^{\alpha}\,,\,\varphi_0^{\beta}\,\rangle$$

$$\eta^{-1}_{\alpha\beta}\ ;\ \sum_{\gamma} \eta^{\alpha\gamma}\,\eta^{-1}_{\gamma\beta} = \delta^{\alpha}_{\beta}$$

$$\frac{\partial}{\partial\tau}\tilde{f}^{(1)} = A\,\tilde{f}^{(1)} + F$$

$$\Rightarrow\ \tilde{f}^{(1)}(\tau) = e^{(\tau-\tau_0)A}\left\{\underline{f^{(1)}(\tau_0)} + \underline{A^{-1}\,\bar{Q}\,F}\right\} + (\tau-\tau_0)\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F.$$

The initial value yet not determined

fast motion to be avoided

$$\bar{P} \equiv f^{\mathrm{eq}}\,P\,f^{\mathrm{eq}-1},\qquad \bar{Q} \equiv f^{\mathrm{eq}}\,Q\,f^{\mathrm{eq}-1}.$$

$$f^{\mathrm{eq}}_{pq} \equiv f^{\mathrm{eq}}_p\,\delta_{pq}$$

$$\Rightarrow\ \tilde{f}^{(1)}(\tau) = (\tau-\tau_0)\,\bar{P}\,F - A^{-1}\,\bar{Q}\,F$$

eliminated by the choice

Modification of the manifold: $f^{(1)}(\tau_0) = -A^{-1}\,\bar{Q}\,F$

Second order solutions

$$\frac{\partial}{\partial \tau}\tilde{f}^{(2)} = A\,\tilde{f}^{(2)} + I$$ with $$I_p \equiv \frac{1}{p\cdot \boldsymbol{a}_p}\, p\cdot \boldsymbol{\nabla}\left[A^{-1}\,\bar{Q}\,F\right]_p$$

$$\Rightarrow \tilde{f}^{(2)}(\tau) = e^{(\tau-\tau_0)A}\left\{f^{(2)}(\tau_0) + A^{-1}\,\bar{Q}\,I\right\} + (\tau-\tau_0)\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I$$

The initial value not yet determined

fast motion

$$\Rightarrow \tilde{f}^{(2)}(\tau) = (\tau-\tau_0)\,\bar{P}\,I - A^{-1}\,\bar{Q}\,I.$$

eliminated by the choice

- Modification of the invariant manifold in the 2nd order; $f^{(2)}(\tau_0) = -A^{-1}\,\bar{Q}\,I,$

Application of RG/E equation to derive slow dynamics

Collecting all the terms, we have;

- Invariant manifold (hydro dynamical coordinates) as the initial value:

$$f(\tau_0) = f^{\rm eq} + \varepsilon\Big(-A^{-1}\bar{Q}\,F\Big) + \varepsilon^2\Big(-A^{-1}\bar{Q}\,I\Big) + O(\varepsilon^3),$$

- The perturbative solution with secular terms:

$$\tilde{f}(\tau) = f^{\rm eq} + \varepsilon\Big(\underline{(\tau-\tau_0)}\,\bar{P}\,F - A^{-1}\bar{Q}\,F\Big) + \varepsilon^2\Big(\underline{(\tau-\tau_0)}\,\bar{P}\,I - A^{-1}\bar{Q}\,I\Big) + O(\varepsilon^3).$$

RG/E equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau_0}\tilde{f}_p(\tau\,,\,\sigma\,;\,\tau_0)\Big|_{\tau_0=\tau} = 0,$$

The meaning of $\tau_0 = \tau$ ⇨ found to be the coarse graining condition

The novel feature in the relativistic case;

Choice of the flow a_p^μ ; **eg.** $a_p^\mu = u^\mu$

$$\partial_\mu J^{\mu\alpha}_{\rm hydro} = 0,$$

$$J^{\mu\alpha}_{\rm hydro} \equiv \sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} \left\{ f^{\rm eq}_p - \left[A^{-1} \bar{Q} F\right]_p \right\} = J^{\mu\alpha}_0 + \Delta J^{\mu\alpha},$$

$$J^{\mu\alpha}_0 \equiv \sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} f^{\rm eq}_p$$

$$\Delta J^{\mu\alpha} \equiv -\sum_p \frac{1}{p^0} p^\mu \varphi^\alpha_{0p} \left[A^{-1} \bar{Q} F\right]_p$$

→ produce the dissipative terms!

The distribution function;

$$f(\tau_0) = f^{\rm eq} - A^{-1} \bar{Q} F - A^{-2} \bar{Q} H - A^{-1} \bar{Q} I$$

Notice that the distribution function as the solution is represented solely by the hydrodynamic quantities!

A generic form of the flow vector

$$\boldsymbol{a}_p^{\mu} = \frac{1}{p \cdot u}\Big((p \cdot u)\,\cos\theta + m\,\sin\theta\Big)\,u^{\mu} \equiv \theta_p^{\mu}$$

θ : a parameter

- $\boldsymbol{\Delta}_p^{\mu\nu} = g^{\mu\nu} - u^{\mu}\,u^{\nu} \equiv \Delta^{\mu\nu}$, $\Delta^{\mu}{}_{\rho}\Delta^{\rho\nu} = \Delta^{\mu\nu}$
- $\boldsymbol{D} = u^{\mu}\,\partial_{\mu} \equiv D$, $\boldsymbol{\nabla}^{\mu} = \Delta^{\mu\nu}\,\partial_{\nu} \equiv \nabla^{\mu}$
- $\langle\,\varphi\,,\,\psi\,\rangle = \sum_p \frac{1}{p^0}\Big((p \cdot u)\,\cos\theta + m\,\sin\theta\Big)\,f_p^{\mathrm{eq}}\,\varphi_p\,\psi_p \equiv \langle\,\varphi\,,\,\psi\,\rangle_{\theta}$

- **Projection op. onto space-like traceless second-rank tensor;**

$$P^{\mu\nu\rho\sigma} \equiv \frac{1}{2}\left(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\sigma} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)$$

$$P^{\mu\nu\alpha\beta}P_{\alpha\beta}{}^{\rho\sigma} = P^{\mu\nu\rho\sigma}$$

Examples

$$\theta = 0$$

$$\Leftrightarrow \boldsymbol{a}_p^{\mu} = u^{\mu}$$

$$\partial_\mu J^{\mu\alpha}_{\text{hydro.}} = 0 \qquad p \equiv nT$$

$$\Delta J^{\mu\alpha} = \begin{cases} -\zeta\,\Delta^{\mu\nu} X + 2\,\eta\, X^{\mu\nu} & \alpha = \nu \\ -T\,\lambda\, z\,\hat{h}^{-1}\, X^{\mu} & \alpha = 4. \end{cases}$$

→ satisfies the Landau constraints

$$u_\mu u_\nu \delta T^{\mu\nu} = 0, u_\mu \Delta_{\sigma\nu}\delta T^{\mu\nu} = 0$$

$$u_\mu \delta N^{\mu} = 0$$

$$X \equiv -\nabla_\mu u^\mu,$$

$$X_\mu \equiv \nabla_\mu \ln T - \hat{h}^{-1}\nabla_\mu \ln(nT),$$

$$X_{\mu\nu} \equiv \frac{1}{2}\left(\Delta_{\mu\rho}\Delta_{\nu\sigma} + \Delta_{\mu\sigma}\Delta_{\nu\rho} - \frac{2}{3}\Delta_{\mu\nu}\Delta_{\rho\sigma}\right)\nabla^\rho u^\sigma.$$

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - (p + \zeta X)\,\Delta^{\mu\nu} + 2\,\eta\, X^{\mu\nu}$$

$$N^\mu = n\,u^\mu - \lambda\,\frac{nT}{\epsilon + p} X^\mu.$$

Landau frame and Landau eq.!

with the microscopic expressions for the transport coefficients;

Bulk viscosity $$\zeta \equiv -\frac{1}{T}\sum_{pq}\frac{1}{p^0}f_p^{\rm eq}\,\Pi_p\,\mathcal{L}_{pq}^{-1}\,\Pi_q$$

Heat conductivity $$\lambda \equiv -\frac{1}{3}\frac{1}{T^2}\sum_{pq}\frac{1}{p^0}f_p^{\rm eq}\,Q_p^{\mu}\,\mathcal{L}_{pq}^{-1}\,Q_{\mu q}$$

Shear viscosity $$\eta \equiv -\frac{1}{10}\frac{1}{T}\sum_{pq}\frac{1}{p^0}f_p^{\rm eq}\,\Pi_p^{\mu\nu}\,\mathcal{L}_{pq}^{-1}\,\Pi_{\mu\nu q}$$

$$\mathcal{L}_{pq} \equiv (p\cdot\theta_p)\,L_{pq}$$ ⟸ θ_p -independent

c.f. $$L_{pq} = -\frac{1}{p\cdot a_p}\frac{1}{2!}\sum_{p_1}\frac{1}{p_1^0}\sum_{p_2}\frac{1}{p_2^0}\sum_{p_3}\frac{1}{p_3^0}\,\omega(p\,,\,p_1|p_2\,,\,p_3)\,f_{p_1}^{\rm eq}\left(\delta_{pq}+\delta_{p_1q}-\delta_{p_2q}-\delta_{p_3q}\right)$$

$(a_p{}^{\mu} = \theta_p{}^{\mu})$

In a Kubo-type form;

$$\zeta \equiv \frac{1}{T}\int_0^\infty {\rm d}s\,\langle\,\Pi(0)\,,\,\Pi(s)\,\rangle_{\rm eq},$$

$$\lambda \equiv -\frac{1}{3}\frac{1}{T^2}\int_0^\infty {\rm d}s\,\langle\,Q^{\mu}(0)\,,\,Q_{\mu}(s)\,\rangle_{\rm eq},$$

$$\eta \equiv \frac{1}{10}\frac{1}{T}\int_0^\infty {\rm d}s\,\langle\,\Pi^{\mu\nu}(0)\,,\,\Pi_{\mu\nu}(s)\,\rangle_{\rm eq}.$$

$$\left[\Pi(s)\right]_p \equiv \sum_q \left[{\rm e}^{s\,\mathcal{L}}\right]_{pq}\Pi_q$$

$$\langle\,\varphi\,,\,\psi\,\rangle_{\rm eq} \equiv \sum_p \frac{1}{p^0}f_p^{\rm eq}\,\varphi_p\,\psi_p$$

C.f. Bulk viscosity may play a role in determining the acceleration of the expansion of the universe, and hence the dark energy!

Eckart (particle-flow) frame:

Landau equation: $a_p^\mu = u^\mu$

Setting $a_p^\mu = \frac{m}{p\cdot u} u^\mu$

$$T^{\mu\nu} = (\epsilon + 3\,\zeta\,\tilde{X})\,u^\mu\,u^\nu - (p + \zeta\,\tilde{X})\,\Delta^{\mu\nu} + \lambda T\,u^\mu\,\tilde{X}^\nu + \lambda T\,u^\nu\,\tilde{X}^\mu + 2\,\eta\,X^{\mu\nu}$$

$$N^\mu = m\,n\,u^\mu$$

i.e., $\delta N^\mu = 0.$

with

$$\tilde{X} \equiv -\{1/3\,(4/3-\gamma)^{-1}\}^2\,\nabla\cdot u$$

$$\tilde{X}^\mu \equiv \nabla^\mu \ln T$$

(i) This satisfies the GMS constraints but not the Eckart's.
(ii) Notice that only the space-like derivative is incorporated.
(iii) This form is different from Eckart's and Grad-Marle-Stewart's, both of which involve the time-like derivative.

Eckart's constraints :
$$\begin{cases} 1.\ u_\mu\,u_\nu\,\delta T^{\mu\nu} = 0, \\ 2.\ u_\mu\,\delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\,\delta N^\nu = 0, \end{cases}$$

⟺

Grad-Marle-Stewart constraints
$$\begin{cases} 5.\ T^\mu{}_\mu = 0, \\ 2.\ u_\mu\,\delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\,\delta N^\nu = 0. \end{cases}$$

c.f. Grad-Marle-Stewart equation;

$$\delta T^{\mu\nu} = -3\,(3T^{-1}\,C_T+1)^{-1}\,\zeta\,u^\mu\,u^\nu\,\nabla\cdot u + u^\mu\,T\,\lambda\left(\frac{1}{T}\,\nabla^\nu T - Du^\nu\right) + u^\nu\,T\,\lambda\left(\frac{1}{T}\,\nabla^\mu T - Du^\mu\right)$$
$$+\,2\,\eta\,\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\,\Delta^{\mu\nu}\,\nabla\cdot u\right) + (3T^{-1}\,C_T+1)^{-1}\,\zeta\,\Delta^{\mu\nu}\,\nabla\cdot u,$$
$$\delta N^\mu = 0.$$

Conditions of fit v.s. orthogonality condition

Preliminaries:

Collision operator $L_{pq} \equiv f_p^{\text{eq}-1} A_{pq} f_q^{\text{eq}}$ $\quad A_{pq} \equiv \frac{1}{p \cdot \boldsymbol{a}_p} \frac{\partial}{\partial f_q} C[f]_p \Big|_{f=f^{\text{eq}}}$

L has 5 zero modes:

$$L \varphi_0^\alpha = 0 \qquad \varphi_{0p}^\alpha = \begin{cases} p^\mu & \alpha = \mu, \\ m & \alpha = 4 \end{cases}$$

The dissipative part; $-[A^{-1} \dot{Q} F]_p = f_p^{\text{eq}} \phi_p$

with $\phi_p \equiv -[L^{-1} Q f^{\text{eq}-1} F]_p$

due to the Q operator.

$$\langle \varphi_0^\alpha, \phi \rangle = 0 \ \text{ for } \ \alpha = 0, 1, 2, 3, 4$$

where $\langle \varphi, \psi \rangle \equiv \sum_p \frac{1}{p^0} (p \cdot \boldsymbol{a}_p) f_p^{\text{eq}} \varphi_p \psi_p$

$\Delta J^{\mu\alpha} \equiv -\sum_p \frac{1}{p^0}\, p^\mu\, \varphi_{0p}^\alpha \left[A^{-1}\bar{Q}\,F\right]_p$ $\langle \varphi\,,\,\psi\rangle \equiv \sum_p \frac{1}{p^0}\,(p\cdot a_p)\, f_p^{\rm eq}\, \varphi_p\, \psi_p.$

The orthogonality condition due to the projection operator exactly corresponds to the constraints for the dissipative part of the energy-momentum tensor and the particle current!

(A) $a_p^\mu = u^\mu$, i.e., Landau frame,

$$\langle \varphi_0^\alpha\,,\,\phi\rangle = 0 \quad \Rightarrow \quad \sum_p \frac{1}{p^0}\,(p\cdot u)\, f_p^{\rm eq}\, \varphi_p^\alpha\, \phi_p = 0$$

$p\cdot u = p^\mu u_\mu$

Matching condition!

$$\begin{cases} u_\nu\,\delta J^{\mu\nu} = 0 \;\Longrightarrow\; u_\mu\, u_\nu\, \delta J^{\mu\nu} = 0,\;\; \Delta_{\mu\rho}\, u_\nu\, \delta J^{\mu\nu} = 0, \\ u_\mu\,\delta J^{\mu 4} = 0, \end{cases}$$

(B) $a_p^\mu = \frac{m}{p\cdot u}u^\mu$, i.e., the Eckart frame, → $(p\cdot a_p) = \text{const.}$,

$$\langle \varphi_0^\alpha\,,\,\phi\rangle = 0 \quad \Rightarrow \quad \sum_p \frac{1}{p^0}\, m\, f_p^{\rm eq}\, \varphi_p^\alpha\, \phi_p = 0$$

$\alpha = 0, 1, 2, 3,$ $\quad \delta J^{\mu 4} = 0 \;\Longrightarrow\; u_\mu\,\delta J$

$\alpha = 4,$ $\quad \delta J^\mu{}_\mu = 0 \quad m^2 =$

Eckart's constraints :

$$\begin{cases} 1.\;\; u_\mu\, u_\nu\, \delta T^{\mu\nu} = 0, \\ 2.\;\; u_\mu\, \delta N^\mu = 0, \\ 3.\;\; \Delta_{\mu\nu}\, \delta N^\nu = 0, \end{cases}$$

(C) there exists no a_p^μ meeting the Eckart's constr…, 1, 2 and 3

Constraints 2, 3 ⟹ $(p\cdot a_p) = \text{const.}$,

Constraint 1 ⟹ $(p\cdot a_p) = \text{const.} \times (p\cdot u)^2.$

Contradiction!

See next page.

(C) there exists no a_p^μ meeting the Eckart's constraints, 1, 2 and 3

$$\text{Eckart's constraints :} \quad \begin{cases} 1.\ u_\mu\, u_\nu\, \delta T^{\mu\nu} = 0, \\ 2.\ u_\mu\, \delta N^\mu = 0, \\ 3.\ \Delta_{\mu\nu}\, \delta N^\nu = 0, \end{cases}$$

Constraints 2, 3 ⟹ $(p \cdot a_p) = \text{const.},$

Constraint 1 ⟹ $(p \cdot a_p) = \text{const.} \times (p \cdot u)^2.$

Contradiction!

Which equation is better, Stewart et al's or ours?

The linear stability analysis around the thermal equilibrium state.

c.f. Ladau equation is stable. (Hiscock and Lindblom ('85))

The stability of the equations in the "Eckart(particle)" frame

K.Tsumura and T.K. ;
Phys. Lett. B 668, 425 (2008).;
arXiv:1107.1519

See also, Y. Minami and T.K.,
Prog. Theor. Phys.122, 881 (2010)

The stability of the solutions in the particle frame:

K.Tsumura and T.K. (2008)

(i) The Eckart and Grad-Marle-Stewart equations gives an instability, which has been known, and is now found to be attributed to the fluctuation-induced dissipation, proportional to Du^{μ}

(ii) Our equation (TKO equation) seems to be stable, being dependent on the values of the transport coefficients and the EOS.

The numerical analysis shows that, the solution to our equation is stable at least for rarefied gasses.

A comment:

The stability of our equations derived with the RG method is proved to be stable without recourse to any numerical calculations; this is a consequence of the positive-definiteness of the inner product. (K. Tsumura and T.K., (2011))

Summary of second-half part

- Eckart equation, which and a causal extension of which are widely used, is not compatible with the underlying Relativistic Boltsmann equation.
- The RG method gives a consistent fluid dynamical equation for the particle (Eckart) frame as well as other frames, which is new and has no time-like derivative for thermal forces.
- The linear analysis shows that the new equation in the Eckart (particle) frame can be stable in contrast to the Eckart and (Grad)-Marle-Stewart equations which involve dissipative terms proportional to Du^{μ} .
- The RG method is a mechanical way for the construction of the invariant manifold of the dynamics and can be applied to derive a causal fluid dynamics, a la Grad 14-moment method. (K. Tsumura and T.K. , in prep.)
- According to the present analysis, even the causal (Israel-Stewart) equation which is an extension of Eckart equation should be modified.
- There are still many fundamental isseus to clarify for establishing the relativistic fluid dynamics for a viscous fluid.

Brief Summary

1. 孤立量子系におけるエントロピー生成を記述する枠組みとして、伏見関数を用いることを提案した。
2. 不安定量子系においては、Husimi-Wehrl エントロピーの増大率は古典系のリャプーノフ指数(Kolmogorov-Sinaiエントロピー)によって与えられる。
3. 古典Yang-Mills系はカオス系であり、ランダムな初期状態から出発しても、リャプーノフ指数は増大し飽和する。
4. 「カラー凝縮+乱雑ゆらぎ」とする初期状態から出発しても、全体としての傾向は変わらない:初期時間と後期において特性は少し異なる。
5. 運動学的方程式から出発して散逸を含む相対論的流体方程式を導出した。
   1次，2次の方程式　――＞新しいモーメント法

(K. Tsumura and T.K., in preparation)

Back Ups

Generic example with zero modes

S.Ei, K. Fujii & T.K. Ann. Phys.('00)

国広悌二、物理学会誌2010年9月号

$$\partial_t \mathbf{u} = A\mathbf{u} + \epsilon \mathbf{F}(\mathbf{u})$$

$|\epsilon| < 1$

$$\mathbf{u}(t;t_0) = \mathbf{u}_0(t;t_0) + \epsilon \mathbf{u}_1(t;t_0) + \epsilon^2 \mathbf{u}_2(t;t_0) + \cdots$$

$$\begin{aligned} \mathbf{W}(t_0) &= \mathbf{W}_0(t_0) + \epsilon \mathbf{W}_1(t_0) + \epsilon^2 \mathbf{W}_2(t_0) + \cdots, \\ &= \mathbf{W}_0(t_0) + \boldsymbol{\rho}(t_0), \end{aligned}$$

$$\begin{aligned} (\partial_t - A)\mathbf{u}_0 &= 0, \\ (\partial_t - A)\mathbf{u}_1 &= \mathbf{F}(\mathbf{u}_0), \\ (\partial_t - A)\mathbf{u}_2 &= \mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1, \end{aligned}$$

$$(\mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1)_i = \sum_{j=1}^{n} \{\partial(F'(\mathbf{u}_0))_i / \partial(u_0)_j\} (u_1)_j$$

When A has semi-simple zero eigenvalues.

$$A\mathrm{U}_i = 0, \quad (i = 1, 2, \ldots, m).$$

We suppose that other eigenvalues have negative real parts;

$$A\mathrm{U}_\alpha = \lambda_\alpha \mathrm{U}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n),$$

where $\mathrm{Re}\lambda_\alpha < 0$. One may assume without loss of generality that U_i's and U_α's are linearly independent.

The adjoint operator $A^\dagger$ has the same eigenvalues as A has;

$$A^\dagger \tilde{\mathrm{U}}_i = 0, \quad (i = 1, 2, \ldots, m),$$
$$A^\dagger \tilde{\mathrm{U}}_\alpha = \lambda_\alpha^* \tilde{\mathrm{U}}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n).$$

Def. P **the projection onto the kernel** $\ker A$

$$P + Q = 1$$

Since we are interested in the asymptotic state as $t \to \infty$, we may assume that the lowest-order initial value belongs to $\ker A$:

$$\mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i = \mathbf{W}_0[\boldsymbol{C}]. \Longleftrightarrow \mathrm{M}_0$$

$$\mathbf{u}_0(t;t_0) = \mathrm{e}^{(t-t_0)A}\mathbf{W}_0(t_0) = \textstyle\sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i.$$

Parameterized with m variables,
Instead of n!

$$\boldsymbol{C} = {}^t(C_1, C_2, \cdots, C_m)$$

$$\begin{aligned}\mathbf{u}_1(t;t_0) = {} & \mathrm{e}^{(t-t_0)A}[\mathbf{W}_1(t_0) + A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0))] \\ & +(t-t_0)P\mathbf{F}(\mathbf{W}_0(t_0)) - A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)).\end{aligned}$$

The would-be rapidly changing terms can be eliminated by the choice;

$$\mathbf{W}_1(t_0) = -A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)), \qquad P\mathbf{W}_1(t_0) = 0$$

Then, the secular term appears only the P space;

$$\mathbf{u}_1(t;t_0) = (t-t_0)P\mathbf{F} - A^{-1}Q\mathbf{F}$$

a deformation of the manifold ρ

Deformed (invariant) slow manifold: $M_1 = \{\mathbf{u} | \mathbf{u} = \mathbf{W}_0 - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0)\}$

$$\mathbf{u}(t; t_0) = \mathbf{W}_0 + \epsilon\{(t - t_0) P\mathbf{F} - A^{-1} Q \mathbf{F}\}$$

A set of locally divergent functions parameterized by t_0 !

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$ gives the envelope, which is globally valid:

$$\dot{\mathbf{W}}_0(t) = \epsilon P \mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \sum_{i=1}^{m} C_i(t) \mathbf{U}_i - \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C]).$$

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

Extension; (a) A Is not semi-simple. (2) Higher orders.

(Ei, Fujii and T.K. Ann.Phys.('00))

Layered pulse dynamics for TDGL and NLS.

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$

gives the envelope, which is globally valid:

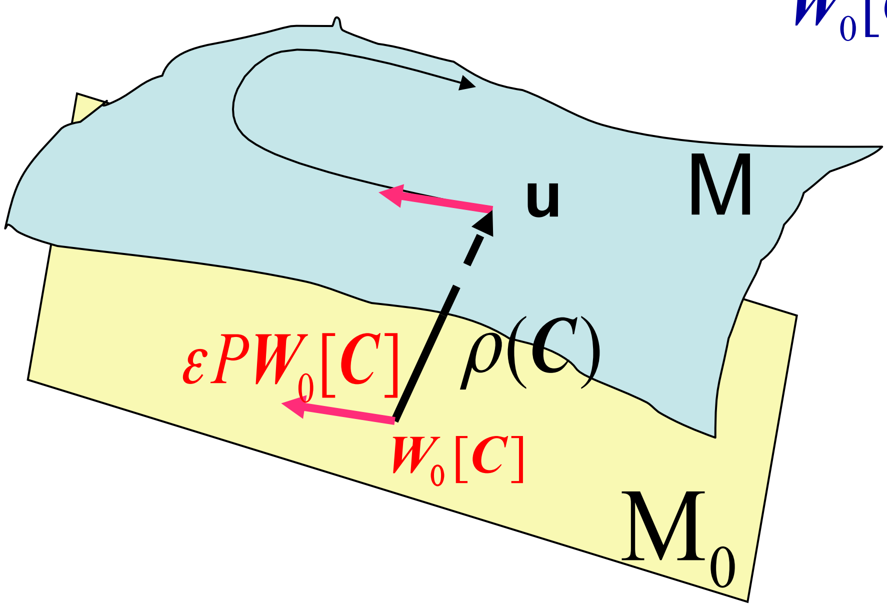
$$\dot{\mathbf{W}}_0(t) = \epsilon P \mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \underbrace{\sum_{i=1}^{m} C_i(t) \mathbf{U}_i}_{W_0[C]} \underbrace{- \epsilon A^{-1} Q \mathbf{F}(\mathbf{W}_0[C])}_{\rho(C)}$$



c.f. Polchinski theorem
in renormalization theory
In QFT.